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Random n-Normed Linear Space

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Abstract

The primary purpose of this paper is to introduce the concept of random n-normed linear space as a generalization of n-normed space which introduce by Gunawan and Mashadi [4] and random 2-normed space which introduce by Ioan Goleţ [6].

Keywords: *random 2-normed space; random n-normed space; n-normed space.*

1 Introduction and Preliminaries

In [2] and [3], S. Gähler introduced an attractive theory of 2-norm and n-norm on a linear space, since these were studied in many papers [7] and [1].

In [4] H. Gunawan and M. Mashadi gave a simple way to derive an (n-1)-norm from the n-norm and realized that any n-normed space is an (n-1)-normed space.

In [9]., K. Menger introduced the notion of probabilistic metric spaces. The idea of K. Menger was to use distribution function instead of non negative real numbers as values of the metric. The concept of random normed spaces were introduced by Serstnev [10] and [8]. We begin with some basic definitions.

Definition 1.1. [2]. Let X be a real linear space of dimension greater than 1 and let $\|\bullet, \bullet\|$be a real valued function on $X \times X$ satisfying the following conditions:

(1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(2) $\|x, y\| = \|y, x\|$,
(3) $\|\alpha x, y\| \leq |\alpha| \, \|x, y\|$, where $\alpha \in \mathbb{R}$,

(4) $\|x, y+z\| \leq \|x, y\| + \|x, z\|$,

$\|\bullet, \bullet\|$ is called a 2-norm on X and the pair $(X, \|\bullet, \bullet\|)$ is called a 2-normed linear space.

Definition 1.2.[4]. Let $n \in \mathbb{N}$ (natural numbers) and X be a real vector space of dimension $d \geq n$. A real valued function $\|\bullet, \bullet, ..., \bullet\|$ on $\underbrace{X \times ... \times X}_{n} = X^n$, satisfying the following properties:

(1) $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,

(2) $\|x_1, x_2, ..., x_n\|$ is invariant under any permutation of $x_1, x_2, ..., x_n$,

(3) $\|x_1, x_2, ..., \alpha x_n\| = |\alpha| \, \|x_1, x_2, ..., x_n\|$, where $\alpha \in \mathbb{R}$ (set of real numbers),

(4) $\|x_1, x_2, ..., x_{n-1}, y+z\| \leq \|x_1, x_2, ..., x_{n-1}, y\| + \|x_1, x_2, ..., x_{n-1}, z\|$,

$\|\bullet, \bullet, ..., \bullet\|$ is called an n-norm on X and the pair $(X, \|\bullet, \bullet, ..., \bullet\|)$ is called an n-normed linear space.

Definition 1.3.[11]. A distance distribution function (briefly, a d.d.f.), is a function F defined from extended interval $[0, +\infty]$ into the unit interval $I = [0, 1]$, that is non decreasing, left continuous on $(0, +\infty)$ such that $F(0) = 0$ and $F(+\infty) = 1$.

The family of all d.d.f.'s will be denoted by Δ^+ We denote

$$D^+ = \left\{ F \in \Delta^+ \backslash \lim_{x \to \infty} F(x) = 1 \right\}.$$

By setting $F \leq G$ whenever $F(x) \leq G(x)$, $\forall x \in \mathbb{R}^+$, one introduces a natural ordering in D^+. If $a \in \mathbb{R}^+$, then ε_a will be an element of D^+, defined by $\varepsilon_a(t) = 0$ if $t \leq a$ and $\varepsilon_a(t) = 1$ if $t > a$. It is obvious that $\varepsilon_a \geq F$ if $t > a$ for all $F \in D^+$. The set D^+ will endowed with the natural topology defined by the modified Lĕvy metric d_L[11].

A t-norm T is a two place function $T : I \times I \longrightarrow I$ which is associative, commutative, non decreasing in each place and such that $T(a, 1) = a$, for all $a \in [0, 1]$. [11].

Definition 1.4. [6]. Let L be a linear space of dimension greater than one, T a triangle function, and let v be a mapping from $L \times L$ into D^+. If the following conditions are satisfied:

$(P2-N1)$ $v_{x,y} = \varepsilon_0$ if x and y are linearly dependent,

$(P2-N2)$ $v_{x,y} \neq \varepsilon_0$ if x and y are linearly independent,

$(P2-N3)$ $v_{x,y} = v_{y,x}$, for every x and y in L,

$(P2-N4)$ $v_{\alpha x,y}(t) = v_{y,x}\left(\frac{t}{|\alpha|}\right)$, for every $t > 0$, $\alpha \neq 0$ and x, $y \in L$,

$(P2-N5)$ $v_{x+y,z} \geq T(v_{x,z}, v_{y,z})$, whenever x, y, $z \in L$,

then v is called a random 2-norm on L and (L, v, T) is called a random 2-normed space.

