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AN APPLICATION OF THE RUSCHEWEYH DERIVATIVES

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ABSTRACT. Let $D^{\alpha}f(z)$ be the Ruscheweyh derivative defined by using the Hadamard product of $f(z)$ and $z/(1 - z)^{1+\alpha}$. Certain new classes S_{α}^{*} and K_{α} are introduced by virtue of the Ruscheweyh derivative. The object of the present paper is to establish several interesting properties of S_{α}^{*} and K_{α}. Further, some results for integral operator $J_c(f)$ of $f(z)$ are shown.

KEY WORDS AND PHRASES. Ruscheweyh derivative, Hadamard product, starlike function, convex function, integral operator.
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1. INTRODUCTION. Let A denote the class of functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_{n+1} z^{n+1} \qquad (a_1 = 1) \tag{1.1}$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. Let S denote the subclass of of A consisting of univalent functions in the unit disk U. A function $f(z)$ belonging to A is said to be starlike with respect to the origin in the unit disk U if it satisfies

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \tag{1.2}$$

for all $z \in U$. We denote by S^{*} the class of all starlike functions with respect to the origin in the unit disk U. A function $f(z)$ belonging to A is said to be convex in the unit disk $\mathcal{U}$ if it satisfies

$$\text{Re}\,\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \tag{1.3}$$

for all $z \in \mathcal{U}$. We denote by $\mathcal{K}$ the class of all convex functions in the unit disk $\mathcal{U}$. We note that $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$ and that

$$\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}.$$

Let $f_j(z)$ $(j = 1,2)$ in $\mathcal{A}$ be given by

$$f_j(z) = \sum_{n=0}^{\infty} a_{n+1,j} z^{n+1} \qquad (a_{1,j} = 1).$$

Then the <u>Hadamard product</u> (or <u>convolution product</u>) $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = \sum_{n=0}^{\infty} a_{n+1,1} a_{n+1,2} z^{n+1}. \tag{1.5}$$

By the Hadamard product, we define

$$D^{\alpha} f(z) = \frac{z}{(1 - z)^{1+\alpha}} * f(z) \qquad (\alpha \geqq -1) \tag{1.6}$$

for $f(z) \in \mathcal{A}$. The symbol $D^{\alpha} f(z)$ was introduced by Ruscheweyh [1], and is called the <u>Ruscheweyh derivative</u> of $f(z)$.

To derive our results, we have to recall here the following lemmas.

LEMMA 1 ([2]). Let $\phi(z)$ and $g(z)$ be analytic in the unit disk $\mathcal{U}$ and satisfy $\phi(0) = g(0) = 0$, $\phi'(0) \neq 0$, $g'(0) \neq 0$. Suppose that for each σ $(|\sigma| = 1)$ and δ $(|\delta| = 1)$, we have

$$\phi(z) * \frac{1 + \delta\sigma z}{1 - \sigma z} g(z) \neq 0 \qquad (0 < |z| < 1) \tag{1.7}$$

Then for each function $F(z)$ analytic in the unit disk $\mathcal{U}$ and satisfying $\text{Re}\,\{F(z)\} > 0$ $(z \in \mathcal{U})$, we have

$$\text{Re}\,\left\{\frac{\phi * G(z)}{\phi * g(z)}\right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.8}$$

where $G(z) = F \cdot g(z)$.

LEMMA 2 ([3]). Let $w(z)$ be regular in the unit disk $\mathcal{U}$, with $w(0) = 0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r$ $(0 \leqq r < 1)$ at a point z_0, we can write

$$z_0 w'(z_0) = m w(z_0),$$

where m is real and $m \geqq 1$.

LEMMA 3 ([4]). For a real number α $(\alpha > -1)$, we have

$$z(D^{\alpha} f(z))' = (\alpha + 1) D^{\alpha+1} f(z) - \alpha D^{\alpha} f(z). \tag{1.9}$$

REMARK. Note that (1.9) holds true for $\alpha = -1$.

