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A Note On L^1-Convergence of the Sine and Cosine Trigonometric Series with Semi-Convex Coefficients

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Abstract

We introduce new modified cosine and sine sums and study their L^1-convergence to the sine and cosine trigonometric series respectively and deduce a result of Bala and Ram [1] as a corollary.

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1 Introduction and Preliminaries

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1}$$

and

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{2}$$

be cosine and sine trigonometric series respectively with their partial sums denoted by $S_n^c(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$, $S_n^s(x) = \sum_{k=1}^{n} a_k \sin kx$ and let $f(x) = \lim_{n\to\infty} S_n^c(x)$, $g(x) = \lim_{n\to\infty} S_n^s(x)$.

For convenience, in the following of this paper we shall assume that $a_{-1} = a_0 = 0$.

A sequence (a_n) is said to be semi-convex if $a_n \to 0, n \to \infty$ and

$$\sum_{n=1}^{\infty} n|\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, \tag{3}$$

where $\Delta^2 a_k = \Delta(\Delta a_k) = a_k - 2a_{k+1} + a_{k+2}$.

R. Bala and B. Ram [1] have proved that for series (1) with semi-convex null coefficients the following theorem holds true.

Theorem A. *If (a_n) is a semi-convex null sequence, then for the convergence of the cosine series (1) in the metric L^1, it is necessary and sufficient that $a_{n-1}\log n = o(1), n \to \infty$.*

Garret and Stanojević [2] have introduced modified cosine sums

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx.$$

Garret and Stanojević [3], Ram [4], Singh and Sharma [5], and Kaur and Bhatia [6], [7] studied the L^1-convergence of this cosine sum under different sets of conditions on the coefficients a_n.

Kumari and Ram [8] introduced new modified cosine and sine sums as

$$h_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

$$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \sin kx$$

and have studied their L^1-convergence under the condition that the coefficients a_n belong to different classes of sequences. Likewise, they deduced some results about L^1-convergence of cosine and sine series as corollaries.

N. Hooda, B. Ram and S. S. Bhatia [9] introduced new modified cosine sums as

$$r_n(x) = \frac{1}{2}\left(a_1 + \sum_{k=0}^{n} \Delta^2 a_k\right) + \sum_{k=1}^{n}\left(a_{k+1} + \sum_{j=k}^{n} \Delta^2 a_j\right) \cos kx$$

and studied the L^1-convergence of these cosine sums.

Later on, K. Kaur [10] introduced new modified sine sums as

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta a_{j-1} - \Delta a_{j+1}) \sin kx,$$

and studied the L^1-convergence of this modified sine sum with semi-convex coefficients proving the following theorem.

Theorem B. *Let (a_n) be a semi-convex null sequence, then $K_n(x)$ converges to $f(x)$ in L^1-norm.*

We introduce here new modified cosine and sine sums as

$$H_n(x) = -\frac{1}{2\sin x}\sum_{k=0}^{n}\sum_{j=k}^{n}\Delta\left[(a_{j-1}-a_{j+1})\cos jx\right]$$

$$G_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}\Delta\left[(a_{j-1}-a_{j+1})\sin jx\right].$$

The main goal of the present work is to study the L^1-convergence of these new modified sine and cosine sums with semi-convex coefficients and to deduce Theorem A as a corollary.

As usually with $D_n(x)$ and $\tilde{D}_n(x)$ we shall denote the Dirichlet and its conjugate kernels respectively, defined by

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n}\cos kx, \quad \tilde{D}_n(x) = \sum_{k=1}^{n}\sin kx.$$

Everywhere in this paper the constants in the O-expressions denote positive constants and they may be different in different relations.

