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Differential Subordination and Superordination with Multiplier Transformation

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Abstract

In [4] the authors introduced a multiplier transformation operator $\mathcal{D}^j_{b,\lambda}f$. In the present investigation, we obtain some Differential Subordination and Superordination results involving this operator for certain normalized analytic functions in the open unit disk. These results are obtained by investigating classes of admissible functions. Sandwich-type results are also obtained.

Keywords: *Analytic functions, Univalent functions, Multiplier transformation, Differential subordination, Differential superordination.*

1 Introduction

Denote by $\mathbb{U}$ the unit disk of the complex plane:

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{H}(\mathbb{U})$ be the space of analytic function in $\mathbb{U}$.

Let

$$\mathcal{A}_n = \{f \in \mathcal{H}(\mathbb{U}),\, f(z) = z + a_{n+1}z^{n+1} + \cdots, \quad (z \in \mathbb{U})\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(\mathbb{U}),\, f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \cdots, \quad (z \in \mathbb{U})\}.$$

If functions f and F are analytic in $\mathbb{U}$, then we say that f is subordinate to F, and write $f \prec F$, if there exists a Schwarz function w analytic in $\mathbb{U}$ with $|w(z)| < 1$ and $w(0) = 0$ such that $f(z) = F(w(z))$ in $\mathbb{U}$. Furthermore, if the function F is univalent in $\mathbb{U}$, then $f(z) \prec F(z)\,(z \in \mathbb{U}) \Leftrightarrow f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

A function f, analytic in $\mathbb{U}$, is said to be convex if it is univalent and $f(\mathbb{U})$ is convex.
Let $p, h \in \mathcal{H}(\mathbb{U})$ and let $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. If p and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the (second-order) differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \quad (z \in \mathbb{U}) \tag{1.1}$$

then p is called a solution of the differential superordination (1.1). (If f subordinate to F, then F is superordinate to f).

An analytic function q is called a subordinant of the differential superodination, if $q \prec p$ for all p satisfying (1.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant. (Note that the best subordinant is unique up to a rotation of $\mathbb{U}$). Recently Miller and Mocanu [12] obtained conditions on h, q and ψ for which the following implication holds:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \Rightarrow\; q(z) \prec p(z) \qquad (z \in \mathbb{U}).$$

We now state the following definition.

Definition 1.1 *[4] Let the function f in $\mathcal{A}$, then for $j \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$ and $\lambda > -1$, we define the following operator:*

$$\mathcal{D}^j_{b,\lambda} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+b}{1+b}\right)^j C(\lambda, k) a_k z^k, \quad (z \in \mathbb{U}),$$

where $C(\lambda, k) = \binom{k+\lambda-1}{\lambda}$.

Obviously, we observe that

$$\mathcal{D}^j_{b,\lambda}(\mathcal{D}^m_{b,\lambda}(z)) = \mathcal{D}^{j+m}_{b,\lambda} f(z) \quad (j, m \in \mathbb{C},\, b \in \mathbb{C} \setminus \mathbb{Z}^-,\, \lambda > -1; z \in \mathbb{U}).$$

It is clear that $\mathcal{D}^j_{b,\lambda}$ is multiplier transformation. For $j \in \mathbb{Z}$, $b = 1$ and $\lambda = 0$ the operators $\mathcal{D}^j_{1,0} \equiv I_j$ were studied by Uralegaddi and Somanatha [1], and for $j \in \mathbb{Z}$, $\lambda = 0$ the operators $\mathcal{D}^j_{b,0} \equiv I^j_b$ are closely related to the multiplier transformations studied by Flett [14], also, for $j = -1, \lambda = 0$, the operators $\mathcal{D}^{-1}_{b,0} \equiv I_b$ is the integral operator studied by Owa and Srivastava [10]. And for any negative real number j and $b = 1$, $\lambda = 0$ the operators $\mathcal{D}^{j}_{1,0} \equiv I^j$ is the multiplier transformation studied by Jung et al. [2], and for any nonnegative integer j and $b = \lambda = 0$, the operators $\mathcal{D}^{j}_{0,0} \equiv S^j$ is the differential operator defined by Sălăgean [3]. Furthermore , for $j = 0$ and $\lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the operators $\mathcal{D}^{0}_{b,\lambda} \equiv R^\lambda$ is the differential operator defined by Ruscheweyh [13]. For $j, \lambda \in \mathbb{N}_0$ and $b = 0$ the operator $\mathcal{D}^{j}_{0,\lambda} \equiv \mathfrak{D}^{j}_{\lambda}$ is the differential operator defined by the authors [5]. Finally, for different choices of j, b and λ we obtain several operators investigated earlier by other authors see, for example [8],[7] and [6].