LEMMA 4 ([5]). Let $\phi(u,v)$ be a complex function, $\phi: D \to \mathbb{C}\times\mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that ϕ satisfies the following conditions

(i) $\phi(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $\operatorname{Re}\{\phi(1,0)\} > 0$;

(iii) $\operatorname{Re}\{\phi(iu_2, v_1)\} \leqq 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \leqq -(1 + u_2^2)/2$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be regular in the unit disk U, such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If $\operatorname{Re}\{\phi(p(z), zp'(z))\} > 0$ $(z \in U)$, then $\operatorname{Re} p(z) > 0$ for $z \in U$.

2. PROPERTIES OF $D^\alpha f(z)$. Applying Lemma 1, we prove

THEOREM 1. Let $f(z)$ be in the class S^* and satisfy the condition $D^\alpha f(z) \neq 0$ $(0 < |z| < 1)$ for $\alpha \geqq -1$. Then $D^\alpha f(z)$ is also in the class S^*.

PROOF. We note that

$$\frac{z(D^\alpha f(z))'}{D^\alpha f(z)} = \frac{D^\alpha(zf'(z))}{D^\alpha f(z)} = \frac{\frac{z}{(1-z)^{1+\alpha}} * (zf'(z))}{\frac{z}{(1-z)^{1+\alpha}} * f(z)}. \tag{2.1}$$

Setting $\delta = -1$, $\phi(z) = z/(1-z)^{1+\alpha}$, $g(z) = f(z)$, and $F(z) = zf'(z)/f(z)$ in Lemma 1, we have

$$\operatorname{Re}\left\{\frac{z(D^\alpha f(z))'}{D^\alpha f(z)}\right\} > 0 \qquad (z \in U), \tag{2.2}$$

which implies $D^\alpha f(z) \in S^*$.

THEOREM 2. Let $f(z)$ be in the class K and satisfy the condition $D^\alpha(zf'(z)) \neq 0$ $(0 < |z| < 1)$ for $\alpha \geqq -1$. Then $D^\alpha f(z)$ is also in the class K.

PROOF. Since $f(z) \in K$ if and only if $zf'(z) \in S^*$, Theorem 1 derives $z(D^\alpha f(z))' = D^\alpha(zf'(z)) \in S^*$. Hence we have $D^\alpha f(z) \in K$.

3. THE CLASSES S_α^* AND K_α. In view of Theorems 1 and 2, we can introduce the following classes;

$$S_\alpha^* = \{f(z) \in A:\ D^\alpha f(z) \in S^*,\ \alpha \geqq -1\}$$

and

$$K_\alpha = \{f(z) \in A:\ D^\alpha f(z) \in K,\ \alpha \geqq -1\}.$$

Now, we derive:

THEOREM 3. For $\alpha \geqq 0$, we have $S_{\alpha+1}^* \subset S_\alpha^*$.

PROOF. For $f(z) \in S_{\alpha+1}^*$, we define the function $w(z)$ by

$$\frac{z(D^\alpha f(z))'}{D^\alpha f(z)} = \frac{1 + w(z)}{1 - w(z)} \qquad (w(z) \neq 1). \tag{3.1}$$

Then, with Lemma 3, we have

$$\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)} = \frac{1}{\alpha+1}\left\{\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)} + \alpha\right\}$$

$$= \frac{(1+\alpha) + (1-\alpha)w(z)}{(1+\alpha)(1-w(z))}. \tag{3.2}$$

Differentiating both sides of (3.2) logarithmically, it follows that

$$\frac{z(D^{\alpha+1}f(z))'}{D^{\alpha+1}f(z)} = \frac{1+w(z)}{1-w(z)} + \frac{2zw'(z)}{(1-w(z))\{(1+\alpha)+(1-\alpha)w(z)\}}. \tag{3.3}$$