2 Main Results

Regarding the series (2) we prove the following result:

Theorem 1. *Let (a_n) be a semi-convex null sequence, then $H_n(x)$ converges to $g(x)$ in L^1-norm.*

Proof. We have

$$\begin{aligned}
H_n(x) &= -\frac{1}{2\sin x}\sum_{k=0}^{n}\sum_{j=k}^{n}\Delta\left[(a_{j-1}-a_{j+1})\cos jx\right]\\
&= -\frac{1}{2\sin x}\sum_{k=0}^{n}\left[(a_{k-1}-a_{k+1})\cos kx-(a_n-a_{n+2})\cos(n+1)x\right]\\
&= -\frac{1}{2\sin x}\sum_{k=0}^{n}(a_{k-1}-a_{k+1})\cos kx\\
&\qquad +(n+1)\left(a_n-a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}\\
&= -\frac{1}{2\sin x}\sum_{k=0}^{n}(\Delta a_{k-1}+\Delta a_k)\cos kx\\
&\qquad +(n+1)\left(a_n-a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}. \qquad (4)
\end{aligned}$$

Applying Abel's transformation in (4) (see [11], p. 17), we have

$$\begin{aligned}
H_n(x) &= -\frac{1}{2\sin x}\Bigg\{\sum_{k=0}^{n-1}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)\left(D_k(x)+\frac{1}{2}\right)\\
&\qquad +\left(\Delta a_{n-1}+\Delta a_n\right)\left(D_n(x)+\frac{1}{2}\right)\Bigg\}\\
&\qquad +(n+1)\left(a_n-a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}\\
&= -\frac{1}{2\sin x}\Bigg\{\sum_{k=0}^{n}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)D_k(x)\\
&\qquad -\left(\Delta a_{n-1}-\Delta a_{n+1}\right)D_n(x)\\
&\qquad +\frac{1}{2}\sum_{k=0}^{n-1}\left(\Delta a_{k-1}-\Delta a_{k+1}\right)+\frac{\Delta a_{n-1}+\Delta a_n}{2}\Bigg\}\\
&\qquad +(n+1)\left(a_n-a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}\\
&= -\frac{1}{2\sin x}\sum_{k=0}^{n}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)D_k(x)\\
&\qquad +(n+1)\left(a_n-a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}. \qquad (5)
\end{aligned}$$

On the other side we have

$$\begin{aligned}
S_n^s(x) &= \frac{1}{\sin x}\sum_{k=1}^{n} a_k \sin kx \sin x \\
&= -\frac{1}{2\sin x}\sum_{k=1}^{n} a_k\left[\cos(k+1)x-\cos(k-1)x\right] \\
&= -\frac{1}{2\sin x}\sum_{k=0}^{n}(a_{k-1}-a_{k+1})\cos kx \\
&\qquad\qquad -a_{n+1}\frac{\cos nx}{2\sin x}-a_n\frac{\cos(n+1)x}{2\sin x} \\
&= -\frac{1}{2\sin x}\sum_{k=0}^{n}(\Delta a_{k-1}+\Delta a_k)\cos kx \\
&\qquad\qquad -a_{n+1}\frac{\cos nx}{2\sin x}-a_n\frac{\cos(n+1)x}{2\sin x}. \qquad (6)
\end{aligned}$$

Applying Abel's transformation to the equality (6) we get

$$\begin{aligned}
S_n^s(x) &= -\frac{1}{2\sin x}\left\{\sum_{k=0}^{n-1}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)\left(D_k(x)+\frac{1}{2}\right)\right. \\
&\qquad \left. +\left(\Delta a_{n-1}+\Delta a_n\right)\left(D_n(x)+\frac{1}{2}\right)\right\} \\
&\qquad -a_{n+1}\frac{\cos nx}{2\sin x}-a_n\frac{\cos(n+1)x}{2\sin x} \\
&= -\frac{1}{2\sin x}\left\{\sum_{k=0}^{n}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)D_k(x)+\frac{1}{2}\sum_{k=0}^{n-1}\left(\Delta a_{k-1}-\Delta a_{k+1}\right)\right. \\
&\qquad \left. +\left(\Delta a_n+\Delta a_{n+1}\right)D_n(x)+\frac{\Delta a_{n-1}+\Delta a_n}{2}\right\} \\
&\qquad -a_{n+1}\frac{\cos nx}{2\sin x}-a_n\frac{\cos(n+1)x}{2\sin x} \\
&= -\frac{1}{2\sin x}\sum_{k=0}^{n}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)D_k(x) \\
&\qquad -\left(a_n-a_{n+2}\right)\frac{D_n(x)}{2\sin x}-a_{n+1}\frac{\cos nx}{2\sin x}-a_n\frac{\cos(n+1)x}{2\sin x}. \qquad (7)
\end{aligned}$$

