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Spline Collocation Methods for Solving Second Order Neutral Delay Differential Equations

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Abstract

The aim of this paper is to solve the second order neutral delay differential equations (NDDEs) based on seventh C^3-spline collocation methods with three parameters $c_1, c_2, c_3 \in (0, 1)$. It is shown that the proposed methods for second order NDDEs possess a convergence rate of order seven if :

$$1 - c_1 - c_2 - c_3 + c_1c_2 + c_1c_3 + c_2c_3 - 2c_1c_2c_3 \leq 0.$$

Numerical results illustrating the behavior of the methods when faced with some difficult problems are presented and the numerical results are compared to those obtained by other methods.

Keywords: *Second order neutral delay differential equations, Spline collocation methods, Error analysis and order of convergence.*

1 Introduction

The purpose of this paper is to investigate the existence, uniqueness, error analysis and order of convergence of C^3-spline methods [4], [5] when applied to the numerical solution of the second order neutral delay differential equations (NDDEs):

$$\left.\begin{array}{ll} y''(t) = f(t, y(t), y(\tau(t)), y'(\tau(t))), & t_0 \leq t \leq t_f, \\ y(t) = \phi(t), \qquad y'(t) = \phi'(t), & t \leq t_0, \end{array}\right\} \qquad (1)$$

where $f \in C^7([t_0, t_f] \times R \times R \times R)$ is Lipschitiz continuous with respect to y. The function $\tau(t) \leq t, \quad t \in [t_0, t_f]$, is usually called the delay function. For some $t \geq t_0$ it can be seen that $t - \tau(t) < t_0$ an initial function $\phi(t)$ is needed for the wellposedness of the problems rather than a simple initial value y_0, as happens for ordinary differential equations (ODEs).

Spline collocation methods for solving second order neutral delay differential equations are studied in [1], [2]. Quintic C^2-spline methods with three points for solving ordinary initial value problems were studied in [8]. More detailed analysis for both the convergence and absolute stability was also given. Spline collocation methods with four points for solving first and second order ordinary differential equations were presented in [4], [5].

2 Description of the methods

Consider the initial value problem (1) for second order NDDEs. The spline methods use four-collocation points $t_{i-1+c_j} = t_{i-1} + c_j h, \quad j = 1(1)4$, in each subinterval $[t_{i-1}, t_i]$, $i = 1(1)N$, with

$$0 < c_1 < c_2 < c_3 < 1 \tag{2}$$

and $h = (t_f - t_0)/N$ is the constant stepsize, where $c_4 = 1$, $t_N = t_f$.

Denote by $t_i = t_0 + ih$, $i = 0(1)N$, the grid points of the uniform partition of $[t_0, t_f]$ into subintervals $I_i = [t_{i-1}, t_i]$, $i = 1(1)N$.

A seventh C^3-spline function $S(t)$ can be represented on each I_i, [4] by

$$\begin{aligned} S(t) &= \xi'^4\Big[(20\xi^3 + 10\xi^2 + 4\xi + 1)S_{i-1}^{(0)} + (10\xi^3 + 4\xi^2 + \xi)S_{i-1}^{(1)} \\ &+ (2\xi^3 + \frac{1}{2}\xi^2)S_{i-1}^{(2)} + (\frac{1}{6}\xi^3)S_{i-1}^{(3)}\Big] + \xi^4\Big[(20\xi'^3 + 10\xi'^2 + 4\xi' + 1)S_i^{(0)} \\ &- (10\xi'^3 + 4\xi'^2 + \xi')S_i^{(1)} + (2\xi'^3 + \frac{1}{2}\xi'^2)S_i^{(2)} - (\frac{1}{6}\xi'^3)S_i^{(3)}\Big], \end{aligned} \tag{3}$$

where

$$\xi = \frac{(t - t_{i-1})}{h} \in [0, 1], \quad \xi' = 1 - \xi$$

and

$$\left.\begin{aligned} &S_i^{(0)} = S(t_i), \quad S_i^{(1)} = hS'(t_i), \quad S_i^{(2)} = h^2 S''(t_i), \\ &S_i^{(3)} = h^3 S'''(t_i), \qquad i = 0(1)N. \end{aligned}\right\} \tag{4}$$

