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Quenching Semidiscretizations in Time of a Nonlocal Parabolic Problem With Neumann Boundary Condition

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Abstract

In this paper, under some conditions, we show that the solution of a semidiscrete form of a nonlocal parabolic problem quenches in a finite time and estimate its semidiscrete quenching time. We also prove that the semidiscrete quenching time converges to the real one when the mesh size goes to zero. Finally, we give some numerical results to illustrate our analysis.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial value problem

$$u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + (1-u)^{-p} \quad \text{in} \quad \overline{\Omega}\times(0,T), \quad (1)$$

$$u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \overline{\Omega}, \quad (2)$$

where $p = \text{const} > 0$, $J : \mathbb{R}^N \to \mathbb{R}$ is a kernel which is nonnegative and bounded in $\mathbb{R}^N$. In addition, J is symmetric $(J(z) = J(-z))$ and $\int_{\mathbb{R}^N} J(z)dz = 1$. The

initial datum $u_0 \in C^0(\overline{\Omega})$, $0 \leq u_0(x) < 1$, $x \in \overline{\Omega}$.
Here, $(0, T)$ is the maximal time interval on which the solution u exists. The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely,

$$\lim_{t \to T} \|u(\cdot, t)\|_\infty = 1,$$

where $\|u(\cdot, t)\|_\infty = \sup_{x \in \overline{\Omega}} |u(x, t)|$. In this last case, we say that the solution u quenches in a finite time, and the time T is called the quenching time of the solution u. Recently, nonlocal diffusion has been the subject of investigation of many authors (see, [1]-[7], [10]-[12], [14]-[18], [20], and the references cited therein). Nonlocal evolution equations of the form

$$u_t = \int_{\mathbb{R}^N} J(x - y)(u(y, t) - u(x, t))dy,$$

and variations of it, have been used by several authors to model diffusion processes (see, [3], [4], [17]). The solution $u(x, t)$ can be interpreted as the density of a single population at the point x, at the time t, and $J(x-y)$ as the probability distribution of jumping from location y to location x. Then, the convolution $(J*u)(x, t) = \int_{\mathbb{R}^N} J(x-y)u(y, t)dy$ is the rate at which individuals are arriving to position x from all other places, and $-u(x, t) = -\int_{\mathbb{R}^N} J(x - y)u(y, t)dy$ is the rate at which they are leaving location x to travel to any other site (see, [17]). Let us notice that the reaction term $(1 - u)^{-p}$ in the equation (1) can be rewritten as follows

$$(1 - u(x, t))^{-p} = \int_{\mathbb{R}^N} J(x - y)(1 - u(x, t))^{-p}dy.$$

Therefore, in view of the above equality, the reaction term $(1 - u)^{-p}$ can be interpreted as a force that increases the rate at which individuals are arriving to location x from all other places. Due to the presence of the term $(1 - u)^{-p}$, we shall see later the quenching of the density $u(x, t)$. On the other hand, the integral in (1) is taken over Ω. Thus, there is no individuals that enter or leave the domain Ω. It is the reason why in the title of the paper, we have added Neumann boundary condition. In the current paper, we are interested in the numerical study of the phenomenon of quenching using a semidiscrete form of (1)-(2). Let us notice that, setting $v = 1 - u$, the problem (1)-(2) is equivalent to

$$v_t(x, t) = \int_\Omega J(x - y)(v(y, t) - v(x, t))dy - v^{-p} \quad \text{in} \quad \overline{\Omega} \times (0, T), \tag{3}$$

$$v(x, 0) = \varphi(x) \geq 0 \quad \text{in} \quad \overline{\Omega}, \tag{4}$$

where $\varphi(x) = 1 - u_0(x)$. Consequently, the solution u of (1)-(2) quenches at the time T if and only if the solution v of (3)-(4) quenches at the time T, that is,

$$\lim_{t\to T} v_{\min}(t) = 0,$$

where $v_{\min}(t) = \min_{\overline{\Omega}} v(x,t)$. We start by the construction of an explicit adaptive scheme as follows. Approximate the solution v of (3)-(4) by the solution U_n of the following semidiscrete equations

$$\delta_t U_n(x) = \int_\Omega J(x-y)(U_n(y) - U_n(x))dy - (U_n(x))^{-p} \quad \text{in} \quad \overline{\Omega}, \tag{5}$$

$$U_n(0) = \varphi(x) \quad \text{in} \quad \overline{\Omega}, \tag{6}$$

where $n \geq 0$, and

$$\delta_t U_n(x) = \frac{U_{n+1}(x) - U_n(x)}{\Delta t_n}.$$

In order to permit the semidiscrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the step so that we take

$$\Delta t_n = \min\{\Delta t, \tau U_{nmin}^{p+1}\},$$

where $U_{nmin} = \min_{x\in\overline{\Omega}} U_n(x)$, $\tau \in (0, 1/2)$ and $\Delta t \in (0, 1/2)$ is a parameter. Let us notice that the restriction on the time step ensures the positivity of the semidiscrete solution.
To facilitate our discussion, let us define the notion of semidiscrete quenching time.