Suppose that for $z_0 \in U$

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq \pm 1). \tag{3.4}$$

Then it follows from Lemma 2 that

$$z_0 w'(z_0) = m w(z_0),$$

where m is real and $m \geqq 1$. Setting $w(z_0) = e^{i\theta_0}$, we obtain

$$\mathrm{Re}\left\{\frac{z_0(D^{\alpha+1}f(z_0))'}{D^{\alpha+1}f(z_0)}\right\}$$

$$= \mathrm{Re}\left\{\frac{1+w(z_0)}{1-w(z_0)}\right\} + \mathrm{Re}\left\{\frac{2mw(z_0)}{(1-w(z_0))\{(1+\alpha)+(1-\alpha)w(z_0)\}}\right\}$$

$$= -\frac{m\alpha(1-\cos\theta_0)}{M} \leqq 0, \tag{3.5}$$

where $M = \{\alpha(1-\cos\theta_0) + (1-\alpha)\sin^2\theta_0\}^2 + \{\alpha + (1-\alpha)\cos\theta_0\}^2\sin^2\theta_0$.
This contradicts the hypothesis that $f(z) \in S^*_{\alpha+1}$. Therefore, $w(z)$ has to satisfy that $|w(z)| < 1$ for all $z \in U$. Thus we have

$$\mathrm{Re}\left\{\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)}\right\} = \mathrm{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0, \tag{3.6}$$

which implies $f(z) \in S^*_{\alpha}$.

THEOREM 4. For $\alpha \geqq 0$, we have

$$\bigcap_{\alpha} S^*_{\alpha} = \{id\},$$

where id is the identity function $f(z) = z$.

PROOF. Note that $D^{\alpha}z = z$ for all α, and that

$$\operatorname{Re}\left\{\frac{z(D^{\alpha}z)'}{D^{\alpha}z}\right\} = 1 > 0 \qquad (z \in U)$$

for all α. Consequently, we conclude that $\mathrm{id} \in S_{\alpha}^{*}$ for all α.

For the converse, we assume that the function $f(z)$ belonging to A is in the class $\bigcap_{\alpha} S_{\alpha}^{*}$. Then we have

$$D^{\alpha}f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} a_{n+1} z^{n+1} \in S^{*}$$

for all $\alpha \geqq 0$. It is well known that

$$|a_{n+1}| \leqq n + 1 \qquad (n \geqq 1)$$

for $f(z) \in S^{*}$. This implies

$$\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} |a_{n+1}| \leqq n + 1 \qquad (n \geqq 1), \tag{3.7}$$

or

$$|a_{n+1}| \leqq \frac{(n+1)!\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \qquad (n \geqq 1) \tag{3.8}$$

for all $\alpha \geqq 0$. Therefore, we have $f(z) = z$.

By virtue of Theorem 3, we prove:

THEOREM 5. For $\alpha \geqq 0$, we have $K_{\alpha+1} \subset K_{\alpha}$.

PROOF. By Theorem 3, it follows that

$$\begin{aligned} f(z) \in K_{\alpha+1} &\Longleftrightarrow D^{\alpha+1}f(z) \in K \\ &\Longleftrightarrow z(D^{\alpha+1}f(z))' \in S^{*} \\ &\Longleftrightarrow D^{\alpha+1}(zf'(z)) \in S^{*} \\ &\Longleftrightarrow zf'(z) \in S_{\alpha+1}^{*} \\ &\Longrightarrow zf'(z) \in S_{\alpha}^{*} \\ &\Longleftrightarrow D^{\alpha}(zf'(z)) \in S^{*} \\ &\Longleftrightarrow z(D^{\alpha}f(z))' \in S^{*} \end{aligned}$$

This asserts the result of the theorem.

THEOREM 6. FOR $\alpha \geqq 0$, we have

$$\bigcap_{\alpha} K_\alpha = \{id\},$$

where id is the identity function $f(z) = z$.

The proof of Theorem 6 is similar to that of Theorem 4.

Furthermore, an application of Lemma 4 to the classes S^*_α and K_α gives:

THEOREM 7. Let $f(z)$ be in the class S^*_α with $\alpha \geqq -1$. Then

$$\operatorname{Re}\left\{\left(\frac{D^\alpha f(z)}{z}\right)^{\beta-1}\right\} > \frac{1}{2\beta - 1} \qquad (z \in U), \tag{3.9}$$

where $1 < \beta \leqq 3/2$.