Differentiating (3) two times, we have

$$\begin{aligned} hS'(t) &= \xi'^3\Big[(-140\xi^3)S_{i-1}^{(0)} + (-70\xi^3 + 6\xi^2 + 3\xi + 1)S_{i-1}^{(1)} \\ &+ (-14\xi^3 + 3\xi^2 + \xi)S_{i-1}^{(2)} + (\frac{-7}{6}\xi^3 + \frac{1}{2}\xi^2)S_{i-1}^{(3)}\Big] \\ &+ \xi^3\Big[(140\xi'^3)S_i^{(0)} + (-70\xi'^3 + 6\xi'^2 + 3\xi' + 1)S_i^{(1)} \\ &- (-14\xi'^3 + 3\xi'^2 + \xi')S_i^{(2)} + (\frac{-7}{6}\xi'^3 + \frac{1}{2}\xi'^2)S_i^{(3)}\Big], \end{aligned} \tag{5}$$

$$\begin{aligned} h^2 S''(t) &= \xi'^2 \Big[(840\xi^3 - 420\xi^2) S_{i-1}^{(0)} + (420\xi^3 - 240\xi^2) S_{i-1}^{(1)} \\ &+ (84\xi^3 - 57\xi^2 + 2\xi + 1) S_{i-1}^{(2)} + (7\xi^3 - 6\xi^2 + \xi) S_{i-1}^{(3)} \Big] \\ &+ \xi^2 \Big[(840\xi'^3 - 420\xi'^2) S_i^{(0)} - (420\xi'^3 - 240\xi'^2) S_i^{(1)} \\ &+ (84\xi'^3 - 57\xi'^2 + 2\xi' + 1) S_i^{(2)} - (7\xi'^3 - 6\xi'^2 + \xi') S_i^{(3)} \Big]. \qquad (6) \end{aligned}$$

We formally apply these methods to (1), for $S(t)$ to be satisfied by the four collocation conditions:

$$\begin{aligned} S''(t_{i-1+c_j}) = f(t_{i-1+c_j}, S(t_{i-1+c_j}), S(\tau(t_{i-1+c_j})), S'(\tau(t_{i-1+c_j}))), \\ j = 1(1)4 \end{aligned} \qquad (7)$$

in each subinterval $[t_{i-1}, t_i]$. More precisely, denoting:

$$f_{i-1+\varphi} \equiv f(t_{i-1+\varphi}, S(t_{i-1+\varphi}), S(\tau(t_{i-1+\varphi})), S'(\tau(t_{i-1+\varphi}))),$$

$0 \le \varphi \le 1$, and $c'_j = 1 - c_j$, we can write (7) as follows:

$$\begin{aligned} & c_j^2 \Big[(840c_j'^3 - 420c_j'^2) S_i^{(0)} + (240c_j'^2 - 420c_j'^3) S_i^{(1)} \\ & + (84c_j'^3 - 57c_j'^2 + 2c'_j + 1) S_i^{(2)} + (-7c_j'^3 + 6c_j'^2 - c'_j) S_i^{(3)} \Big] \\ & = c_j'^2 \Big[(420c_j^2 - 840c_j^3) S_{i-1}^{(0)} + (240c_j^2 - 420c_j^3) S_{i-1}^{(1)} \\ & - (84c_j^3 - 57c_j^2 + 2c_j + 1) S_{i-1}^{(2)} - (7c_j^3 - 6c_j^2 + c_j) S_{i-1}^{(3)} \Big] \\ & + h^2 f_{i-1+c_j}, \quad j = 1(1)4. \qquad (8) \end{aligned}$$