2 Random n-normed space

By generalizing Definition 1.2., we obtain a satisfactory notion of random n-normed space as follows.

Definition 2.1. Let L be a linear space of a dimension greater than one over a real field. T a triangle function and let v be a mapping from $\underbrace{L \times L \times ... \times L}_{n} \times \mathbb{R}$($\mathbb{R}$, set of real numbers) into D^+. If the following conditions are satisfied:

$(R-n-N1)$ $v_{(x_1,x_2,...,x_n)} = \varepsilon_0 \Leftrightarrow x_1, x_2, ..., x_n$ are linearly dependent,

$(R-n-N2)$ $v_{(x_1,x_2,...,x_n)}$ is invariant under any permutation of $x_1, x_2, ..., x_n$,

$(R-n-N3)$ $v_{(x_1,x_2,...,\alpha x_n)}(t) = v_{(x_1,x_2,...,x_n)}\left(\frac{t}{|\alpha|}\right)$, for every $t > 0$, $\alpha \neq 0$, $\alpha \in \mathbb{R}$,

$(R-n-N4)$ $v_{(x_1,x_2,...,x_n+x'_n)} \geq T\left(v_{(x_1,x_2,...,x_n)}, v_{(x_1,x_2,...,x'_n)}\right)$,

then (L, v, T) is called a random n-normed linear space (briefly R-n-NLS).

Remark 2.2. From $(R-n-N3)$, it follows that in an R-n-NLS and $(R-n-N4)$ for all $t, s \in \mathbb{R}$ with $t > 0$

$$v_{(x_1,x_2,...,\alpha x_i,...,x_n)}(t) = v_{(x_1,x_2,...,x_i,...,x_n)}\left(\frac{t}{|\alpha|}\right), \text{ if } \alpha \neq 0,$$

$$v_{\left(x_1,x_2,...,x_i+x'_i,...,x_n\right)}(s+t) \geq T\left(v_{(x_1,x_2,...,x_i,...,x_n)}(s), v_{\left(x_1,x_2,...,x'_i,...,x_n\right)}(t)\right).$$

The following example agrees with our notion of R-n-NLS.

Example 2.3. Let $(L, \|\bullet, \bullet, ., ., \bullet\|)$ be an n-normed space. Defined

$$v_{(x_1,x_2,...,x_n)}(t) = \begin{cases} 0, & \text{when } t \leq \|x_1, x_2, ..., x_n\|, \\ 1, & \text{when } \|x_1, x_2, ..., x_n\| < t. \end{cases}$$

Then $(L, v, T = \min)$ is an R-n-LNS.

Proof: $(R-n-N1)$ For all $t \in \mathbb{R}$,
$v_{(x_1,x_2,...,x_n)}(t) = \varepsilon_0 \Leftrightarrow \|x_1, x_2, ..., x_n\| < t \Leftrightarrow \|x_1, x_2, ..., x_n\| = 0 \Leftrightarrow x_1, x_2, ..., x_n$ are linearly dependent.

$(R-n-N2)$ As $\|x_1, x_2, ..., x_n\|$ is invariant under any permutation of $x_1, x_2, ..., x_n$, it follow that $v_{(x_1,x_2,...,x_n)}$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

$(R-n-N3)$ For all $t \in \mathbb{R}$ with $t > 0$ and $\alpha \in \mathbb{R} \backslash \{0\}$

$$v_{(x_1,x_2,...,\alpha x_n)}(t) = 0 \Leftrightarrow t \leq \|x_1, x_2, ..., \alpha x_n\| \Leftrightarrow \frac{t}{|\alpha|} \leq \|x_1, x_2, ..., x_n\|$$

$$\Leftrightarrow v_{(x_1,x_2,...,x_n)}\left(\frac{t}{|\alpha|}\right) = 0, \text{and}$$

$$v_{(x_1,x_2,...,\alpha x_n)}(t) = 1 \Leftrightarrow \|x_1, x_2, ..., \alpha x_n\| < t$$
$$\Leftrightarrow |\alpha| \|x_1, x_2, ..., x_n\| < t \Leftrightarrow \|x_1, x_2, ..., x_n\| < \tfrac{t}{|\alpha|}$$
$$\Leftrightarrow v_{(x_1,x_2,...,x_n)}\left(\tfrac{t}{|\alpha|}\right) = 1.$$

Thus $v_{(x_1,x_2,...,\alpha x_n)}(t) = v_{(x_1,x_2,...,x_n)}\left(\frac{t}{|\alpha|}\right)$.