In order to prove the original results we shall need the following definition and theorems.

Definition 1.2 *[11, Definition 2.2b p.21] Denote by Q, the set of all functions q that are analytic and injective on $\overline{\mathbb{U}} \setminus E(q)$, where*

$$E(q) = \left\{ \zeta \in \partial\mathbb{U} \,:\, \lim_{z \to \zeta} q(z) = \infty \right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(f)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$ and $Q(1) = Q_1$.

Definition 1.3 *[11, Definition 2.3a p.27] Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; z) \notin \Omega$ whenever $r = q(\zeta)$, $s = k\zeta q'(\zeta)$, and*

$$\Re\left\{ \frac{t}{s} + 1 \right\} \geq k\, \Re\left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\}, \qquad (z \in \mathbb{U},\, \zeta \in \partial\mathbb{U} \setminus E(q),\, k \geq n).$$

We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In particular when $q(z) = M\frac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, then $q(\mathbb{U}) = \mathbb{U}_M = \{w : |w| < M\}$, $q(0) = a$, $E(q) = \emptyset$ and $q \in Q(a)$. In this case, we set $\Psi_n[\Omega, M, a] = \Psi_n[\Omega, q]$, and in the special case when $\Omega = \mathbb{U}_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 1.4 *[12, Definition 3 p.817] Let Ω be a set in $\mathbb{C}$, $q(z) \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; \zeta) \in \Omega$ whenever $r = q(z)$, $s = \frac{zq'(z)}{m}$, and*

$$\Re\left\{ \frac{t}{s} + 1 \right\} \leq \frac{1}{m}\, \Re\left\{ \frac{zq''(z)}{q'(z)} + 1 \right\}, \qquad (z \in \mathbb{U},\, \zeta \in \partial\mathbb{U},\, 1 \leq n \leq m).$$

In particular, we write $\Psi'_1[\Omega, q]$ *as* $\Psi'[\Omega, q]$.

Theorem 1.5 *[11, Theorem 2.3b 3 p.28] Let* $\psi \in \Psi_n[\Omega, q]$ *with* $q(0) = a$. *If the analytic function* $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$ *satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega,$$

then $p(z) \prec q(z)$.

Theorem 1.6 *[12, Theorem 1 p.818] Let* $\psi \in \Psi'_n[\Omega, q]$ *with* $q(0) = a$. *If* $p(z) \in Q(a)$ *and* $\psi(p(z), zp'(z), z^2p''(z); z)$ *is univalent in* $\mathbb{U}$, *then*

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathbb{U}\}$$

implies $q(z) \prec p(z)$.

In the present paper, we shall use the method of differential subordination introduced by Miller and Mocanu [11, Theorem 2.3b 3 p.28] and [12, Theorem 1 p.818] to derive certain properties of multiplier transformation $\mathcal{D}^j_{b,\lambda} f$. Additionally, the corresponding differential superordination problem is investigated and several sandwich-type results are obtained.

2 Subordination Results

First, the following class of admissible functions is required in our first result.

Definition 2.1 *Let* Ω *be a set in* $\mathbb{C}$ *and* $q(z) \in Q_1 \cap \mathcal{H}[q(0), 1]$. *The class of admissible functions* $\Phi_n[\Omega, q]$ *consists of those functions* $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ *that satisfy the admissibility condition*

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \qquad v = \frac{k\zeta q'(\zeta) + (\lambda + 1)q(\zeta)}{\lambda + 1}$$

$$\Re\left\{\frac{(\lambda + 2)(w - u)}{v - u} - (2\lambda + 3)\right\} \geq k\,\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$
$$(z \in \mathbb{U},\ \zeta \in \partial\mathbb{U} \setminus E(q),\ k \geq 1).$$

Now , we will derive our first result.