Definition 1.1 *We say that the semidiscrete solution U_n of (5)-(6) quenches in a finite time if* $\lim_{n\to\infty} U_{nmin} = 0$*, and the series* $\sum_{n=0}^{\infty} \Delta t_n$ *converges. The quantity* $\sum_{n=0}^{\infty} \Delta t_n$ *is called the semidiscrete quenching time of the semidiscrete solution U_n.*

In the present paper, under some conditions, we show that the semidiscrete solution quenches in a finite time and estimate its semidiscrete quenching time. We also show that the semidiscrete quenching time converges to the real one when the mesh size goes to zero. A similar result has been obtained by Le Roux in [21]-[22], and the same author and Mainge in [23] within the framework of the phenomenon of blow-up for local parabolic problems (we say that a solution blows up in a finite if it reaches infinity in a finite time). One may also consult the papers [25] and [26] for numerical studies of the phenomenon of quenching where semidiscretizations in space have been utilized. The remainder of the paper is organized as follows. In the next section, we reveal certain properties of the continuous problem. In the third section, we exhibit some features of the semidiscrete scheme. In the fourth section, under some assumptions, we demonstrate that the semidiscrete solution quenches in a finite time, and estimate its semidiscrete quenching time. In the fifth section, the convergence of the semidiscrete quenching time is analyzed, and finally, in the last section, we show some numerical experiments to illustrate our analysis.

2 Local existence

In this section, we shall establish the existence and uniqueness of solutions of (1)-(2) in $\Omega \times (0,T)$ for all small T. Some results about quenching are also given.
Let t_0 be fixed, and define the function space $Y_{t_0} = \{u; u \in C([0,t_0], C(\overline{\Omega}))\}$ equipped with the norm defined by $\|u\|_{Y_{t_0}} = \max_{0\leq t\leq t_0} \|u\|_\infty$ for $u \in Y_{t_0}$. It is easy to see that Y_{t_0} is a Banach space. Introduce the set

$$X_{t_0} = \{u; u \in Y_{t_0}, \|u\|_{Y_{t_0}} \leq b_0\},$$

where $b_0 = \frac{\|u_0\|_\infty+1}{2}$. We observe that X_{t_0} is a nonempty bounded closed convex subset of Y_{t_0}. Define the map R as follows

$$R : X_{t_0} \to X_{t_0},$$

$$R(v)(x,t) = u_0(x)+\int_0^t \int_\Omega J(x-y)(v(y,s)-v(x,s))dyds+\int_0^t (1-v(x,s))^{-p}ds.$$

Theorem 2.1 *Assume that $u_0 \in Y_{t_0}$. Then R maps X_{t_0} into X_{t_0}, and R is strictly contractive if t_0 is approximately small relative to $\|u_0\|_\infty$.*

Proof. Due to the fact that $\int_\Omega J(x-y)dy \leq \int_{\mathbb{R}^N} J(x-y)dy = 1$, a straightforward computation reveals that

$$|R(v)(x,t) - u_0(x)| \leq 2\|v\|_{Y_{t_0}} t + (1 - \|v\|_{Y_{t_0}})^{-p} t,$$

which implies that

$$\|R(v)\|_{Y_{t_0}} \leq \|u_0\|_\infty + 2b_0 t_0 + (1 - b_0)^{-p} t_0.$$

If

$$t_0 \leq \frac{b_0 - \|u_0\|_\infty}{2b_0 + (1 - b_0)^{-p}}, \tag{7}$$

then

$$\|R(v)\|_{Y_{t_0}} \leq b_0.$$

Therefore, if (7) holds, then R maps X_{t_0} into X_{t_0}. Now, we are going to prove that the map R is strictly contractive. Let $t_0 > 0$ and let $v, z \in X_{t_0}$. Setting $\alpha = v - z$, we discover that

$$|(R(v) - R(z))(x,t)| \leq |\int_0^t \int_\Omega J(x-y)(\alpha(y,s) - \alpha(x,s))dyds|$$

$$+|\int_0^t ((1 - v(x,s))^{-p} - (1 - z(x,s))^{-p})ds|.$$

Use Taylor's expansion to obtain

$$|(R(v) - R(z))(x,t)| \leq 2\|\alpha\|_{Y_{t_0}} t + t\|v - z\|_{Y_{t_0}} p(1 - \|\beta\|_{Y_{t_0}})^{-p-1},$$

where β is an intermediate value between v and z. We deduce that

$$\|R(v) - R(z)\|_{Y_{t_0}} \leq 2\|\alpha\|_{Y_{t_0}} t_0 + t_0\|v - z\|_{Y_{t_0}} p(1 - \|\beta\|_{Y_{t_0}})^{-p-1},$$

which implies that

$$\|R(v) - R(z)\|_{Y_{t_0}} \leq (2t_0 + t_0 p(1 - b_0)^{-p-1})\|v - z\|_{Y_{t_0}}.$$

If

$$t_0 \leq \frac{1}{4 + 2p(1 - b_0)^{-p-1}}, \tag{8}$$

then $\|R(v) - R(z)\|_{Y_{t_0}} \leq \frac{1}{2}\|v - z\|_{Y_{t_0}}$. Hence, we see that $R(v)$ is a strict contraction in Y_{t_0} and the proof is complete. □

It follows from the contraction mapping principle that for appropriately chosen t_0, R has a unique fixed point $u(x,t) \in Y_{t_0}$ which is a solution of (1)-(2). If $\|u\|_{Y_{t_0}} < 1$, then taking as initial data $u(x,t) \in C(\overline{\Omega})$ and arguing as before, it is possible to extend the solution up to some interval $[0, t_1)$ for certain $t_1 > t_0$. The following lemma is a version of the maximum principle for nonlocal problems.

