

**ON THE GENERAL SOLUTION OF A  
 FUNCTIONAL EQUATION CONNECTED TO SUM  
 FORM INFORMATION MEASURES ON OPEN DOMAIN — III**

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(Received August 6, 1985 and in revised form April 30, 1986)

**ABSTRACT.** In this series, this paper is devoted to the study of a functional equation connected with the characterization of weighted entropy and weighted entropy of degree  $\beta$ . Here, we find the general solution of the functional equation (2) on an open domain, without using 0-probability and 1-probability.

**KEY WORDS AND PHRASES.** Functional equation, weighted entropy, weighted entropy of degree  $\beta$ , open domain, sum form.

**1980 MATHEMATICS SUBJECT CLASSIFICATION CODE:** 94A15, 94A17, 39B40.

1. INTRODUCTION.

Let  $\Gamma_n^0 = \{P = (p_1, p_2, \dots, p_n) \mid 0 < p_j < 1, \sum_{k=1}^n p_k = 1\}$  and  $\Gamma_n$  be the closure of  $\Gamma_n^0$ . Let  $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ , where  $\mathbb{R}$  is the set of real numbers. Let  $(\Omega, A, \mu)$  be a probability space and let us consider an experiment that is a finite measurable partition  $\{A_1, A_2, \dots, A_n\}$ , ( $n > 1$ ) of  $\Omega$ . The weighted entropy of such an experiment is defined by Belis and Guiasu [1] as

$$H_n^1(P, U) = - \sum_{k=1}^n u_k p_k \log p_k$$

where  $p_k = \mu(A_k)$  is the objective probability of the event  $A_k$ ,

$P = (p_1, p_2, \dots, p_n) \in \Gamma_n$  and  $U = (u_1, u_2, \dots, u_n) \in \mathbb{R}_+^n$ . The weighed entropy of degree  $\beta$  ( $\beta \in \mathbb{R} - \{1\}$ ) of an experiment is defined by Emptoz [2] as

$$H_n^\beta(P, U) = (1-2^{1-\beta})^{-1} \sum_{k=1}^n u_k (p_k - p_k^\beta).$$

The measures  $H_n^1(P, U)$  satisfy the following functional equation (see Kannappan [3])

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j, u_i v_j) = \sum_{i=1}^n p_i u_i \cdot \sum_{j=1}^m f(q_j, v_j) + \sum_{j=1}^m q_j v_j \cdot \sum_{i=1}^n f(p_i, u_i) \quad (1.1)$$

for all  $P \in \Gamma_n$ ,  $Q \in \Gamma_m$ ,  $u_i, v_j \in \mathbb{R}_+$ . A generalization of (1) is the following:

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j, u_i v_j) = \sum_{i=1}^n p_i^\alpha u_i \cdot \sum_{j=1}^m f(q_j, v_j) + \sum_{j=1}^m q_j^\beta v_j \cdot \sum_{i=1}^n f(p_i, u_i), \quad (1.2)$$

where  $P \in \Gamma_n$ ,  $Q \in \Gamma_m$ ,  $(u_1, u_2, \dots, u_n) \in \mathbb{R}_+^n$ ,  $(v_1, v_2, \dots, v_m) \in \mathbb{R}_+^m$ ,  $\alpha, \beta \in \mathbb{R} - \{0, 1\}$ . The measurable solution of (1.2) for  $\alpha = 1$  was given by Kannappan in [3]. In a recent paper of Kannappan and Sahoo [4], measurable solution of a more general functional equation than (1.2) was given using the result of this paper. In this paper, we determine the general solution of (1.2) where  $P \in \Gamma_n^0$ ,  $Q \in \Gamma_m^0$ ,  $(u_1, u_2, \dots, u_n) \in \mathbb{R}_+^n$ ,  $(v_1, v_2, \dots, v_m) \in \mathbb{R}_+^m$ ,  $\alpha, \beta \in \mathbb{R} - \{0, 1\}$  and  $m, n$  (fixed and)  $\geq 3$ , on an open domain.

## 2. SOLUTION OF (1.2) ON AN OPEN DOMAIN

We need the following result in this sequel.