Substituting $S_i^{(2)} = h^2 f_i$, $S_{i-1}^{(2)} = h^2 f_{i-1}$, into (8) and dividing by $c_j^2 c_j'^2$, we get the equivalent recurrence formulae:

$$\left. \begin{aligned} \underline{S}_i &= A \underline{S}_{i-1} + h^2 B \underline{f}_i, \\ S_i^{(2)} &= h^2 f_i, \quad i = 1(1)N \end{aligned} \right\} \qquad (9)$$

where $A = MW$, $B = MH$ and

$$M = \begin{bmatrix} 840c'_1 - 420 & 240 - 420c'_1 & 6 - 7c'_1 - \frac{1}{c'_1} \\ 840c'_2 - 420 & 240 - 420c'_2 & 6 - 7c'_2 - \frac{1}{c'_2} \\ 840c'_3 - 420 & 240 - 420c'_3 & 6 - 7c'_3 - \frac{1}{c'_3} \end{bmatrix}^{-1},$$

$$W = \begin{bmatrix} 420 - 840c_1 & 240 - 420c_1 & 6 - 7c_1 - \frac{1}{c_1} \\ 420 - 840c_2 & 240 - 420c_2 & 6 - 7c_2 - \frac{1}{c_2} \\ 420 - 840c_3 & 240 - 420c_3 & 6 - 7c_3 - \frac{1}{c_3} \end{bmatrix},$$

$$H = \begin{bmatrix} 57-84c_1-\frac{2}{c_1}-\frac{1}{c_1^2} & \frac{1}{c_1^2c_1'^2} & 0 & 0 & 57-84c_1'-\frac{2}{c_1'}-\frac{1}{c_1'^2} \\ 57-84c_2-\frac{2}{c_2}-\frac{1}{c_2^2} & 0 & \frac{1}{c_2^2c_2'^2} & 0 & 57-84c_2'-\frac{2}{c_2'}-\frac{1}{c_2'^2} \\ 57-84c_3-\frac{2}{c_3}-\frac{1}{c_3^2} & 0 & 0 & \frac{1}{c_3^2c_3'^2} & 57-84c_3'-\frac{2}{c_3'}-\frac{1}{c_3'^2} \end{bmatrix},$$

$\underline{S}_i = (S_i^{(0)}, S_i^{(1)}, S_i^{(3)})^T, \qquad \underline{f}_i = (f_{i-1}, f_{i-1+c_1}, f_{i-1+c_2}, f_{i-1+c_3}, f_i)^T.$

It is easy to observe that $S(\tau(t_{i-1+c_j})) = \phi(\tau(t_{i-1+c_j}))$ when $\tau(t_{i-1+c_j}) \le t_0$, and if $\tau(t_{i-1+c_j}) \in [t_{k-1}, t_k]$, $k = 1(1)N$, then $S(\tau(t_{i-1+c_j}))$ can be calculated from (3):

$$\begin{aligned} S(\tau(t_{i-1+c_j})) &= c_j'^4\Big[(20c_j^3+10c_j^2+4c_j+1)S_{k-1}^{(0)} + (10c_j^3+4c_j^2+c_j)S_{k-1}^{(1)} \\ &+ (2c_j^3+\frac{1}{2}c_j^2)S_{k-1}^{(2)} + (\frac{1}{6}c_j^3)S_{k-1}^{(3)}\Big] + c_j^4\Big[(20c_j'^3+10c_j'^2+4c_j'+1)S_k^{(0)} \\ &- (10c_j'^3+4c_j'^2+c_j')S_k^{(1)} + (2c_j'^3+\frac{1}{2}c_j'^2)S_k^{(2)} - (\frac{1}{6}c_j'^3)S_k^{(3)}\Big], \\ & j = 1(1)4. \end{aligned}$$