PROOF. We define the function $p(z)$ by

$$A\left(\frac{D^\alpha f(z)}{z}\right)^{\beta-1} = p(z) + (A - 1), \tag{3.10}$$

where $A = 1 + 1/2(\beta-1)$. Differentiating (3.10) logarithmically, we have

$$\frac{z(D^\alpha f(z))'}{D^\alpha f(z)} = \frac{1}{\beta - 1} \quad \frac{zp'(z)}{p(z) + (A - 1)} + 1. \tag{3.11}$$

Since $f(z) \in S^*_\alpha$, it follows that

$$\operatorname{Re}\left\{\frac{1}{\beta - 1} \cdot \frac{zp'(z)}{p(z) + (A - 1)} + 1\right\} > 0 \qquad (z \in U). \tag{3.12}$$

Let $p(z) = u = u_1 + iu_2$ and $zp'(z) = v = v_1 + iv_2$, and define the function $\phi(u,v)$ by

$$\phi(u,v) = \frac{1}{\beta - 1} \cdot \frac{v}{u + (A - 1)} + 1. \tag{3.13}$$

Then $\phi(u,v)$ is continuous in $D = (C-\{1-A\}) \times C$, and together with $(1,0) \in D$ and $\operatorname{Re}\{\phi(1,0)\} = 1 > 0$. Moreover, for all $(iu_2, v_1) \in D$ such that $v_1 \leqq -(1 + u_2^2)/2$, we can show that

$$\operatorname{Re}\{\phi(iu_2,v_1)\} = \frac{1}{\beta - 1} \operatorname{Re}\left\{\frac{v_1}{iu_2 + (A - 1)}\right\} + 1$$

$$\leqq \frac{-1}{\beta - 1} \cdot \frac{(A - 1)(1 + u_2^2)}{2\{u_2^2 + (A - 1)^2\}} + 1 \leqq 0, \tag{3.14}$$

for $1 < \beta \leqq 3/2$. Hence the function $\phi(u,v)$ satisfies the conditions in Lemma 4. It follows from this fact that $\operatorname{Re} p(z) > 0$ for $z \in U$, that is,

$$\operatorname{Re}\left\{ A\left(\frac{D^\alpha f(z)}{z}\right)^{\beta-1} - (A - 1)\right\} > 0 \qquad (z \in U). \tag{3.15}$$

This completes the assertion of Theorem 7.

Taking $\beta = 3/2$ in Theorem 7, we have:

COROLLARY 1. Let $f(z)$ be in the class S^*_α with $\alpha \geqq -1$. Then

$$\text{Re}\ \left\{\left(\frac{D^{\alpha}f(z)}{z}\right)^{1/2}\right\} > \frac{1}{2} \qquad (z \in \mathcal{U}), \tag{3.16}$$

COROLLARY 2. Let $f(z)$ be in the class $\mathcal{K}_{\alpha}$ with $\alpha \geqq -1$. Then

$$\text{Re}\ \{(D^{\alpha}f(z))'\}^{\beta-1} > \frac{1}{2\beta - 1} \qquad (z \in \mathcal{U}), \tag{3.17}$$

where $1 < \beta \leqq 3/2$.

PROOF. Note that

$$\begin{aligned} f(z) \in \mathcal{K}_{\alpha} &\Longleftrightarrow D^{\alpha}f(z) \in \mathcal{K} \\ &\Longleftrightarrow z(D^{\alpha}f(z))' \in \mathcal{S}^{*} \\ &\Longleftrightarrow D^{\alpha}(zf'(z)) \in \mathcal{S}^{*} \\ &\Longleftrightarrow zf'(z) \in \mathcal{S}^{*}_{\alpha}, \end{aligned}$$

which implies

$$\frac{D^{\alpha}(zf'(z))}{z} = (D^{\alpha}f(z))'.$$

Therefore, we have the corollary with the aid of Theorem 7.