Result 1 [5]. Let  $f, g: ]0, 1[ \rightarrow \mathbb{R}$  be real valued functions and satisfy

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n p_i^\alpha \cdot \sum_{j=1}^m g(q_j) + \sum_{j=1}^m q_j^\beta \cdot \sum_{i=1}^n f(p_i) \quad (2.1)$$

for  $P \in \Gamma_n^0$ ,  $Q \in \Gamma_m^0$ ,  $\alpha, \beta \in \mathbb{R} - \{0, 1\}$  and  $m, n (\geq 3)$  are arbitrary but fixed integers. Then the general solutions of (2.1) are given by

$$\left. \begin{aligned} f(p) &= A(p) + ap^\alpha + bp^\beta, \\ g(p) &= A'(p) + a(p^\alpha - p^\beta) + c \end{aligned} \right\} \text{ for } \alpha \neq \beta$$

and

$$\left. \begin{aligned} f(p) &= A(p) + D(p)p^\alpha + dp^\beta, \\ g(p) &= A'(p) + D(p)p^\alpha + c \end{aligned} \right\} \text{ for } \alpha = \beta$$

where  $a, b, c, d$  are arbitrary constants,  $A, A'$  are additive functions on  $\mathbb{R}$  with  $A(1) = 0$ ,  $A'(1) + mc = 0$  and  $D$  is a real valued function satisfying

$$D(pq) = D(p) + D(q), \quad p, q \in ]0, 1[. \quad (2.2)$$

Now we proceed to determine the general solution of (1.2) on  $]0, 1[$ . Let  $f: ]0, 1[ \times \mathbb{R}_+ \rightarrow \mathbb{R}$  be a real valued function and satisfy the functional equation (1.2) for an arbitrary but fixed pair of positive integers  $m, n (\geq 3)$ , for  $P \in \Gamma_n^0$ ,  $Q \in \Gamma_m^0$ , with  $\alpha, \beta \in \mathbb{R} - \{0, 1\}$ . Letting  $u_i = u$  for all  $i = 1, 2, \dots, n$  and  $v_j = v$  for  $j = 1, 2, \dots, m$  in (1.2), we obtain

$$\sum_{i=1}^n \sum_{j=1}^m \frac{f(p_i q_j, uv)}{uv} = \sum_{i=1}^n p_i^\alpha \cdot \sum_{j=1}^m \frac{f(q_j, v)}{v} + \sum_{j=1}^m q_j^\beta \cdot \sum_{i=1}^n \frac{f(p_i, u)}{u}, \quad (2.3)$$

where  $u, v \in \mathbb{R}_+$ . Putting  $v = 1$  in (2.3), we get

$$\sum_{i=1}^n \sum_{j=1}^m \frac{f(p_i q_j, u)}{u} = \sum_{i=1}^n p_i^\alpha \cdot \sum_{j=1}^m f(q_j, 1) + \sum_{j=1}^m q_j^\beta \cdot \sum_{i=1}^n \frac{f(p_i, u)}{u} \quad (2.4)$$

where  $u, v \in \mathbb{R}_+$ . Putting  $v = 1$  in (2.3), we get

$$\sum_{i=1}^n \sum_{j=1}^m \frac{f(p_i q_j, u)}{u} = \sum_{i=1}^n p_i^\alpha \cdot \sum_{j=1}^m f(q_j, 1) + \sum_{j=1}^m q_j^\beta \cdot \sum_{i=1}^n \frac{f(p_i, u)}{u}$$

for  $u \in \mathbb{R}_+$  and  $P \in \Gamma_n^0$ ,  $Q \in \Gamma_m^0$ . For fixed  $u \in \mathbb{R}_+$ , (2.4) is of the form

(2.1) and hence its general solutions can be obtained from Result 1.

First we consider the case  $\alpha \neq \beta$ . Then from Result 1, we have

$$f(p,u) = A_1(p,u)u + a(u)u^{\alpha} + b(u)u^{\beta} \tag{2.5}$$

where  $a, b: \mathbb{R}_+ \rightarrow \mathbb{R}$  are real valued functions of  $u$  and  $A_1$  is additive in the first variable, with  $A_1(1,u) = 0$ . Letting (2.5) into (2.3), we get

$$\begin{aligned} & (a(uv) - a(v)) \sum_{i=1}^n p_i^{\alpha} \sum_{j=1}^m q_j^{\alpha} + (b(uv) - b(u)) \sum_{i=1}^n p_i^{\beta} \sum_{j=1}^m q_j^{\beta} \\ & - (b(v) + a(u)) \sum_{i=1}^n p_i^{\alpha} \sum_{j=1}^m q_j^{\beta} = 0. \end{aligned} \tag{2.6}$$