Also $S'(\tau(t_{i-1+c_j})) = \phi'(\tau(t_{i-1+c_j}))$ when $\tau(t_{i-1+c_j}) \le t_0$, and if $\tau(t_{i-1+c_j}) \in [t_{k-1}, t_k]$, $k = 1(1)N$, then $S'(\tau(t_{i-1+c_j}))$ can be calculated from (5):

$$\begin{aligned} hS'(\tau(t_{i-1+c_j})) &= c_j'^3\Big[(-140c_j^3)S_{k-1}^{(0)} + (-70c_j^3+6c_j^2+3c_j+1)S_{k-1}^{(1)} \\ &+ (-14c_j^3+3c_j^2+c_j)S_{k-1}^{(2)} + (\frac{-7}{6}c_j^3+\frac{1}{2}c_j^2)S_{k-1}^{(3)}\Big] \\ &+ c_j^3\Big[(140c_j'^3)S_k^{(0)} + (-70c_j'^3+6c_j'^2+3c_j'+1)S_k^{(1)} \\ &- (-14c_j'^3+3c_j'^2+c_j')S_k^{(2)} + (\frac{-7}{6}c_j'^3+\frac{1}{2}c_j'^2)S_k^{(3)}\Big], \\ & j = 1(1)4. \end{aligned}$$

If $0 < c_1 < c_2 < c_3 < 1$, then A is nonsingular, [5] because

$$|M^{-1}| = \frac{(c_1-1)(c_2-1)(c_3-1)}{25200(c_2-c_1)(c_3-c_1)(c_3-c_2)} \neq 0,$$

$$|A| = \frac{(c_1-1)(c_2-1)(1-c_3)}{c_1c_2c_3} \neq 0.$$

Notice that $\underline{f}_i$ depends on $\underline{S}_i, \underline{S}_{i-1}$ via (9). One proves in standard way, that (9) possesses a unique solution $\underline{S}_i$ for arbitrary given $\underline{S}_{i-1}$ if $h \in [0, h_0]$ with $h_0 > 0$ depending on c_1, c_2, c_3 and the Lipschitz constant of f.

3 Error analysis and order of convergence

In this section we consider the convergence of the methods (4) and (7) with initial function $\phi(t)$. To find a numerical approximation $S(t)$ to the exact solution y, define $S(t) = \phi(t)$ and $S'(t) = \phi'(t)$ for $t \le t_0$. The spline methods produce function values $S(t_i)$ as approximation to $y(t_i)$. The unknown values $y(\tau(t))$, $y'(\tau(t))$ may be replaced by $S(\tau(t))$, $S'(\tau(t))$ respectively (cf. [7]). Then the problem is reduced to the numerical solution of an initial value problem of ordinary differential equations. This case has been studied in [5].

Thus, we consider the case

$$y''(t) = f(t, y(t), y(\tau(t)), y'(\tau(t))), \qquad \tau(t) \ge t_0.$$

Theorem 3.1 *Let $0 < c_1 < c_2 < c_3 < 1$ then, the methods (4) and (7) are stable if and only if*

$$1 - c_1 - c_2 - c_3 + c_1c_2 + c_1c_3 + c_2c_3 - 2c_1c_2c_3 \le 0 \tag{10}$$

For the proof of this theorem see [5]. The relation (10) is satisfied for all values c_i, $i = 1, 2, 3$ listed in Table 1.

Table 1: Some intervals, which satisfy (10)

$0.1 \le c_1$	$0.7500000 < c_2 < c_3 < 1$
$0.2 \le c_1$	$0.6666667 < c_2 < c_3 < 1$
$0.3 \le c_1$	$0.6043561 < c_2 < c_3 < 1$
$0.4 \le c_1$	$0.5505103 < c_2 < c_3 < 1$
$0.5 \le c_1$	$0.5000000 < c_2 < c_3 < 1$

Theorem 3.2 *Let $f \in C^7([t_0, t_f] \times R \times R \times R)$, then the methods (4) and (7) are consistent and are of order seven.*