Lemma 2.2 *Let $a \in C^0(\overline{\Omega} \times [0,T))$, and let $u \in C^{0,1}(\overline{\Omega} \times [0,T))$ satisfying the following inequalities*

$$u_t - \int_\Omega J(x-y)(u(y,t) - u(x,t))dy + a(x,t)u(x,t) \geq 0 \quad in \quad \overline{\Omega} \times (0,T), \tag{9}$$

$$u(x,0) \geq 0 \quad in \quad \overline{\Omega}. \tag{10}$$

Then, we have $u(x,t) \geq 0$ in $\overline{\Omega} \times (0,T)$.

Proof. Let T_0 be any positive quantity satisfying $T_0 < T$. Since $a(x,t)$ is bounded in $\overline{\Omega}\times[0,T_0]$, then there exists λ such that $a(x,t)-\lambda > 0$ in $\overline{\Omega}\times[0,T]$. Define $z(x,t) = e^{\lambda t}u(x,t)$ and let $m = \min_{x\in\overline{\Omega},t\in[0,T_0]} z(x,t)$. Due to the fact that z is continuous in $\overline{\Omega}\times[0,T_0]$, then it achieves its minimum in $\overline{\Omega}\times[0,T_0]$. Consequently, there exists $(x_0,t_0)\in\overline{\Omega}\times[0,T_0]$ such that $m = z(x_0,t_0)$. We get $z(x_0,t_0)\le z(x_0,t)$ for $t\le t_0$ and $z(x_0,t_0)\le z(y,t_0)$ for $y\in\Omega$. This implies that

$$z_t(x_0,t_0)\le 0, \quad \int_\Omega J(x_0-y)(z(y,t_0)-z(x_0,t_0))dy \ge 0. \tag{11}$$

With the aid of the first inequality of the lemma, it is not hard to see that

$$z_t(x_0,t_0) - \int_\Omega J(x_0-y)(z(y,t_0)-z(x_0,t_0))dy + (a(x_0,t_0)-\lambda)z(x_0,t_0) \ge 0.$$

We deduce from (9) that $(a(x_0,t_0)-\lambda)z(x_0,t_0 \ge 0$. Since $a(x_0,t_0)-\lambda > 0$, we get $z(x_0,t_0)\ge 0$. This implies that $u(x,t)\ge 0$ in $\overline{\Omega}\times[0,T_0]$, and the proof is complete. □

An immediate consequence of the above lemma is that the solution u of (1)-(2) is nonnegative in $\overline{\Omega}\times(0,T)$ because the initial datum $u_0(x)$ is nonnegative in $\overline{\Omega}$.

Now, let us give a result about quenching which says that the solution u of (1)-(2) always quenches in a finite time. This assertion is stated in the theorem below.

Theorem 2.3 *The solution u of (1)-(2) quenches in a finite time, and its quenching time T_h satisfies the following estimate*

$$T \le \frac{(1-A)^{p+1}}{p+1},$$

where $A = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx$.

Proof. Since $(0,T_h)$ is the maximal time interval of existence of the solution u, our aim is to show that T_h is finite and satisfies the above inequality. Due to the fact that the initial datum $u_0(x)$ is nonnegative in $\overline{\Omega}$, we know from Lemma 2.1 that the solution $u(x,t)$ of (1)-(2) is nonnegative in $\overline{\Omega}\times(0,T)$. Integrating both sides of (1) over $(0,t)$, we find that

$$u(x,t)-u_0(x) = \int_0^t\int_\Omega J(x-y)(u(y,s)-u(x,s))dyds$$

$$+\int_0^t (1-u(x,s))^{-p}ds \quad \text{for} \quad t\in(0,T). \tag{12}$$

Integrate again in the x variable and apply Fubini's theorem to obtain

$$\int_\Omega u(x,t)dx - \int_\Omega u_0(x)dx = \int_0^t (\int_\Omega (1-u(x,s))^{-p} dxds \quad \text{for} \quad t \in (0,T). \quad (13)$$

Set

$$w(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx \quad \text{for} \quad t \in [0,T).$$

Taking the derivative of w in t and using (13), we arrive at

$$w'(t) = \int_\Omega \frac{1}{|\Omega|}(1-u(x,s))^{-p} dx \quad \text{for} \quad t \in (0,T).$$

