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Involute-Evolute Curve Couples in the Euclidean 4-Space

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Abstract

***In this work, for regular involute-evolute curve couples, it is proven that evolute's Frenet apparatus can be formed by involute's apparatus in four dimensional Euclidean space when involute curve has constant Frenet curvatures. By this way, another orthonormal frame of the same space is obtained via the method expressed in* [5]. *Moreover, it is observed that evolute curve cannot be an inclined curve. In an analogous way, we use method of* [6].**

Keywords: *Classical Differential Geometry, Euclidean space, Involute-Evolute curve couples, Inclined curves.*

1 Introduction

The idea of a string involute is due to C. Huygens (1658), who is also known for his work in optics. He discovered involutes while trying to build a more accurate clock (see [1]). The involute of a given curve is a well-known concept in Euclidean-3 space E^3 (see [3]).

It is well-known that, if a curve differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve, is called ***Frenet apparatus*** of the curves.

In recent years, the theory of degenerate submanifolds has been treated by researchers and some classical differential geometry topics have been extended to

Lorentz manifolds. For instance, in [6], the authors extended and studied space-like involute-evolute curves in Minkowski space-time.

In the presented paper, in an analogous way as in [6], we calculate Frenet apparatus of the evolute curve by apparatus of involute *W*-curve. We use the method expressed in [5]. Thereafter, we prove that evolute of an involute cannot be an inclined curve in four dimensional Euclidean space.

2 Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E^4 are briefly presented (A more complete elementary treatment can be found in [3]).

Let $\vec{\alpha}: I \subset R \to E^4$ be an arbitrary curve in the Euclidean space E^4. Recall that the curve $\vec{\alpha}$ is said to be of unit speed (or parametrized by arclength function s) if $\langle \vec{\alpha}', \vec{\alpha}' \rangle = 1$, where $\langle .,. \rangle$ is the standard scalar (inner) product of E^4 given by

$$\langle \vec{a}, \vec{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 \tag{1}$$

for each $\vec{a} = (a_1, a_2, a_3, a_4)$, $\vec{b} = (b_1, b_2, b_3, b_4) \in E^4$. In particular, the norm of a vector $\vec{a} \in E^4$ is given by $\|\vec{a}\| = \sqrt{\langle \vec{a}, \vec{a} \rangle}$.

Denote by $\{\vec{T}(s), \vec{N}(s), \vec{B}(s), \vec{E}(s)\}$ the moving Frenet frame along the unit speed curve $\vec{\alpha}$. Then the Frenet formulas are given by (see [2])

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \\ \vec{E}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \\ \vec{E} \end{bmatrix}. \tag{2}$$

Here $\vec{T}, \vec{N}, \vec{B}$ and $\vec{E}$ are called, respectively, the tangent, the normal, the binormal and the trinormal vector fields of the curves. And the functions $\kappa(s), \tau(s)$ and $\sigma(s)$ are called, respectively, the first, the second and the third curvature of the curve $\vec{\alpha}$. And, recall that a regular curve is called a ***W***-curve if it has constant Frenet curvatures [4]. Let $\vec{\alpha}: I \subset R \to E^4$ be a regular curve. If tangent vector field $\vec{T}$ of $\vec{\alpha}$ forms a constant angle with unit vector $\vec{U}$, this curve is called an inclined curve in E^4. Let $\vec{\phi}$ and $\vec{\xi}$ be unit speed regular curves in E^4 $\vec{\phi}$ is an involute of $\vec{\xi}$ if $\vec{\phi}$ lies on the tangent line to $\vec{\xi}$ at $\vec{\xi}(s_0)$ and the tangents to $\vec{\xi}$, $\vec{\xi}(s_0)$ and $\vec{\phi}$ are perpendicular for each s_0. $\vec{\phi}$ is an evolute of $\vec{\xi}$ if $\vec{\xi}$ is an involute of $\vec{\phi}$. And this curve couple defined by $\vec{\phi} = \vec{\xi} + \mu \vec{T}$.

In the same space, in [5], author presented a vector product and a method to determine Frenet apparatus of the regular curves as follows.

Definition 2.1 *Let* $\vec{a} = (a_1, a_2, a_3, a_4)$, $\vec{b} = (b_1, b_2, b_3, b_4)$ *and* $\vec{c} = (c_1, c_2, c_3, c_4)$ *be vectors in* E^4*.The vector product of* $\vec{a}, \vec{b}$ *and* $\vec{c}$ *is defined by the determinant*

$$\vec{a} \wedge \vec{b} \wedge \vec{c} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix},$$

where

$$\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 = \vec{e}_4, \; \vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_4 = \vec{e}_1,$$
$$\vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_1 = \vec{e}_2, \vec{e}_4 \wedge \vec{e}_1 \wedge \vec{e}_2 = \vec{e}_3, \; \vec{e}_3 \wedge \vec{e}_2 \wedge \vec{e}_1 = \vec{e}_4.$$