Theorem 2.2 *Let $\phi \in \Phi_n[\Omega, q]$. If $f(z) \in \mathcal{A}$ satisfies*

$$\left\{\phi((\mathcal{D}^j_{b,\lambda}f(z))', (\mathcal{D}^j_{b,\lambda+1}f(z))', (\mathcal{D}^j_{b,\lambda+2}f(z))'; z) : z \in \mathbb{U}\right\} \subset \Omega, \tag{2.1}$$

then

$$(\mathcal{D}^j_{b,\lambda}f(z))' \prec q(z).$$

Proof. Define the analytic function p in $\mathbb{U}$ by

$$p(z) = (\mathcal{D}^j_{b,\lambda}f(z))'. \tag{2.2}$$

In view of the relation

$$z(\mathcal{D}^j_{b,\lambda}f(z))' = (\lambda+1)\mathcal{D}^j_{b,\lambda+1}f(z) - \lambda\mathcal{D}^j_{b,\lambda}f(z), \tag{2.3}$$

from (2.2), we get

$$(\mathcal{D}^j_{b,\lambda+1}f(z))' = \frac{zp'(z) + (\lambda+1)p(z)}{\lambda+1}. \tag{2.4}$$

Further, a simple computation shows that

$$(\mathcal{D}^j_{b,\lambda+2}f(z))' = \frac{z^2p''(z) + 2(\lambda+2)zp'(z) + (\lambda+1)(\lambda+2)p(z)}{(\lambda+2)(\lambda+1)}. \tag{2.5}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$\begin{aligned} &u(r,s,t) = r, \;\; v(r,s,t) = \frac{s+(\lambda+1)r}{\lambda+1}, \\ &w(r,s,t) = \frac{t+2(\lambda+2)s+(\lambda+1)(\lambda+2)r}{(\lambda+2)(\lambda+1)}. \end{aligned} \tag{2.6}$$

Let

$$\begin{aligned} \psi(r,s,t;z) &= \phi(u,v,w;z) \\ &= \phi\left(r, \frac{s+(\lambda+1)r}{\lambda+1}, \frac{t+2(\lambda+2)s+(\lambda+1)(\lambda+2)r}{(\lambda+2)(\lambda+1)}; z\right) \end{aligned} \tag{2.7}$$

The proof shall make use of Theorem 1.5. Using equations (2.2), (2.4) and (2.5), from (2.7), we obtain

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big) = \phi\big((\mathcal{D}^j_{b,\lambda}f(z))', (\mathcal{D}^j_{b,\lambda+1}f(z))', (\mathcal{D}^j_{b,\lambda+2}f(z))'; z\big). \tag{2.8}$$

Hence (2.1) becomes

$$\begin{aligned} &\phi\big((\mathcal{D}^j_{b,\lambda}f(z))', (\mathcal{D}^j_{b,\lambda+1}f(z))', (\mathcal{D}^j_{b,\lambda+2}f(z))'; z\big) \\ &\qquad = \psi\big(p(z), zp'(z), z^2p''(z); z\big) \in \Omega. \end{aligned} \tag{2.9}$$

The proof is complete if it can be shown that the admissibility condition for $\phi \in \Phi_n[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.3.
Note that

$$\frac{t}{s} + 1 = \frac{(\lambda+2)(w-u)}{v-u} - (2\lambda+3),$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 1.5, $p(z) \prec q(z)$, or $(\mathcal{D}^j_{b,\lambda} f(z))' \prec q(z)$.

We next consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case the class $\Phi_n[h(\mathbb{U}), q]$ is written as $\Phi_n[h, q]$.

The following result is an immediate consequence of Theorem 2.2.

Theorem 2.3 *Let $\phi \in \Phi_n[h, q]$ with $q(0) = 1$. If $f(z) \in \mathcal{A}$ satisfies*

$$\phi\Big((\mathcal{D}^j_{b,\lambda} f(z))', (\mathcal{D}^j_{b,\lambda+1} f(z))', (\mathcal{D}^j_{b,\lambda+2} f(z))'; z\Big) \prec h(z), \tag{2.10}$$

then

$$(\mathcal{D}^j_{b,\lambda} f(z))' \prec q(z).$$

Our next result is an extension of Theorem 2.2 to the case where the behavior of q on $\partial\mathbb{U}$ is not known.