$(R-n-N4)$ For all $s, t \in \mathbb{R}$

$$v_{(x_1,x_2,...,x_n+x_n')}(s+t) = 0 \Leftrightarrow s+t \leq \|x_1, x_2, ..., x_n + x_n'\|$$
$$\leq \|x_1, x_2, ..., x_n\| + \|x_1, x_2, ..., x_n'\|.$$

If $\|x_1, x_2, ..., x_n\| < S$ then $\|x_1, x_2, ..., x_n\| \nless t$. That is , if $v_{(x_1,x_2,...,x_n)}(s) = 1$ then $v_{(x_1,x_2,...,x_n')}(t) = 0$.

Thus $v_{(x_1,x_2,...,x_n+x_n')}(s+t) = 0 \Rightarrow \min\left(v_{(x_1,x_2,...,x_n)}(s), v_{(x_1,x_2,...,x_n')}(t)\right) = 0$,

Similarly, $v_{(x_1,x_2,...,x_n+x_n')}(s+t) \geq \min\left(v_{(x_1,x_2,...,x_n)}(s), v_{(x_1,x_2,...,x_n')}(t)\right)$.

Example 2.4. Let $(L, \|., ., ., ., .\|)$ be an n-normed space as in Def 2.2. Defined

$$v_{(x_1,x_2,...,x_n)}(t) = \begin{cases} 0, & \text{when } t \leq 0, \\ \frac{t}{t+\|x_1,x_2,...,x_n\|}, & \text{when } t > 0. \end{cases}$$

Then $(L, v, T = \min)$ is a R-n-LNS.

Proof: $(R-n-N1)$ For all $t \in \mathbb{R}$,with $t > 0$.

$$\begin{aligned} v_{(x_1,x_2,...,x_n)}(t) &= \varepsilon_0 \Leftrightarrow \frac{t}{t + \|x_1, x_2, ..., x_n\|} = 1 \\ &\Leftrightarrow t = t + \|x_1, x_2, ..., x_n\| \Leftrightarrow \|x_1, x_2, ..., x_n\| = 0 \\ &\Leftrightarrow x_1, x_2, ..., x_n \text{ are linearly dependent.} \end{aligned}$$

$(R-n-N2)$ As $\|x_1, x_2, ..., x_n\|$ is invariant under any permutation of $x_1, x_2, ..., x_n$,

$$\begin{aligned} v_{(x_1,x_2,...,x_n)}(t) &= \varepsilon_0 \Leftrightarrow \frac{t}{t + \|x_1, x_2, ..., x_n\|} = \frac{t}{t + \|x_1, x_2, ..., x_n, x_{n-1}\|} \\ &= v_{(x_1,x_2,...,x_n,x_{n-1})}(t) = ... \end{aligned}$$

So that $v_{(x_1,x_2,...,x_n)}$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

$(R-n-N3)$ For all $t \in \mathbb{R}$ with $t > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned} v_{(x_1,x_2,...,x_n)}\left(\frac{t}{|\alpha|}\right) &= \frac{\frac{t}{|\alpha|}}{\frac{t}{|\alpha|}+\|x_1,x_2,...,x_n\|} \\ &= \frac{\frac{t}{|\alpha|}}{\frac{t+|\alpha|\|x_1,x_2,...,x_n\|}{|\alpha|}} = \frac{t}{t+|\alpha|\,\|x_1,x_2,...,\alpha x_n\|} \\ &= \frac{t}{t+\|x_1,x_2,...,\alpha x_n\|} = v_{(x_1,x_2,...,\alpha x_n)}(t). \end{aligned}$$

Thus $v_{(x_1,x_2,...,\alpha x_n)}(t) = v_{(x_1,x_2,...,x_n)}\left(\frac{t}{|\alpha|}\right)$.