Proof. Let $\tau(t_{i-1+c_j}) \in [t_{k-1}, t_k]$, then we have the discretization error

$$\underline{d}_i = \begin{bmatrix} y(t_i) \\ hy'(t_i) \\ h^3y'''(t_i) \end{bmatrix} - A \begin{bmatrix} y(t_{i-1}) \\ hy'(t_{i-1}) \\ h^3y'''(t_{i-1}) \end{bmatrix}$$

$$-h^2B \begin{bmatrix} f(t_{i-1}, p_i(t_{i-1}), p_k(t_{k-1}), p'_k(t_{k-1})) \\ f(t_{i-1+c_1}, p_i(t_{i-1+c_1}), p_k(t_{k-1+c_1}), p'_k(t_{k-1+c_1})) \\ f(t_{i-1+c_2}, p_i(t_{i-1+c_2}), p_k(t_{k-1+c_2}), p'_k(t_{k-1+c_2})) \\ f(t_{i-1+c_3}, p_i(t_{i-1+c_3}), p_k(t_{k-1+c_3}), p'_k(t_{k-1+c_3})) \\ f(t_i, p_i(t_i), p_k(t_k), p'_k(t_k)) \end{bmatrix},$$

$i = 1(1)N, \quad k \leq i$, where

$$\begin{aligned} p_i(t) &= \xi'^4\Big[(20\xi^3 + 10\xi^2 + 4\xi + 1)y(t_{i-1}) + (10\xi^3 + 4\xi^2 + \xi)y'(t_{i-1})h \\ &+ (2\xi^3 + \frac{1}{2}\xi^2)y''(t_{i-1})h^2 + (\frac{1}{6}\xi^3)y'''(t_{i-1})h^3\Big] \\ &+ \xi^4\Big[(20\xi'^3 + 10\xi'^2 + 4\xi' + 1)y(t_i) - (10\xi'^3 + 4\xi'^2 + \xi')y'(t_i)h \\ &+ (2\xi'^3 + \frac{1}{2}\xi'^2)y''(t_i)h^2 - (\frac{1}{6}\xi'^3)y'''(t_i)h^3\Big] \end{aligned}$$

is the seventh Hermite interpolation polynomial which interpolates y, y', y'', y''' at $t = t_{i-1}$ and $t = t_i$, $i = 1(1)N$. And

$$\begin{aligned} hp_i'(t) &= \xi'^3\Big[(-140\xi^3)y(t_{i-1}) + (-70\xi^3 + 6\xi^2 + 3\xi + 1)y'(t_{i-1})h \\ &+ (-14\xi^3 + 3\xi^2 + \xi)y''(t_{i-1})h^2 + (\frac{-7}{6}\xi^3 + \frac{1}{2}\xi^2)y'''(t_{i-1})h^3\Big] \\ &+ \xi^3\Big[(140\xi'^3)y(t_i) + (-70\xi'^3 + 6\xi'^2 + 3\xi' + 1)y'(t_i)h \\ &- (-14\xi'^3 + 3\xi'^2 + \xi')y''(t_i)h^2 + (\frac{-7}{6}\xi'^3 + \frac{1}{2}\xi'^2)y'''(t_i)h^3\Big] \end{aligned}$$

for each subinterval $I_i = [t_{i-1}, t_i]$.

Since

$$|p_i(t) - y(t)| \leq Lh^8, \qquad t \in I_i,\ n = 1(1)N,$$

it follows that

$$\underline{d}_i = \tilde{\underline{d}}_i + O(h^9), \qquad i = 1(1)N,$$

where

$$\tilde{\underline{d}}_i = \begin{bmatrix} y(t_i) \\ hy'(t_i) \\ h^3y'''(t_i) \end{bmatrix} - A \begin{bmatrix} y(t_{i-1}) \\ hy'(t_{i-1}) \\ h^3y'''(t_{i-1}) \end{bmatrix} - hB^2 \begin{bmatrix} y''(t_{i-1}) \\ y''(t_{i-1+c_1}) \\ y''(t_{i-1+c_2}) \\ y''(t_{i-1+c_3}) \\ y''(t_i) \end{bmatrix}.$$

Now using Taylor's expansion

$$y(t) = q_7(t) + \frac{h^8}{8!}y^{(8)}(t_{i-1})\xi^8 + O(h^9), \quad t \in [t_{i-1}, t_i], \quad y \in C^9[t_0, t_f]$$

and observing that the methods are exact for polynomials of degree ≤ 7 (that means for $y \equiv q_7$ we have $\tilde{\underline{d}} = \underline{d}_i = 0$) we deduce, according to Lemma 8.11 (cf. (8.16)) in [6], that the methods are thus consistent and are of order seven for all c_1, c_2, c_3 listed in Table 1.