Corollary 2.4 *Let $\Omega \subset \mathbb{C}$ and let q be univalent in $\mathbb{U}$, $q(0) = 1$. Let $\phi \in \Phi_n[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}$ and*

$$\phi\Big((\mathcal{D}^j_{b,\lambda} f(z))', (\mathcal{D}^j_{b,\lambda+1} f(z))', (\mathcal{D}^j_{b,\lambda+2} f(z))'; z\Big) \in \Omega,$$

then

$$(\mathcal{D}^j_{b,\lambda} f(z))' \prec q(z).$$

Proof. Theorem 2.2 yields $(\mathcal{D}^j_{b,\lambda} f(z))' \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$.

Theorem 2.5 *Let h and q be univalent function in $\mathbb{U}$, with $q(0) = 1$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$, satisfy one of the following conditions:*
(i) $\phi \in \Phi_n[h, q_\rho]$, for some $\rho \in (0, 1)$, or
(ii) there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_n[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.
If $f \in \mathcal{A}$ satisfies (2.10), then $(\mathcal{D}^j_{b,\lambda} f(z))' \prec q(z)$.

Proof. By using the same methods given by [11], we have
(i) By applying Theorem 2.2 we obtain $(\mathcal{D}^j_{b,\lambda}f(z))' \prec q_\rho(z)$. Since $q_\rho(z) \prec q(z)$ we deduce $(\mathcal{D}^j_{b,\lambda}f(z))' \prec q(z)$.
(ii) If we let $(\mathcal{D}^j_{b,\lambda}f_\rho(z))' = (\mathcal{D}^j_{b,\lambda}f(\rho z))'$, then

$$\begin{aligned}&\phi\Big((\mathcal{D}^j_{b,\lambda}f_\rho(z))', (\mathcal{D}^j_{b,\lambda+1}f_\rho(z))', (\mathcal{D}^j_{b,\lambda+2}f_\rho(z))';_\rho z\Big)\\ &= \phi\Big((\mathcal{D}^j_{b,\lambda}f(\rho z))', (\mathcal{D}^j_{b,\lambda+1}f(\rho z))', (\mathcal{D}^j_{b,\lambda+2}f(\rho z))'; \rho z\Big) \in h_\rho(\mathbb{U}).\end{aligned}$$

By using Theorem 2.2 and the comment associated with (2.9) with $w(z) = \rho z$, we obtain $(\mathcal{D}^j_{b,\lambda}f_\rho(z))' \prec q_\rho(z)$, for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$, we obtain $(\mathcal{D}^j_{b,\lambda}f(z))' \prec q(z)$.

The next two theorems yield best dominants of the differential subordination (2.10).

Theorem 2.6 *Let h be univalent in $\mathbb{U}$, and $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. Suppose the differential equation*

$$\phi\left(q(z), \frac{zq'(z) + (\lambda+1)q(z)}{\lambda+1}, \frac{z^2q''(z) + 2(\lambda+2)zq'(z) + (\lambda+2)(\lambda+1)q(z)}{(\lambda+2)(\lambda+1)}; z\right) = h(z) \tag{2.11}$$

has a solution q with $q(0) = 1$ and satisfy one of the following conditions:
(i) $q(z) \in Q$ and $\phi \in \Phi_n[h, q]$,
(ii) $q(z)$ is univalent in $\mathbb{U}$ and $\phi \in \Phi_n[h, q_\rho]$, for some $\rho \in (0, 1)$, or
(iii) $q(z)$ is univalent in $\mathbb{U}$ and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_n[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.
If $f(z) \in \mathcal{A}$ satisfies (2.10), then $(\mathcal{D}^j_{b,\lambda}f(z))' \prec q(z)$ and $q(z)$ is the best dominant.

Proof. By applying Theorems 2.3 and 2.4, we deduce that q is a dominant of (2.10). Since q satisfies (2.11), it is a solution of (2.10) and therefore q will be dominated by all dominants of (2.10). Hence q will be the best dominant of (2.10).