4. INTEGRAL OPERATOR $J_c(f)$. We define the integral operator $J_c(f)$ by

$$J_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \qquad (c > -1) \tag{4.1}$$

for $f(z) \in \mathcal{A}$. The operator $J_c(f)$ when $c \in \mathbb{N} = \{1, 2, 3, \ldots\}$ was studied by Bernardi [6]. In particular, the operator $J_1(f)$ was studied by Libera [7] and Livingston [8].

THEOREM 8. Let $f(z)$ be in the class $\mathcal{S}^{*}_{\alpha}$ with $\alpha \geqq 0$. Then $J_{\alpha}(f)$ is also in the class $\mathcal{S}^{*}_{\alpha}$.

PROOF. Define the function $w(z)$ by

$$\frac{z(D^{\alpha}J_{\alpha}(f))'}{D^{\alpha}J_{\alpha}(f)} = \frac{1 + w(z)}{1 - w(z)} \qquad (w(z) \neq 1). \tag{4.2}$$

Then, by taking the differentiation of both sides logarithmically, we have

$$\frac{z^2(D^{\alpha}J_{\alpha}(f))'' + z(D^{\alpha}J_{\alpha}(f))'}{z(D^{\alpha}J_{\alpha}(f))'} - \frac{z(D^{\alpha}J_{\alpha}(f))'}{D^{\alpha}J_{\alpha}(f)} = \frac{2zw'(z)}{(1 + w(z))(1 - w(z))} \tag{4.3}$$

Since

$$z(z(D^{\alpha}f(z))')' = z^2(D^{\alpha}f(z))'' + z(D^{\alpha}f(z))', \tag{4.4}$$

we can see that

$$z^2(D^{\alpha}f(z))'' = (\alpha + 1)z(D^{\alpha+1}f(z))' - (\alpha + 1)z(D^{\alpha}f(z))' \tag{4.5}$$

by Lemma 3. Furthermore, it follows from the definition of $J_\alpha(f)$ that

$$D^\alpha f(z) = D^{\alpha+1} J_\alpha(f). \tag{4.6}$$

By using (4.5) and (4.6), we have

$$\begin{aligned} z^2(D^\alpha J_\alpha(f))'' &= (\alpha + 1)z(D^{\alpha+1} J_\alpha(f))' - (\alpha + 1)z(D^\alpha J_\alpha(f))' \\ &= (\alpha + 1)z(D^\alpha f(z))' - (\alpha + 1)z(D^\alpha J_\alpha(f))'. \end{aligned} \tag{4.7}$$

With the aid of Lemma 3, we have

$$z(D^\alpha J_\alpha(f))' = (\alpha + 1)D^\alpha f(z) - \alpha D^\alpha J_\alpha(f). \tag{4.8}$$

Consequently, from (4.3), we obtain

$$\frac{(\alpha + 1)z(D^\alpha f(z))'}{z(D^\alpha J_\alpha(f))'} - \frac{(\alpha + 1)D^\alpha f(z)}{D^\alpha J_\alpha(f)} = \frac{2zw'(z)}{(1 + w(z))(1 - w(z))}, \tag{4.9}$$

or

$$\frac{(\alpha + 1)D^\alpha f(z)}{z(D^\alpha J_\alpha(f))'} \left\{\frac{z(D^\alpha f(z))'}{D^\alpha f(z)} - \frac{z(D^\alpha J_\alpha(f))'}{D^\alpha J_\alpha(f)}\right\} = \frac{2zw'(z)}{(1 + w(z))(1 - w(z))}. \tag{4.10}$$

Since (4.8) implies

$$\frac{(\alpha + 1)D^\alpha f(z)}{z(D^\alpha J_\alpha(f))'} = 1 + \frac{\alpha D^\alpha J_\alpha(f)}{z(D^\alpha J_\alpha(f))'} = \frac{(1 + \alpha) + (1 - \alpha)w(z)}{1 + w(z)}, \tag{4.11}$$

it follows from (4.10) that

$$\frac{z(D^\alpha f(z))'}{D^\alpha f(z)} = \frac{1 + w(z)}{1 - w(z)} + \frac{2zw'(z)}{(1 - w(z))\{(1 + \alpha) + (1 - \alpha)w(z)\}}. \tag{4.12}$$