$(R-n-N4)$ For all $s,t\in\mathbb{R}$,
If $s+t<0$, $s=t=0$, and $s+t>0$; $(s>0, t<0$ or $s<0, t>0)$, then

$$v_{(x_1,x_2,...,x_n+x_n')}(s+t) \geq Min\left(v_{(x_1,x_2,...,x_n)}(s), v_{(x_1,x_2,...,x_n')}(t)\right).$$

If $s>0, t>0, s+t>$ then assume that

$$\begin{aligned} v_{(x_1,x_2,...,x_n')}(t) &\leq v_{(x_1,x_2,...,x_n)}(s) \Rightarrow \frac{t}{t+\|x_1,x_2,...,x_n'\|} \leq \frac{s}{s+\|x_1,x_2,...,x_n\|} \\ &\Rightarrow t\left(s+\|x_1,x_2,...,x_n\|\right) \leq s\left(t+\|x_1,x_2,...,x_n'\|\right) \\ &\Rightarrow t\,\|x_1,x_2,...,x_n\| \leq s\,\|x_1,x_2,...,x_n'\| \\ &\Rightarrow \|x_1,x_2,...,x_n\| \leq \frac{s}{t}\,\|x_1,x_2,...,x_n'\| \end{aligned}$$

Therefore,

$$\begin{aligned} \|x_1,x_2,...,x_n\|+\|x_1,x_2,...,x_n'\| &\leq \frac{s}{t}\|x_1,x_2,...,x_n'\|+\|x_1,x_2,...,x_n'\| \\ &\leq \left(\frac{s}{t}+1\right)\|x_1,x_2,...,x_n'\| \\ &= \left(\frac{s+t}{t}\right)\|x_1,x_2,...,x_n'\|. \end{aligned}$$

But,

$$\begin{aligned} \|x_1,x_2,...,x_n+x_n'\| &\leq \|x_1,x_2,...,x_n\|+\|x_1,x_2,...,x_n'\| \\ &\leq \left(\frac{s+t}{t}\right)\|x_1,x_2,...,x_n'\| \end{aligned}$$

Then,

$$1 + \frac{\|x_1, x_2, ..., x_n + x'_n\|}{s+t} \leq 1 + \frac{\|x_1, x_2, ..., x'_n\|}{t}$$

$$\Rightarrow \quad \frac{s+t}{s+t+\|x_1, x_2, ..., x_n + x'_n\|} \leq \frac{t}{t+\|x_1, x_2, ..., x'_n\|}$$
$$\Rightarrow \quad v_{(x_1,x_2,...,x_n+x'_n)}(s+t) \geq Min\left(v_{(x_1,x_2,...,x_n)}(s), v_{(x_1,x_2,...,x'_n)}(t)\right).$$

3 Open Problem

It can be easily to introduce the notions of convergent sequence, Cauchy sequence, completeness, and bounded sets in random n-normed linear space.

References

[1] C. Diminie and A.G. White, Non expansive mappings in linear 2-normed space. *Math. Japonica* 21 (1976) 197–200.

[2] S. Gähler, Lineare 2-normierte Raume. *Math. Nachr.* 28 (1964), 1-43.

[3] S. Gähler, Untersuchungen uber verallgemenerte m-metrische Raume, I, II, III. *Math. Nachr.* 40 (1969), 165-189.

[4] H. Gunawan and M. Mashadi, On n-normed space. *Int. J. Math. Math. Sci.* 27 (2001), no. 10, 631–639.

[5] I. Goleţ, On probabilistic 2-normed spaces. *Novi Sad J. Math.* 35 (2005), 95-102.

[6] I. Goleţ, Random 2-normed spaces. *Sem. on Probab. Theory Appl. Univ. of Timişoara* 84 (1988), 1-18.

[7] K. Iseki, On non-expansive mapping in strictly covex linear 2-normed space. *Math. Sem. Note Kobe University* 3 (1975) 125-129.

[8] I. Jebril, Mohd. S. Md. Noorani, and A. Saari, An example of a probabilistic metric space not induced from a random normed space. *Bull. Malays Math. Sci. Soc.* 2 (26) (2003) 93-99.

[9] K. Menger, Statistical metrics. *Proc. Nat. Acad. Sci. USA* 28 (1942) 535–537.

[10] A.N. Serstnev, Random normed space. *Problems of completeness Kazan Gos. Univ. Ucen. Zap.* 122 (1962) 3-20.

[11] B. Schweizer and A. Sklar, Probabilistic metric space. *New York, Amsterdam, Oxford: North Holland* (1983).