Theorem 2.7 *Let the function h be univalent in $\mathbb{U}$ and let $\phi : \mathbb{C}^3 \to \mathbb{C}$. Suppose that the differential equation*

$$\begin{aligned}&\phi\left(q(z), \frac{nzq'(z) + (\lambda+1)q(z)}{\lambda+1},\right.\\ &\left.\frac{n(n-1)zq'(z) + n^2z^2q''(z) + 2(\lambda+2)nzq'(z) + (\lambda+2)(\lambda+1)q(z)}{(\lambda+2)(\lambda+1)}\right) = h(z)\end{aligned} \tag{2.12}$$

has a solution q with $q(0)=1$, and one of the following conditions is satisfied:
(i) $q(z)\in Q$ and $\phi\in\Phi_n[h,q]$,
(ii) $q(z)$ is univalent in $\mathbb{U}$ and $\phi\in\Phi_n[h,q_\rho]$, for some $\rho\in(0,1)$, or
(iii) $q(z)$ is univalent in $\mathbb{U}$ and there exists $\rho_0\in(0,1)$ such that $\phi\in\Phi_n[h_\rho,q_\rho]$, for all $\rho\in(\rho_0,1)$.
If $f(z)\in\mathcal{A}$, $\phi\big((\mathcal{D}^j_{b,\lambda}f(z))',(\mathcal{D}^j_{b,\lambda+1}f(z))',(\mathcal{D}^j_{b,\lambda+2}f(z))'\big)$ is analytic in $\mathbb{U}$ and $(\mathcal{D}^j_{b,\lambda}f(z))'$ satisfies

$$\phi\big((\mathcal{D}^j_{b,\lambda}f(z))',(\mathcal{D}^j_{b,\lambda+1}f(z))',(\mathcal{D}^j_{b,\lambda+2}f(z))'\big)\prec h(z), \tag{2.13}$$

then $(\mathcal{D}^j_{b,\lambda}f(z))'\prec q(z)$ and $q(z)$ is the best $(1,n)-$dominant.

Proof. By applying Theorems 2.3 and 2.4 we deduce that q is dominant of (2.13). If we let $(\mathcal{D}^j_{b,\lambda}f(z))'=q(z^n)$, then $(\mathcal{D}^j_{b,\lambda+1}f(z))'=\frac{nzq'(z)+(\lambda+1)q(z)}{\lambda+1}$ and $(\mathcal{D}^j_{b,\lambda+2}f(z))'=\frac{n(n-1)zq'(z)+n^2z^2q''(z)+2(\lambda+2)nzq'(z)+(\lambda+2)(\lambda+1)q(z)}{(\lambda+2)(\lambda+1)}$. Therefore from (2.12), we obtain

$$\phi\big((\mathcal{D}^j_{b,\lambda}f(z))',(\mathcal{D}^j_{b,\lambda+1}f(z))',(\mathcal{D}^j_{b,\lambda+2}f(z))'\big)=h(z^n)\prec h(z).$$

Since $(\mathcal{D}^j_{b,\lambda}f(\mathbb{U}))'=q(\mathbb{U})$, we conclude that q is the $(1,n)-$best dominant.

In the particular case $q(z)=1+Mz,\ M>0$, and in view of Definition 2.1, the class of admissible functions $\Phi_n[\Omega,q]$, denoted by $\Phi_n[\Omega,M]$, can be expressed in the following form:

Definition 2.8 *Let Ω be a set in $\mathbb{C}$ and $M>0$. The class of admissible functions $\Phi_n[\Omega,M]$ consists of those functions $\phi:\mathbb{C}^3\times\mathbb{U}\to\mathbb{C}$, such that:*

$$\phi\left(1+Me^{i\theta},1+\frac{(K+\lambda+1)Me^{i\theta}}{\lambda+1},\right.$$
$$\left.1+\frac{L+[2(\lambda+2)K+(\lambda+2)(\lambda+1)]Me^{i\theta}}{(\lambda+2)(\lambda+1)};z\right)\notin\Omega, \tag{2.14}$$

whenever $K\geq nM,\ \Re[Le^{-i\theta}]\geq(n-1)K$, $z\in\mathbb{U}$ and $\theta\in\mathbb{R}$.