It follows from Jensen's inequality that $w'(t) \geq (1-w(t))^{-p}$ for $t \in (0,T)$, or equivalently

$$(1-w)^p dw \geq dt \quad \text{for} \quad t \in (0,T). \quad (14)$$

Integrate the above inequality over $(0,T)$ to obtain

$$T \leq \frac{(1-w(0))^{p+1}}{p+1}.$$

Since the quantity on the right hand side of the above inequality is finite, we deduce that u quenches in a finite time at the time T which obeys the above inequality. Use the fact that $w(0) = A$ to complete the rest of the proof. □

3 Properties of the semidiscrete scheme

In this section, we give some results about the semidiscrete maximum principle of nonlocal problems for our subsequent use.
The lemma below is a semidiscrete version of the maximum principle for nonlocal parabolic problems

Lemma 3.1 *For $n \geq 0$, let $U_n, a_n \in C^0(\overline{\Omega})$ be such that*

$$\delta_t U_n(x) \geq \int_\Omega J(x-y)(U_n(y)-U_n(x))dy + a_n(x)U_n(x) \quad in \quad \overline{\Omega}, \quad n \geq 0,$$

$$U_0(x) \geq 0 \quad in \quad \overline{\Omega}.$$

Then, we have $U_n(x) \geq 0$ in $\overline{\Omega}$, $n > 0$ when $\Delta t_n \leq \frac{1}{1+\|a_n\|_\infty}$.

Proof. If $U_n(x) \geq 0$ in $\overline{\Omega}$, then a straightforward computation reveals that

$$U_{n+1}(x) \geq U_n(x)(1 - \Delta t_n - \|a_n\|_\infty \Delta t_n) \quad \text{in} \quad \overline{\Omega}, \quad n \geq 0. \tag{15}$$

To obtain the above inequality, we have used the fact that

$$\int_\Omega J(x-y)U_n(y)dy \geq 0 \quad \text{in} \quad \overline{\Omega}, \text{ and } \int_\Omega J(x-y)dy \leq \int_{\mathbb{R}^N} J(x-y)dy = 1.$$

Making use of (15) and an argument of recursion, we easily check that $U_{n+1}(x) \geq 0$ in $\overline{\Omega}$, $n \geq 0$. This finishes the proof. □
An immediate consequence of the above result is the following comparison lemma. Its proof is straightforward.

Lemma 3.2 *For $n \geq 0$, let U_n, V_n and $a_n \in C^0(\overline{\Omega})$ be such that*

$$\delta_t U_n(x) - \int_\Omega J(x-y)(U_n(y) - U_n(x))dy + a_n(x)U_n(x)$$

$$\geq \delta_t V_n(x) - \int_\Omega J(x-y)(V_n(y) - V_n(x))dy + a_n(x)V_n(x) \quad in \quad \overline{\Omega}, \quad n \geq 0,$$

$$U_0(x) \geq V_0(x) \quad in \quad \overline{\Omega}.$$

Then, we have $U_n(x) \geq V_n(x)$ in $\overline{\Omega}$, $n > 0$ when $\Delta t_n \leq \frac{1}{1+\|a_n\|_\infty}$.

Remark 3.3 *Set $Z_n(x) = U_n(x) - \|\varphi\|_\infty$ where U_n is the solution of (5)-(6). A straightforward computation reveals that*

$$\delta_t Z_n(x) \leq \int_\Omega J(x-y)(Z_n(y) - Z_n(x))dy \quad in \quad \overline{\Omega}, \quad n \geq 0,$$

$$Z_0(x) \leq 0 \quad in \quad \overline{\Omega}.$$

It follows from Lemma 2.1 that $U_n(x) \leq \|\varphi\|_\infty$ in $\overline{\Omega}$, $n \geq 0$.

4 The semidiscrete quenching time

In this section, we show that the semidiscrete solution quenches in a finite time and estimate its semidiscrete quenching time.
Our result concerning the semidiscrete quenching time is stated in the following theorem.

Theorem 4.1 *The semidiscrete solution U_n of (5)-(6) quenches in a finite time, and its quenching time $T^{\Delta t}$ obeys the following estimate*

$$T^{\Delta t} \leq \frac{\tau \varphi_{\min}^{p+1}}{1-(1-\tau')^{p+1}},$$

where $\tau' = A \min\{\Delta t \varphi_{\min}^{-p-1}, \tau\}$ and $A = 1 - \|\varphi\|_\infty^{p+1}$.

Proof. We know from Remark 3.1 that $\|U_n\|_\infty \leq \|\varphi\|_\infty$. Since $\int_\Omega J(x-y)dy \leq \int_{\mathbb{R}^N} J(x-y)dx = 1$, exploiting (1), we see that

$$\delta_t U_n(x) \leq \|\varphi\|_\infty - (U_n(x))^{-p} \quad \text{in} \quad \overline{\Omega}, \quad n \geq 0,$$

or equivalently

$$\delta_t U_n(x) \leq -(U_n(x))^{-p}(1 - \|\varphi\|_\infty (U_n(x))^p) \quad \text{in} \quad \overline{\Omega}, \quad n \geq 0.$$