Theorem 2.2 *Let* $\vec{\alpha} = \vec{\alpha}(t)$ *be an arbitrary curve in* E^4. *Frenet apparatus of the* $\vec{\alpha}$ *can be calculated by the following equations.*

$$\vec{T} = \frac{\vec{\alpha}'}{\|\vec{\alpha}'\|}, \tag{3}$$

$$\vec{N} = \frac{\|\vec{\alpha}'\|^2 \vec{\alpha}'' - \langle \vec{\alpha}', \vec{\alpha}'' \rangle \vec{\alpha}'}{\left\| \|\vec{\alpha}'\|^2 \vec{\alpha}'' - \langle \vec{\alpha}', \vec{\alpha}'' \rangle \vec{\alpha}' \right\|}, \tag{4}$$

$$\vec{B} = \eta \, \vec{E} \wedge \vec{T} \wedge \vec{N}, \tag{5}$$

$$\vec{E} = \eta \frac{\vec{T} \wedge \vec{N} \wedge \vec{\alpha}'''}{\|\vec{T} \wedge \vec{N} \wedge \vec{\alpha}'''\|}, \tag{6}$$

$$\kappa = \frac{\left\| \|\vec{\alpha}'\|^2 \vec{\alpha}'' - \langle \vec{\alpha}', \vec{\alpha}'' \rangle \vec{\alpha}' \right\|}{\|\vec{\alpha}'\|^4}, \tag{7}$$

$$\tau = \frac{\left\| \vec{T} \wedge \vec{N} \wedge \vec{\alpha}''' \right\| \|\vec{\alpha}'\|}{\left\| \|\vec{\alpha}'\|^2 \vec{\alpha}'' - \langle \vec{\alpha}', \vec{\alpha}'' \rangle \vec{\alpha}' \right\|}, \tag{8}$$

$$\sigma = \frac{\langle \vec{\alpha}^{(IV)}, \vec{E} \rangle}{\left\| \vec{T} \wedge \vec{N} \wedge \vec{\alpha}''' \right\| \|\vec{\alpha}'\|} \tag{9}$$

where η is taken ± 1 to make 1 determinant of $[\vec{T}, \vec{N}, \vec{B}, \vec{E}]$ matrix.

3 Main Results

Theorem 3.1 *Let $\vec{\phi}$ be involute of $\vec{\xi}$ and $\vec{\xi}$ be a W-curve in E^4. The Frenet apparatus of $\vec{\phi}$ $\left(\{\vec{T}_\phi, \vec{N}_\phi, \vec{B}_\phi, \vec{E}_\phi, \kappa_\phi, \tau_\phi, \sigma_\phi\}\right)$ can be formed by apparatus of $\vec{\xi}$ $\left(\{\vec{T}, \vec{N}, \vec{B}, \vec{E}, \kappa, \tau, \sigma\}\right)$.*

Proof From definition of involute-evolute curve couples we know

$$\vec{\phi} = \vec{\xi} + \mu \vec{T}. \tag{10}$$

Differentiating both sides with respect to s, we obtain

$$\frac{d\vec{\phi}}{ds_\phi} \frac{ds_\phi}{ds} = \vec{T} + \frac{d\mu}{ds} \vec{T} + \mu\kappa\vec{N}. \tag{11}$$

Definition of such kind curves yields [3], $\vec{T}_\phi \perp \vec{T}$. Therefore, we have

$$\frac{d\mu}{ds} + 1 = 0. \tag{12}$$

Hence $\mu = c - s$. Then, we write

$$\vec{\phi} = \vec{\xi} + (c - s)\vec{T} \tag{13}$$

and

$$\vec{T}_\phi \frac{ds_\phi}{ds} = (c - s)\kappa\vec{N}. \tag{14}$$

Taking the norm of both sides (here $.$ denotes derivative according to s)

$$\vec{T}_\phi = \vec{N} \tag{15}$$

and

$$\left\| \dot{\vec{\phi}} \right\| = (c - s)\kappa. \tag{16}$$

Considering the presented method, we calculate differentiations of (14) four times. We write, respectively,

$$\ddot{\vec{\phi}} = -\kappa^2 (c-s)\vec{T} - \kappa\vec{N} + \kappa\tau(c-s)\vec{B}. \tag{17}$$

$$\dddot{\vec{\phi}} = \left\{-2\kappa^2\vec{T} - \kappa(c-s)\left[\kappa^2 + \tau^2\right]N - 2\kappa\tau\vec{B} + \kappa\tau\sigma(c-s)\vec{E}\right\}. \tag{18}$$

$$\ddddot{\vec{\phi}} = \left\{\kappa^2 (c-s)\left[\kappa^2 + \tau^2\right]\vec{T} - 2\kappa\left[\kappa^2 + \tau^2\right]\vec{N} - \kappa\tau(c-s)\left[\kappa^2 + \tau^2 + \sigma^2\right]\vec{B} - 2\kappa\tau\sigma\vec{E}\right\}. \tag{19}$$

Considering (4), we form

$$\left\|\dot{\vec{\phi}}\right\|^2 \ddot{\vec{\phi}} - \left\langle \dot{\vec{\phi}}, \ddot{\vec{\phi}} \right\rangle \dot{\vec{\phi}} == \kappa(c-s)\left[-\kappa\vec{T} + \tau\vec{B}\right]. \tag{20}$$