Theorem 3.3 *Let $f \in C^7([t_0, t_f] \times R \times R \times R)$ be Lipschitz continuous. Then the spline approximation $S(t)$ given by (4) and (7) converges to the solution $y(t)$ of (1.1) as $h \to 0$ whenever (10) is fulfilled and*

$$\lim_{h \to 0} h^{-j} S_0^{(j)} = y^{(j)}(t_0), \qquad j = 0(1)3 \tag{11}$$

Furthermore, the convergence order is seven, i.e., we have

$$\left.\begin{array}{ll} \left| y^{(k)}(t_i) - \frac{1}{h^k} S_i^{(k)} \right| \leq L_k h^7, & k = 0, 1, 2 \\ \left| y^{(3)}(t_i) - \frac{1}{h^3} S_i^{(3)} \right| \leq L_3 h^5, & i = 1(1)N \end{array}\right\} \tag{12}$$

whenever the initial values (4) satisfy (12) (with i=0). In addition, the following global error estimates hold true:

$$\left| y^{(j)}(t) - S_i^{(j)}(t) \right| \leq L_j h^{7-j}, \qquad j = 0(1)6, \quad t \in [t_0, t_f]. \tag{13}$$

Proof. Using Lipschits condition, we have

$$\begin{aligned} |y''(t_i) - S''(t_i)| &= \left| f(t_i, y(t_i), y(\tau(t_i)), y'(\tau(t_i))) - f(t_i, S(t_i), S(\tau(t_i)), S'(\tau(t_i))) \right| \\ &\leq L\Big\{ |y(t_i) - S(t_i)| + |y(\tau(t_i)) - S(\tau(t_i))| + |y'(\tau(t_i)) - S'(\tau(t_i))| \Big\} \\ &\leq L\{L_0 h^7 + L_0 h^7 + L_0 h^7\} = L_1 h^7, \end{aligned}$$

where $L_1 = 3LL_0$.

EL-Hawary and Mahmoud [4], [5] gave a more detailed analysis for both the absolute stability properties of the seventh C^3-spline methods. They showed that for $0.888035 \leq c_1 < c_2 < c_3 < 1$ the methods are absolutely stable (see Table 2), and increase regions of absolute stability when $c_1, c_2, c_3 \to 1^-$.

Table 2: Some absolute stability intervals for the proposed methods

$0.5 \leq c_1$	$0.99999 < c_2 < c_3 < 1$
$0.6 \leq c_1$	$0.981118 < c_2 < c_3 < 1$
$0.7 \leq c_1$	$0.956982 < c_2 < c_3 < 1$
$0.8 \leq c_1$	$0.925398 < c_2 < c_3 < 1$
$0.879 \leq c_1$	$0.892286 < c_2 < c_3 < 1$
$0.888 \leq c_1$	$0.888030 < c_2 < c_3 < 1$
$0.888035 \leq c_1$	$0.888035 < c_2 < c_3 < 1$

4 Numerical results

In this section we present the results of some numerical experiments on applying the spline collocation method. Numerical results have been obtained and compared with other methods. All computations in this section were carried by MATLAB 7.