Use the fact that $\|U_n\|_\infty \leq \|\varphi\|_\infty$, $n \geq 0$ to arrive at

$$\delta_t U_n(x) \leq -(U_n(x))^{-p}(1 - \|\varphi\|_\infty^{p+1}) \quad \text{in} \quad \overline{\Omega}, \quad n \geq 0.$$

This estimate may be rewritten as follows

$$U_{n+1}(x) \leq U_n(x) - A\Delta t_n (U_n(x))^{-p} \quad \text{in} \quad \overline{\Omega}, \quad n \geq 0. \tag{16}$$

Let $x_0 \in \overline{\Omega}$ be such that $U_n(x_0) = U_{nmin}$. Replacing x by x_0 in (16), we note that

$$U_{n+1}(x_0) \leq U_{nmin} - A\Delta t_n U_{nmin}^{-p}, \quad n \geq 0,$$

which implies that

$$U_{n+1min} \leq U_{nmin} - A\Delta t_n U_{nmin}^{-p}, \quad n \geq 0, \tag{17}$$

because $U_{n+1}(x_0) \geq U_{n+1min}$. We observe that

$$A\Delta t_n U_{nmin}^{-p-1} = A \min\{\Delta t U_{nmin}^{-p-1}, \tau\}. \tag{18}$$

Exploiting (17), we see that $U_{n+1min} \leq U_{nmin}$, $n \geq 0$, and by induction, we note that $U_{nmin} \leq U_{0min} = \varphi_{\min}$. In view of (18), we discover that

$$A\Delta t_n U_{nmin}^{-p-1} \geq A \min\{\Delta t \varphi_{\min}^{-p-1}, \tau\} = \tau'. \tag{19}$$

Therefore, employing (17), we get

$$U_{n+1min} \leq U_{nmin}(1-\tau'), \quad n \geq 0. \tag{20}$$

Using an argument of recursion, we find that

$$U_{nmin} \leq U_{0min}(1-\tau')^n = \varphi_{\min}(1-\tau')^n, \quad n \geq 0. \tag{21}$$

This implies that U_{nmin} goes to zero as n approaches infinity. Now, let us estimate the semidiscrete quenching time. The restriction on the time step and (21) lead us to

$$\sum_{n=0}^{\infty} \Delta t_n \leq \tau \varphi_{\min}^{p+1} \sum_{n=0}^{\infty} ((1-\tau')^{p+1})^n. \tag{22}$$

Use the fact that the series on the right hand side of the above inequality converges towards $\frac{1}{1-(1-\tau')^{p+1}}$ to complete the rest of the proof. □

Remark 4.2 *Due to (20), an argument of recursion reveals that*

$$U_{nmin} \leq U_{qmin}(1-\tau')^{n-q}, \quad n \geq q.$$

In view of the above estimate, the restriction on the time step allows us to write

$$\sum_{n=q}^{\infty} \Delta t_n \leq \tau U_{qmin}^{p+1} \sum_{n=q}^{\infty} ((1-\tau')^{p+1})^{n-q}.$$

Since the series on the right hand side of the above inequality converges towards $\frac{1}{1-(1-\tau')^{p+1}}$, *we infer that*

$$\sum_{n=q}^{\infty} \Delta t_n \leq \frac{\tau U_{qmin}^{p+1}}{1-(1-\tau')^{p+1}},$$

or equivalently

$$T^{\Delta t} - t_q \leq \frac{\tau U_{qmin}^{p+1}}{1-(1-\tau')^{p+1}}.$$

Apply Taylor's expansion to obtain $(1-\tau')^{p+1} = 1-(p+1)\tau' + o(\tau')$. *This implies that* $\frac{\tau}{1-(1-\tau')^{p+1}} = \frac{\tau}{\tau'((p+1)+o(1))}$. *Due to the fact that* $\tau' = A\min\{\Delta t \varphi_{\min}^{-p-1}, \tau\}$, *if we choose* $\tau = \Delta t$, *then we note that* $\frac{\tau'}{\tau} = A\min\{\varphi_{\min}^{-p-1}, 1\}$, *which implies that* $\frac{\tau}{\tau'} = O(1)$ *with the choice* $\tau = \Delta t$.

In the sequel, we pick $\tau = \Delta t$.

5 Convergence of the semidiscrete quenching time

In this section, under some hypotheses, we prove that the semidiscrete solution quenches in a finite time, and its semidiscrete quenching time converges to the real one when the mesh size goes to zero. In order to obtain this result, we firstly prove that the semidiscrete solution approaches the real one in any interval $\overline{\Omega} \times [0, T-\tau]$ with $\tau \in (0, T)$. This result is stated in the following theorem.

Theorem 5.1 *Assume that the problem (3)-(4) admits a solution* $v \in C^{0,2}(\overline{\Omega}\times [0, T-\tau])$ *with* $\tau \in (0, T)$. *Then, the problem (5)-(6) admits a unique solution* $U_n \in C^0(\overline{\Omega})$ *for* Δt *small enough,* $n \leq J$, *and the following relation holds*

$$\sup_{0\leq n\leq J} \|U_n - u(\cdot, t_n)\|_\infty = O(\Delta t) \quad as \quad \Delta t \to 0,$$

where J *is a positive integer such that* $\sum_{j=0}^{J-1} \Delta t_j \leq T - \tau$, *and* $t_n = \sum_{j=0}^{n-1} \Delta t_j$.