Noting  $\alpha \neq \beta$ , ( $\alpha \neq 1, \beta \neq 1$ ) equating the coefficients of  $\sum_{i=1}^n p_i^{\alpha}$  and  $\sum_{i=1}^n p_i^{\beta}$  (then using the same for  $\sum_{j=1}^m q_j^{\alpha}$  and  $\sum_{j=1}^m q_j^{\beta}$ ) in (2.6), we get

$$a(uv) = a(v), \quad b(uv) = b(u) \quad \text{and} \quad b(v) = -a(u).$$

From these it is easy to see that

$$a(u) = -b(v) = a, \quad \text{constant} \tag{2.7}$$

for all  $u, v \in \mathbb{R}_+$ . Now putting (2.7) into (2.5), we get

$$f(p,u) = A_1(p,u)u + au(p^{\alpha} - p^{\beta}) \tag{2.8}$$

with  $A_1(1,u) = 0$ . Again letting (2.8) into (1.2), we get

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m A_1(p_i q_j, u_i v_j) u_i v_j &= \sum_{j=1}^m A_1(q_j, v_j) v_j \sum_{i=1}^n u_i p_i^{\alpha} + \\ &+ \sum_{i=1}^n A_1(p_i, u_i) u_i \sum_{j=1}^m v_j q_j^{\beta}. \end{aligned} \tag{2.9}$$

Since  $A_1$  is additive in the first variable, by putting  $u_i = 1$  and  $p_i = \frac{1}{n}$  (note that  $\alpha \neq 1$ ), we have

$$\sum_{j=1}^m A_1(q_j, v_j) v_j = 0. \tag{2.10}$$

We let  $v_1 = v_2, \dots, = v_{m-1} = v$  and  $v_m = v'$ , where  $v, v' \in \mathbb{R}_+$ , into (2.10) and obtain

$$\sum_{j=1}^{m-1} A_1(q_j, v) v + A_1(q_m, v') v' = 0.$$

Since  $A_1$  is additive in the first variable, and  $A_1(1, v) = 0$ , we get

$$A_1(q_m, v) v = A_1(q_m, v') v' \tag{2.11}$$

for all  $q_m \in ]0, 1[$ , and  $v, v' \in \mathbb{R}_+$ . From equation (2.11) it is clear that

$$A_1(x,y)y = A(x) \quad (2.12)$$

where  $A$  is an additive function with  $A(1) = 0$ . Now using (2.12) in (2.8), we obtain

$$f(p,u) = A(p) + au(p^\alpha - p^\beta), \quad p \in ]0,1[, u \in \mathbb{R}_+ \quad (2.13)$$

where  $A$  is an additive function on  $\mathbb{R}$  with  $A(1) = 0$  and  $a$  is an arbitrary constant.

Next we consider the case  $\underline{\alpha = \beta}$ . Again the general solution of (2.4) from Result 1 can be obtained as

$$f(p,u) = uA_2(p,u) + D_1(p,u)p^\alpha u + d(u)p^\alpha u \quad (2.14)$$

where  $d: \mathbb{R}_+ \rightarrow \mathbb{R}$  is a real valued function of  $u$  and  $A_2$  is an additive function in the first variable with  $A_2(1,u) = 0$  and  $D_1: ]0,1[ \times \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfies (2.2). Putting (2.14) into (2.4), we get by equating the coefficient of  $\sum_{i=1}^n p_i^\alpha$  (note  $\alpha \neq 1$ )

$$\sum_{j=1}^m [D_1(q_j, u) - D_1(q_j, 1) - d_1] q_j^\alpha = 0. \quad (2.15)$$

Using  $u = 1$  in (2.15), gives  $d_1 = 0$ . Hence (2.15) with  $d_1 = 0$ , by the use of the Result 1 of [5], yields

$$(D_1(x,u) - D_1(x,1))x^\alpha = A_3(x - \frac{1}{m}, u) \quad (2.16)$$

for all  $x \in ]0,1[$  and  $A_3$  is an additive function in the first variable. Since  $D_1$  satisfies (2.2), we get

$$A_3(x - \frac{1}{m}, u)y^\alpha + A_3(y - \frac{1}{m}, u)x^\alpha = A_3(xy - \frac{1}{m}, u). \quad (2.17)$$