Problem 1: Consider the following second order NDDEs (cf. [2])

$$\begin{cases} y''(t) = \left|t - \frac{1}{2}\right| + y'(t-1), \qquad t \in [0,1], \\ \\ y(t) = 1, \quad t \leq 0. \end{cases}$$

The analytical solution is

$$y(t) = \begin{cases} -\frac{1}{6}t^3 + \frac{1}{4}t^2 + 1, \qquad t \in [0, 1/2], \\ \\ \frac{1}{6}t^3 - \frac{1}{4}t^2 + \frac{1}{4}t + \frac{23}{24}, \qquad t \in [1/2, 1]. \end{cases}$$

In Table 3 we give the absolute error between the analytical solution and the numerical results by spline collocation methods and comparison between present method and the spline collocation method given in [2] where $s_d \in S_4, s_d \in C^2$ and $s_c \in S_4, s_c \in C^3$.

Table 3: Test results for problem 1, with $c_1 = 0.7,$ $c_2 = 0.96,\ c_3 = 0.998$

t_i	es_d [2]	es_d [2]	Present method
	h=0.1	h=0.1	h=0.1
0.1	1.0E-04	1.4E-04	4.440E-16
0.2	3.1E-05	9.4E-05	1.776E-15
0.3	2.7E-05	2.9E-04	2.442E-15
0.4	6.8E-05	1.0E-03	2.220E-16
0.5	9.1E-05	2.0E-03	4.662E-15
0.6	1.0E-04	3.3E-03	8.215E-15
0.7	1.4E-04	4.3E-03	1.998E-14
0.8	1.9E-04	5.0E-03	2.930E-14
0.9	2.5E-04	5.4E-03	4.196E-14
1	3.3E-04	5.5E-03	4.263E-14

Problem 2: Consider the following second order NDDEs

$$\begin{cases} y''(t) = -(cos(t) + sin(t))y(t) - (6 + sin(t))y'(t) + sin(t - \frac{\pi}{4})y(t - \frac{\pi}{4}) \\ \qquad\qquad + y'(t - \frac{\pi}{4}) - 5sin(t)exp(cos(t)), \quad t \geq 0, \\ y(t) = exp(cos(t)), \quad t \leq 0. \end{cases}$$

The exact solution of the problem is $y(t) = exp(cos(t))$.

In Table 4 we give the absolute error between the exact solution and the numerical results by present method.

Table 4: Maximum absolute error for the solution of problem 2, with $c_1 = 0.7,\ c_2 = 0.957,\ c_3 = 0.96$

t_i	Present method h=0.2	Present method h=0.1
1	1.159E-09	2.323E-12
2	5.807E-10	5.078E-12
3	4.782E-10	1.241E-12
4	1.763E-09	2.256E-11
5	4.420E-09	7.684E-11
6	1.293E-08	1.976E-10
7	5.370E-09	1.121E-10
8	1.932E-09	6.160E-11
9	2.577E-10	5.444E-11
10	1.256E-09	9.261E-11

Problem 3: Consider the following second order DDEs (cf. [3])

$$\begin{cases} y''(t) = -\frac{1}{2}y(t) + \frac{1}{2}y(t-\pi), & t \in [0,\pi], \\ y(t) = 1 - sin(t), & -\pi \le t \le 0. \end{cases}$$

The exact solution is $y(t) = 1 - sin(t)$.

In Table 5 we compared the absolute error of the present method and cubic spline functions given in [3] at the end point $t = \pi,\ h = \frac{\pi}{10*2^i}(i = 0(1)6)$.

Table 5: Test results for problem 3, with $c_1 = 0.5,\ c_2 = 0.95,\ c_3 = 0.98$

i	cubic spline [3] $h = \pi/(10*2^i)$	Present method $h = \pi/(10*2^i)$
0	1.84E-02	3.738E-09
1	4.62E-03	5.772E-11
2	1.16E-03	1.032E-12
3	2.89E-04	6.771E-13
4	7.23E-05	2.583E-12
5	1.81E-05	1.038E-11
6	4.52E-06	4.233E-11

5 Open Problem

It can be easily to introduce the notions of error analysis and order of convergence, Spline collocation methods for solving nth order neutral delay differential equations.

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