Proof. The problem (5)-(6) admits for each $n \geq 0$, a unique solution $U_n \in C^0(\overline{\Omega})$. Let $N \leq J$ be the greatest integer such that

$$\|U_n - u(\cdot, t_n)\|_\infty < \frac{\alpha}{2} \quad \text{for} \quad n < N. \tag{23}$$

Making use of the fact that (23) holds when $n = 0$, we note that $N \geq 1$. An application of the triangle inequality renders

$$U_{nmin} \leq u_{\min}(t_n) + \|U_n - u(\cdot, t_n)\|_\infty \leq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} \quad \text{for} \quad n < N. \tag{24}$$

Exploit Taylor's expansion to obtain

$$\delta_t u(x, t_n) = u_t(x, t_n) + \frac{\Delta t_n}{2} u_{tt}(x, \widetilde{t_n}) \quad \text{in} \quad \overline{\Omega}, \quad n < N,$$

which implies that

$$\delta_t u(x, t_n) = \int_\Omega J(x-y)(u(y, t_n) - u(x, t_n))dy - (u(x, t_n))^{-p}$$

$$+\frac{\Delta t_n}{2} u_{tt}(x, \widetilde{t_n}) \quad \text{in} \quad \overline{\Omega}, \quad n < N.$$

Introduce the error e_n defined as follows

$$e_n(x) = U_n(x) - u(x, t_n) \quad \text{in} \quad \overline{\Omega}, \quad n < N.$$

Invoking the mean value theorem, it is easy to see that

$$\delta_t e_n(x) = \int_\Omega J(x-y)(e_n(y) - e_n(x))dy + p(\xi_n(x))^{-p-1} e_n(x)$$

$$-\frac{\Delta t_n}{2} u_{tt}(x, \widetilde{t_n}) \quad \text{in} \quad \overline{\Omega}, \quad n < N,$$

where $\xi_n(x)$ is an intermediate value between $u(x, t_n)$ and $U_n(x)$. We infer that there exists a positive constant K such that

$$\delta_t e_n(x) \leq \int_\Omega J(x-y)(e_n(y) - e_n(x))dy + p(\xi_n(x))^{-p-1} e_n(x)$$

$$+K\Delta t \quad \text{in} \quad \overline{\Omega}, \quad n < N, \tag{25}$$

because $u \in C^{0,2}$ and $\Delta t_n = O(\Delta t)$. Introduce the function Z_n defined as follows

$$Z_n(x) = K\Delta t e^{(L+1)t_n} \quad \text{in} \quad \overline{\Omega}, \quad n < N,$$

where $L = p\left(\frac{\alpha}{2}\right)^{-p-1}$. A straightforward computation reveals that

$$\delta_t Z_n \geq \int_\Omega J(x-y)(Z_n(y) - Z_n(x))dy + p(\xi_n(x))^{-p-1} Z_n(x)$$

$$+K\Delta t \quad \text{in} \quad \overline{\Omega}, \quad n < N,$$

$$Z_0(x) \geq e_0(x) \quad \text{in} \quad \overline{\Omega}.$$

We deduce from Lemma 3.2 that

$$Z_n(x) \geq e_n(x) \quad \text{in} \quad \overline{\Omega}, \quad n < N.$$

In the same way, we also show that

$$Z_n(x) \geq -e_n(x) \quad \text{in} \quad \overline{\Omega}, \quad n < N,$$

which implies that

$$|e_n(x)| \leq Z_n(x) \quad \text{in} \quad \overline{\Omega}, \quad n < N,$$

or equivalently

$$\|U_n - u(\cdot, t_n)\|_\infty \leq K\Delta t e^{(L+1)t_n}, \quad n < N. \tag{26}$$

Now, let us reveal that $N = J$. To prove this result, we argue by contradiction. Assume that $N < J$. Replacing n by N in (26), and using (23), we discover that

$$\frac{\alpha}{2} \leq \|U_N - u(\cdot, t_N)\|_\infty \leq K\Delta t e^{(L+1)T}.$$

Since the term on the right hand side of the second inequality goes to zero as Δt tends to zero, we deduce that $\frac{\alpha}{2} \leq 0$, which is impossible. Consequently, $N = J$, and the proof is complete. □

Now, we are in a position to prove the main result of this section.