Putting  $y = \frac{1}{m}$  and using  $A_3(0, u) = 0$  in (2.17), we get

$$A_3(x, u) = c_1 A_3(1, u). \quad (2.18)$$

Since  $A_3$  is additive in the first variable we obtain from (2.18) that  $A_3 \equiv 0$  for  $x \in ]0,1[$ , and all  $u \in \mathbb{R}_+$ . Thus, (2.16) reduces to

$$D_1(x,u) - D_1(x,1) = 0. \quad (2.19)$$

From (2.19), we see that  $D_1$  is independent of  $u$ , i.e.

$$D_1(x,y) = D(x), \quad x \in ]0,1[ \quad (2.20)$$

and since  $D_1$  satisfies (2.2),  $D$  also satisfies (2.2). Using (2.20) in (2.14), we get

$$f(p,u) = uA_2(p,u) + D(p)up^\alpha + d(u)up^\alpha \quad (2.21)$$

where  $A_2$  is additive with  $A_2(1,u) = 0$ . Letting (2.21) into (2.3), we get

$$(d(uv) - d(u) - d(v)) \sum_{i=1}^n \sum_{j=1}^m (p_i q_j)^\alpha = 0 \quad (2.22)$$

for all  $u, v \in \mathbb{R}_+$ . Since  $\sum_{i=1}^n \sum_{j=1}^m (p_i q_j)^\alpha \neq 0$  we obtain

$$d(uv) = d(u) + d(v), \quad u, v \in \mathbb{R}_+. \tag{2.23}$$

Again putting (2.21) into (1.2) and using (2.23) and (2.2), we get

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m A_2(p_i q_j, u_i v_j) u_i v_j &= \sum_{i=1}^n u_i p_i^\alpha \sum_{j=1}^m A_2(q_j, v_j) v_j + \\ &+ \sum_{j=1}^m v_j q_j^\alpha \sum_{i=1}^n A_2(p_i, u_i) u_i. \end{aligned} \tag{2.24}$$

Putting  $u_i = 1$  and  $p_i = \frac{1}{n}$  in (2.4), we obtain

$$\sum_{j=1}^m A_2(q_j, v_j) v_j = 0. \tag{2.25}$$

Note that (2.25) is of the form of (2.10) and hence by a similar argument we get

$$A_2(q, u)u = A(q) \tag{2.26}$$

where  $A$  is additive with  $A(1) = 0$ . Using (2.26) in (2.21), we obtain

$$f(p, u) = A(p) + D(p)u p^\alpha + d(u)u p^\alpha \tag{2.27}$$

where  $A$  is additive on  $\mathbb{R}$  with  $A(1) = 0$  and  $D: ]0, 1[ \rightarrow \mathbb{R}$ ,  $d: \mathbb{R}_+ \rightarrow \mathbb{R}$ , are functions satisfying (2.2) and (2.23) respectively.

Thus we have proved the following theorem.

Theorem. Let  $f: ]0, 1[ \times \mathbb{R}_+ \rightarrow \mathbb{R}$  be a real valued function satisfying (1.2) for arbitrary but fixed pair of  $m, n (\geq 3)$  and  $\alpha, \beta \notin \{0, 1\}$ ,  $P \in \Gamma_n^0$ ,  $Q \in \Gamma_m^0$ . Then  $f$  is given by (2.13) when  $\alpha \neq \beta$  and by (2.27) when  $\alpha = \beta$ .

Corollary. If  $f$  is measurable in the Theorem then

$$f(p, u) = a(p^\alpha - p^\beta) \quad \alpha \neq \beta$$

and

$$f(p, u) = b p^\alpha \log p + c p^\alpha u \log u, \quad \alpha = \beta$$

where  $a, b, c$  are arbitrary constants.

Remark. Because of the occurrence of the parameters  $\alpha, \beta$  as powers,  $f$  is independent of  $m$  and  $n$ .

ACKNOWLEDGEMENT. This work is partially supported by a NSERC of Canada grant.

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