Since (a_n) is semi-convex sequence, then from (3) we have

$$\begin{aligned} |(n+1)(a_n-a_{n+2})| &= (n+1)\left|\sum_{k=n}^{\infty}(\Delta a_k-\Delta a_{k+2})\right| \\ &= (n+1)\left|\sum_{k=n+1}^{\infty}(\Delta a_{k-1}-\Delta a_{k+1})\right| \\ &\le \sum_{k=n+1}^{\infty}k|\Delta^2 a_{k-1}+\Delta^2 a_k| = o(1), n\to\infty. \end{aligned} \tag{8}$$

Using (8) and when we pass on limit as $n\to\infty$ to (5) and (7) we get

$$\begin{aligned} g(x) &= \lim_{n\to\infty} S_n^s(x) \\ &= \lim_{n\to\infty} H_n(x) = -\frac{1}{2\sin x}\sum_{k=0}^{\infty}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)D_k(x). \end{aligned} \tag{9}$$

Applying well-known inequality $D_k(x)\le 1/2+k, k=1,2,\ldots,$ and relations (5), (8) and (9) we have

$$\begin{aligned} &\int_{-\pi}^{\pi}|g(x)-H_n(x)|\,dx = \\ &= \int_{-\pi}^{\pi}\left|\frac{1}{2\sin x}\sum_{k=n+1}^{\infty}\left(\Delta^2 a_{k-1}+\Delta^2 a_k\right)D_k(x)\right. \\ &\quad \left. +(n+1)(a_n-a_{n+2})\frac{\cos(n+1)x}{2\sin x}\right|dx \\ &= O\left(\sum_{k=n+1}^{\infty}k|\Delta^2 a_{k-1}+\Delta^2 a_k|\right) \\ &\quad +O\left(|(n+1)(a_n-a_{n+2})|\right) = o(1), n\to\infty, \end{aligned}$$

which fully proves the Theorem 1.

Corollary 1. *Let (a_n) be a semi-convex null sequence, then the necessary and sufficient condition for L^1-convergence of the series* (2) *is* $a_n\log n = o(1), n\to\infty$.

Proof. Let $\|S_n^s(x)-g(x)\| = o(1), n\to\infty$. We would show that $a_n\log n =$

$o(1), n \to \infty$. Indeed, we have

$$\begin{aligned}
&\left\|g(x) - H_n(x)\right\| + \left\|S_n^s(x) - g(x)\right\| \\
&+\left\|(n+1)\left(a_n - a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}\right\| + \left\|\left(a_n - a_{n+2}\right)\frac{D_n(x)}{2\sin x}\right\| \\
&\qquad\geq \left\|a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x}\right\| \\
&\qquad\geq a_{n+1}\int_{-\pi}^{\pi}\left|\frac{\cos nx}{2\sin x} + \frac{\cos(n+1)x}{2\sin x}\right| dx \\
&\qquad= a_{n+1}\int_{-\pi}^{\pi}\left|\tilde{D}_n(x) - \frac{1}{2}ctg\frac{x}{2}\right| dx \\
&\qquad= a_{n+1}\left(\int_{-\pi}^{\pi}\left|\tilde{D}_n(x)\right| dx - \int_{0}^{\pi}\left|ctg\frac{x}{2}\right| dx\right) \\
&\qquad= O\left(a_{n+1}\log n\right), \qquad (10)
\end{aligned}$$

(see [11], p. 116).