Since, we have the principal normal and the first curvature of the curve

$$\vec{N}_\phi = \frac{-\kappa\vec{T} + \tau\vec{B}}{\sqrt{\kappa^2 + \tau^2}} \tag{21}$$

and

$$\kappa_\phi = \frac{\sqrt{\kappa^2 + \tau^2}}{\left[(c-s)\kappa\right]^3}. \tag{22}$$

Using vector product of $\vec{T}_\phi \wedge \vec{N}_\phi \wedge \dddot{\vec{\phi}}$, we get

$$\vec{T}_\phi \wedge \vec{N}_\phi \wedge \dddot{\vec{\phi}} = \begin{vmatrix} \vec{T} & \vec{N} & \vec{B} & \vec{E} \\ 0 & 1 & 0 & 0 \\ -\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} & 0 & \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} & 0 \\ \dddot{\phi}_1 & \dddot{\phi}_2 & \dddot{\phi}_3 & \dddot{\phi}_4 \end{vmatrix} \tag{23}$$

$$= \frac{\kappa\tau}{\sqrt{\kappa^2 + \tau^2}}\left[\tau\sigma(c-s)\vec{T} + \kappa\sigma(c-s)\vec{B} + 3\kappa\vec{E}\right].$$

We obtain trinormal vector

$$\vec{E}_\phi = \eta \frac{\tau\sigma(c-s)\vec{T} + \kappa\sigma(c-s)\vec{B} + 3\kappa\vec{E}}{\sqrt{\left[\tau\sigma(c-s)\right]^2 + \left[\kappa\sigma(c-s)\right]^2 + 9\kappa^2}}. \tag{24}$$

By this way, we easily have the second curvature

$$\tau_\phi = \sqrt{\frac{\left[\tau\sigma(c-s)\right]^2 + \left[\kappa\sigma(c-s)\right]^2 + 9\kappa^2}{\kappa^2 + \tau^2}}. \tag{25}$$

Considering (9) and (24), we have the third curvature of the curve $\vec{\phi}$ as

$$\sigma_\phi = \frac{-\kappa^2\tau\sigma\left[\sigma^2(c-s)^2-3\right]}{\sqrt{\left[\tau\sigma(c-s)\right]^2+\left[\kappa\sigma(c-s)\right]^2+9\kappa^2}}. \tag{26}$$

Finally, the vector product of $\vec{E}_\phi \wedge \vec{T}_\phi \wedge \vec{N}_\phi$ follows that

$$\vec{B}_\phi = \frac{\eta}{A\sqrt{\kappa^2+\tau^2}}\left\{-3\kappa\tau\vec{T}-3\kappa^2\vec{B}+\sigma(c-s)(\kappa^2+\tau^2)\vec{E}\right\}, \tag{27}$$

where $A=\sqrt{\left[\tau\sigma(c-s)\right]^2+\left[\kappa\sigma(c-s)\right]^2+9\kappa^2}$.

Theorem 3.2 *Let $\vec{\phi}$ and $\vec{\xi}$ be unit speed regular curves in E^4. $\vec{\phi}$ be involute of $\vec{\xi}$. The evolute $\vec{\phi}$ cannot be an inclined curve.*

Proof By the definition of inclined curves, we may write

$$\left\langle \vec{T}_\phi, \vec{U}\right\rangle = \cos\varphi, \tag{28}$$

where $\vec{U}$ is a constant vector and φ is a constant angle. Considering (15), we easily have

$$\left\langle \vec{N}, \vec{U}\right\rangle = \cos\varphi. \tag{29}$$

Differentiating (29), we obtain

$$\left\langle -\kappa\vec{T}+\tau\vec{B}, \vec{U}\right\rangle = 0. \tag{30}$$

Therefore, we may write $\vec{T}\perp U$ and $\vec{B}\perp\vec{U}$. Let us decompose $\vec{U}$ as

$$\vec{U}=u_1\vec{N}+u_2\vec{E}. \tag{31}$$

One more differentiating of (31) and using Frenet equations, we have

$$\vec{U}=0, \tag{32}$$

which is a contradiction. Thus, evolute $\vec{\phi}$ cannot be an inclined curve.

4 Conclusion

Considering obtained equations, we express the following results.

Corollary 4.1 $\left\{\vec{T}_\phi, \vec{N}_\phi, \vec{B}_\phi, \vec{E}_\phi\right\}$ *is an orthonormal frame of E^4.*

Corollary 4.2 *In the case $\vec{\xi}$ is a W-curve, suffice it to say that evolute $\vec{\phi}$ may not be a W-curve.*

5 Open Problem

In this work, we investigate relations between involute-evolute curve couples' Frenet apparatus in the Euclidean 4-space. In the existing literature, one can seek details about Bertrand curves in the Euclidean 3-space. Using method of [5], with an analogous way, relations among Frenet apparatus of Bertrand curves may be determined.

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