By assuming

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq \pm 1)$$

for $z_0 \in U$ and using the same technique as in the proof of Theorem 3, we can show that

$$\text{Re}\left\{\frac{z(D^\alpha J_\alpha(f))'}{D^\alpha J_\alpha(f)}\right\} = \text{Re}\left\{\frac{1 + w(z)}{1 - w(z)}\right\} > 0 \qquad (z \in U). \tag{4.13}$$

Thus we conclude that $J_\alpha(f)$ is in the class S_α^*.

COROLLARY 3. Let f(z) be in the class S_α^* with $\alpha \geq 0$. Then, for $p \in N$,

$$(z\,{}_{p+1}F_p(\alpha+1, \ldots, \alpha+1, 1; \alpha+2, \ldots, \alpha+2; z)) * f(z) \in S_\alpha^*,$$

where ${}_{p+1}F_p(\alpha_1, \ldots, \alpha_{p+1}; \beta_1, \ldots, \beta_p; z)$ denotes the generalized hypergeometric function.

PROOF. It is easy to see that

$$J_\alpha(f) = \frac{\alpha+1}{z^\alpha}\int_0^z t^{\alpha-1}\left(\sum_{n=0}^{\infty} a_{n+1}t^{n+1}\right)dt$$

$$= \sum_{n=0}^{\infty}\left(\frac{\alpha+1}{n+\alpha+1}\right)a_{n+1}z^{n+1}$$

$$= (z\,{}_2F_1(\alpha+1,1;\alpha+2;z))*f(z) \tag{4.14}$$

for $f(z)\in A$. Therefore, by Theorem 8, we have

$$(z\,{}_2F_1(\alpha+1,1;\alpha+2;z))*f(z)\in S_\alpha^*.$$

Repeating the same manner, we conclude that

$$f(z)\in S_\alpha^* \Longrightarrow (z\,{}_2F_1(\alpha+1,1;\alpha+2;z))*f(z)\in S_\alpha^*$$
$$\Longrightarrow (z\,{}_3F_2(\alpha+1,\alpha+1,1;\alpha+2,\alpha+2;z))*f(z)\in S_\alpha^*$$
$$\Longrightarrow (z\,{}_{p+1}F_p(\alpha+1,\cdots,\alpha+1,1;\alpha+2,\cdots,\alpha+2;z))*f(z)\in S_\alpha^*.$$

Finally, we prove

THEOREM 9. Let $f(z)$ be in the class K_α with $\alpha \geqq 0$, Then $J_\alpha(f)$ is also in the class K_α.

PROOF. In view of Theorem 5, we can see that

$$f(z)\in K_\alpha \Longleftrightarrow z(D^\alpha f(z))'\in S^*$$
$$\Longleftrightarrow D^\alpha(zf'(z))\in S^*$$
$$\Longleftrightarrow zf'(z)\in S_\alpha^*$$
$$\Longrightarrow J_\alpha(zf'(z))\in S_\alpha^*$$
$$\Longleftrightarrow D^\alpha(J_\alpha(zf'))\in S^*$$
$$\Longleftrightarrow z(D^\alpha J_\alpha(f))'\in S^*$$
$$\Longleftrightarrow D^\alpha J_\alpha(f)\in K$$
$$\Longleftrightarrow J_\alpha(f)\in K_\alpha,$$

which completes the proof of Theorem 9.

COROLLARY 4. Let $f(z)$ be in the class K_α with $\alpha \geqq 0$. Then, for $p\in N$, $(z\,{}_{p+1}F_p(\alpha+1,\ldots,\alpha+1,1;\alpha+2,\ldots,\alpha+2;z))*f(z)\in K_\alpha$, where ${}_{p+1}F_p(\alpha_1,\ldots,\alpha_{p+1};\beta_1,\ldots,\beta_p;z)$ denotes the generalized hypergeometric function.

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