Since $\left\|g(x) - H_n(x)\right\| = o(1)$ by Theorem 1, $\left\|S_n^s(x) - g(x)\right\| = o(1)$ by assumption of Corollary, third and fourth term at the left side of relation (10) tend to 0 by (8), consequently $a_n \log n = o(1), n \to \infty$.

Conversely, let $a_n \log n = o(1), n \to \infty$. Then by (8)

$$\begin{aligned}
\left\|S_n^s(x) - g(x)\right\| &\leq \left\|H_n(x) - g(x)\right\| + \left\|H_n(x) - S_n^s(x)\right\| \\
&= \left\|H_n(x) - g(x)\right\| + \left\|(n+1)\left(a_n - a_{n+2}\right)\frac{\cos(n+1)x}{2\sin x}\right. \\
&\quad \left. + \left(a_n - a_{n+2}\right)\frac{D_n(x)}{2\sin x} + a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x}\right\| \\
&= o(1) + O\left((n+1)|a_n - a_{n+2}|\right) + O\left(n|a_n - a_{n+2}|\right) \\
&\quad +\left\|a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x}\right\| \\
&= o(1) + \left\|a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x}\right\|. \qquad (11)
\end{aligned}$$

But

$$\begin{aligned}
\int_{-\pi}^{\pi}\left|a_{n+1}\frac{\cos nx}{2\sin x} + a_n\frac{\cos(n+1)x}{2\sin x}\right| dx &\leq a_n\int_{-\pi}^{\pi}\left|\frac{\cos nx + \cos(n+1)x}{2\sin x}\right| dx \\
= a_n\int_{-\pi}^{\pi}\left|\frac{1}{2}ctg\frac{x}{2} - \tilde{D}_n(x)\right| dx &= O\left(a_n\log n\right) = o(1),
\end{aligned}$$

therefore from (11) we get $\left\|S_n^s(x) - g(x)\right\| = o(1), n \to \infty$, which implies the conclusion of the Corollary.

Regarding the series (1) the following statements are true:

Theorem 2. *Let (a_n) be a semi-convex null sequence, then $G_n(x)$ converges to $f(x)$ in L^1-norm.*

Corollary 2. *Let (a_n) be a semi-convex null sequence, then the necessary and sufficient condition for L^1-convergence of the series* (1) *is* $a_n \log n = o(1), n \to \infty$.

The proofs of Theorem 2 and Corollary 2 are similar to the proofs of Theorem 1 and Corollary 2, therefore we omit them.

Remark 1. Kaur K. [10] introduced the modified sine sums $K_n(x)$ and studied L^1-convergence of cosine series (1). Here we shall introduce the modified cosine sums $F_n(x)$, similar with $K_n(x)$, as

$$F_n(x) = -\frac{1}{2\sin x}\sum_{k=0}^{n}\sum_{j=k}^{n}(\Delta a_{j-1} - \Delta a_{j+1})\cos kx,$$

where $a_{-1} = a_0 = 0$.

By a similar argument, we can show that the Theorem B also holds for the series (2), when instead of $K_n(x)$ we use $F_n(x)$.

More precisely, the following statement holds.

Theorem 3. *Let (a_n) be a semi-convex null sequence, then $F_n(x)$ converges to $g(x)$ in L^1-norm.*

Remark 2. The Corollary 1 is a consequence of Theorem 3, as well.

3 Open Problem

In [6] Kaur K. and Bhatia S. S. have defined a new class of sequences, so-colled generalized semi-convex sequences, as follows:

A null sequence (a_n) is said to be generalized semi-convex if

$$\sum_{n=1}^{\infty} n^{\alpha}|\Delta^{\alpha+1}a_{n-1} + \Delta^{\alpha+1}a_n| < \infty, \text{ for } \alpha > 0, (a_0 = 0).$$

Since for $\alpha = 1$ a generalized semi-convex sequence reduces to a semi-convex sequence, then the following problem arises :

If we use a generalized semi-convex sequence instead of a semi-convex sequence, how to define the new modified cosine and sine sums $H_n(x)$ and $G_n(x)$ so that our results in this paper can be as corollaries of new results?

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