Theorem 5.2 *Assume that the problem (3)-(4) has a solution u which quenches in a finite time T such that $v \in C^{0,2}(\overline{\Omega} \times [0, T))$. Then, the solution U_n of (5)-(6) quenches in a finite time, and its semidiscrete quenching time $T^{\Delta t}$ obeys the following relation*

$$\lim_{\Delta t \to 0} T^{\Delta t} = T.$$

Proof. Let $0 < \varepsilon < T/2$. In view of Remark 4.1, we know that $\frac{\tau}{\tau'}$ is bounded. Thus, there exists a positive constant ρ such that

$$\frac{\tau \rho^{p+1}}{1-(1-\tau')^{p+1}} \leq \frac{\varepsilon}{2}. \tag{27}$$

Since u quenches at the time T, there exists a time $T_0 \in (T-\varepsilon/2, T)$ such that

$$0 < u_{\min}(t) < \frac{\rho}{2} \quad \text{for} \quad t \in [T_0, T).$$

Let q be a positive integer such that

$$t_q = \sum_{n=0}^{q-1} \Delta t_n \in [T_0, T).$$

Invoking Theorem 4.1, we know that the problem (5)-(6) admits a unique solution $U_n \in C^0(\overline{\Omega})$ such that $\|U_q - u(\cdot, t_q)\|_\infty \leq \frac{\rho}{2}$. An application of the triangle inequality gives $U_{qmin} \leq u_{\min}(t_q) + \|U_q - u(\cdot, t_q)\|_\infty$, which implies that $U_{qmin} \leq \frac{\rho}{2} + \frac{\rho}{2} = \rho$. It follows from Remark 4.1 and (27) that

$$|T^{\Delta t} - T| \leq |T^{\Delta t} - t_q| + |t_q - T| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This finishes the proof. □

6 Numerical results

In this section, we give some computational experiments to illustrate the theory given in the previous section. We consider the problem (3)-(4) in the case where $\Omega = (-1, 1)$,

$$J(x) = \begin{cases} \frac{3}{2}x^2 & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| \geq 1, \end{cases}$$

$\varphi(x) = \frac{2+\varepsilon\cos(\pi x)}{4}$ with $\varepsilon \in (0,1)$. We start by the construction of some adaptive schemes as follows. Let I be a positive integer and let $h = 2/I$. Define the grid $x_i = -1 + ih$, $0 \leq i \leq I$, and approximate the solution v of (3)-(4) by the solution $U_h^{(n)} = (U_0^{(n)}, \cdots, U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \sum_{j=0}^{I-1} hJ(x_i - x_j)(U_j^{(n)} - U_i^{(n)}) - (U_i^{(n)})^{-p}, \quad 0 \leq i \leq I,$$

$$U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq I,$$

where $\varphi_i = \frac{2+\varepsilon\cos(\pi x_i)}{4}$. In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the time step so that we take

$$\Delta t_n = \min\{h^2, h^2(U_{\min}^{(n)})^{p+1}\}$$

with $U_{\min}^{(n)} = \min_{0\le i\le I} U_i^{(n)}$. Let us notice that the restriction on the time step ensures the positivity of the discrete solution. We also approximate the solution u of (1)-(2) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \sum_{j=0}^{I-1} hJ(x_i - x_j)(U_j^{(n+1)} - U_i^{(n+1)}) - (U_i^{(n)})^{-p}, \quad 0 \le i \le I,$$

$$U_i^{(0)} = \varphi_i, \quad 0 \le i \le I.$$

As in the case of the explicit scheme, here, we also choose

$$\Delta t_n = h^2(U_{\min}^{(n)})^{p+1}.$$

Let us again remark that for the above implicit scheme, existence and positivity of the discrete solution are also guaranteed using standard methods (see, for instance [9]).

We need the following definition.

Definition 6.1 *We say that the discrete solution* $U_h^{(n)}$ *of the explicit scheme or the implicit scheme quenches in a finite time if* $\lim_{n\to\infty} U_{\min} = 0$*, and the series* $\sum_{n=0}^{\infty} \Delta t_n$ *converges. The quantity* $\sum_{n=0}^{\infty} \Delta t_n$ *is called the numerical quenching time of the discrete solution* $U_h^{(n)}$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |t_{n+1} - t_n| \le 10^{-16}.$$

The order (s) of the method is computed from

$$s = \frac{\log((T_{2h} - T_h)/(T_{4h} - T_{2h}))}{\log(2)}.$$

Remark 6.2 *If we consider the problem (3)-(4) in the case where* $u_0(x) = 1/2$*, then using standard methods, one may easily check that the quenching time of the solution* u *is* $T = 0.125$*. We note from Tables 1 to 8 that the numerical quenching time of the discrete solution goes to 0.125 when* ε *diminishes. We observe in passing the continuity of the numerical quenching time.*

In what follows, we also give some plots to illustrate our analysis. In Figures 1-8, we can appreciate that the discrete solution quenches in a finite time at the first node.

Numerical experiments for $p = 1$
First case: $\varepsilon = 1$

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.0317443	927	1.8	-
32	0.0313563	3545	15.5	-
64	0.0312717	13488	136	2.21
128	0.0312546	51131	2162	2.20

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.0317562	927	2.2	-
32	0.0313576	3545	21	-
64	0.0312719	13488	186	2.21
128	0.0312547	51131	1879	2.31

Second case: $\varepsilon = 1/10$

Table 3: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.1139599	967	2	-
32	0.1130846	3711	18	-
64	0.1128777	14154	141	2.08
128	0.1128284	53793	2340	2.07

Table 4: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.1140762	967	2.2	-
32	0.1131010	3711	21.5	-
64	0.1128799	14154	196	2.14
128	0.1128286	53793	2460	2.11

Third case: $\varepsilon = 1/100$

Table 5: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.1248243	970	2	-
32	0.1240248	3723	17.5	-
64	0.1238216	14201	144	1.98
128	0.1237703	53982	1352	1.99

Table 6: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.1249639	970	2.2	-
32	0.1240446	3723	21.3	-
64	0.1238243	14201	196	2.06
128	0.1237706	53982	2380	2.04

Fourth case: $\varepsilon = 1/1000$

Table 7: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.1259359	971	2.5	-
32	0.1251463	3728	18.2	-
64	0.1249438	14205	148	1.17
128	0.1248923	54001	1320	3.95

Table 8: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	CPU time	s
16	0.1260731	971	2.3	-
32	0.1251665	3728	21	-
64	0.1249465	14205	197	2.04
128	0.1248927	54001	2310	2.03

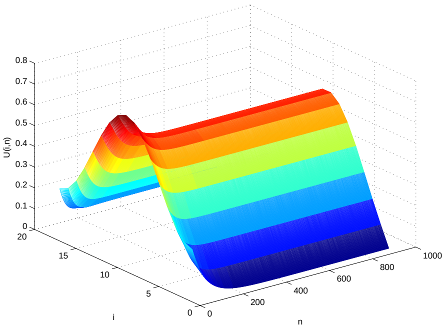

Figure 1: Evolution of the explicit discrete solution, $\varepsilon = 1$

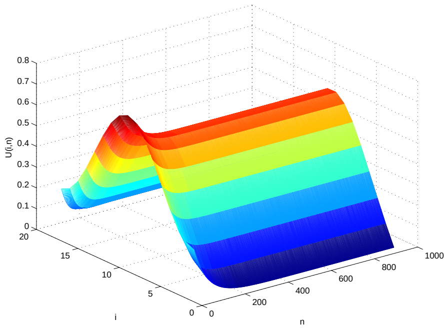

Figure 2: Evolution of the implicit discrete solution, $\varepsilon = 1$

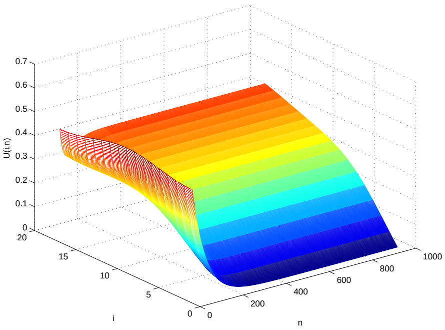

Figure 3: Evolution of the explicit discrete solution, $\varepsilon = 1/10$

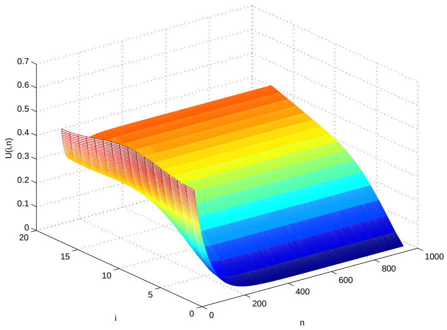

Figure 4: Evolution of the implicit discrete solution, $\varepsilon = 1/10$

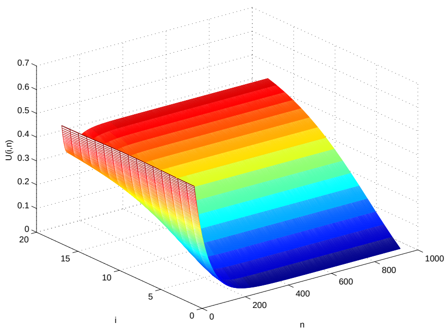

Figure 5: Evolution of the explicit discrete solution, $\varepsilon = 1/100$

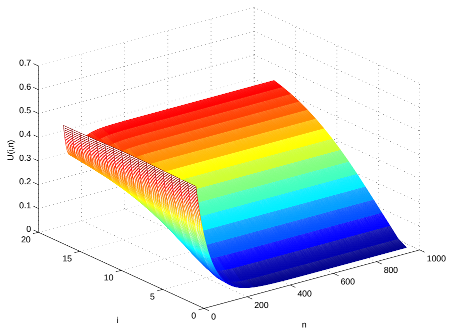

Figure 6: Evolution of the implicit discrete solution, $\varepsilon = 1/100$

7 Conclusion

In the present paper, we have studied the phenomenon of quenching of a nonlocal problem using a semidiscrete scheme. Also, due to the fact that the solution of the above problem increases rapidly when the time t approaches the quenching time T, we have utilized an adaptive scheme which is the scheme appropriate to this kind of problems. Finally, some numerical results are given for a good illustration of the theory developed in the paper.

8 Open Problem

In this paper, we have treated the phenomenon of quenching using a semidiscrete scheme and a particular nonlinearity. In future studies, one may consider a similar problem using a general nonlinearity. On the other hand, to handle the phenomenon of quenching, we have taken into account a semidiscrete scheme. It will be better in the works to come to consider the phenomenon of quenching using full discrete schemes.

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