From above definition and Theorem 2.2 we have

Corollary 2.9 *Let $\phi\in\Phi_n[\Omega,M]$. If $f(z)\in\mathcal{A}$ satisfies*

$$\phi\big((\mathcal{D}^j_{b,\lambda}f(z))',(\mathcal{D}^j_{b,\lambda+1}f(z))',(\mathcal{D}^j_{b,\lambda+2}f(z))'\big)\in\Omega,$$

then $|(\mathcal{D}^j_{b,\lambda}f(z))'-1|<M$.

In the special case $\Omega = q(\mathbb{U}) = \{w : |w-1| < M\}$, the class $\Phi_n[\Omega, M]$ is simply denoted by $\Phi_n[M]$.

Corollary 2.10 *Let $\phi \in \Phi_n[M]$. If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\phi\big((\mathcal{D}^j_{b,\lambda}f(z))', (\mathcal{D}^j_{b,\lambda+1}f(z))', (\mathcal{D}^j_{b,\lambda+2}f(z))'; z\big) - 1\right| < M,$$

then $|(\mathcal{D}^j_{b,\lambda}f(z))' - 1| < M$.

Corollary 2.11 *If $M > 0$ and $f(z) \in \mathcal{A}$ satisfies*

$$\left|(\mathcal{D}^j_{b,\lambda+1}f(z))' - (\mathcal{D}^j_{b,\lambda}f(z))'\right| < \frac{M}{\lambda+1}$$

$|(\mathcal{D}^j_{b,\lambda}f(z))' - 1| < M$.

Proof. This follows from Corollary 2.9 by taking $\phi(u, v, m; z) = v - u$ and $\Omega = h(\mathbb{U})$, where $h(z) = \frac{M}{\lambda+1}z$, $(M > 0)$. To use the Corollary 2.9 we need to show that $\phi \in \Phi_n[\Omega, M]$, that is the admissibility condition (2.14) is satisfied. This follows since

$$\left|\phi\left(1 + Me^{i\theta}, 1 + \frac{(K+\lambda+1)Me^{i\theta}}{\lambda+1}, 1 + \frac{L + [2(\lambda+2)K + (\lambda+2)(\lambda+1)]Me^{i\theta}}{(\lambda+2)(\lambda+1)}; z\right)\right|$$
$$= \frac{K}{\lambda+1} \geq \frac{M}{\lambda+1}$$

whenever $z \in \mathbb{U}$, $K \geq nM$, $\Re[Le^{-i\theta}] \geq (n-1)K$ and $\theta \in \mathbb{R}$. The required result now follows from Corollary 2.9.

Theorem 2.7 shows that the result is sharp. The differential equation

$$\frac{zq'(z)}{\lambda+1} = \frac{M}{\lambda+1}z \qquad (\lambda + 1 < M)$$

has a univalent solution $q(z) = 1 + Mz$. It follows from Theorem 2.7 that $q(z) = 1 + Mz$ is the best dominant.

By taking $b = 0$, $j = 1$ and $\lambda = 0$, Corollary 2.10 shows that for $f \in \mathcal{A}$, if $zf''(z) \prec 1 + Mz$, then $f'(z) \prec 1 + Mz$.

Now we have the following:

Definition 2.12 *Let Ω be a set in $\mathbb{C}$ and $q(z) \in Q \cap \mathcal{H}[q(0), 1]$. The class of adissible functions $\Phi_{n,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w; z) \notin \Omega,$$

whenever

$$u=q(\zeta),\ \ v=\frac{1}{\lambda+2}\Big((\lambda+1)q(\zeta)+1+\frac{k\zeta q'(\zeta)}{q(\zeta)}\Big),\qquad (q(\zeta)\neq 0),$$
$$\Re\left\{\frac{[(\lambda+3)w-(\lambda+2)v-1](\lambda+2)v}{(\lambda+2)v-[(\lambda+1)r+1]}-[2(\lambda+1)r+1-(\lambda+2)v]\right\}$$
$$\geq k\,\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},\quad (z\in\mathbb{U},\,\zeta\in\partial\mathbb{U}\setminus E(q),\ k\geq 1).$$

Theorem 2.13 *Let $\phi\in\Phi_{n,1}[\Omega,q]$. If $f\in\mathcal{A}$ satisfies*

$$\left\{\phi\left(\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'},\frac{(\mathcal{D}^j_{b,\lambda+2}f(z))'}{(\mathcal{D}^j_{b,\lambda+1}f(z))'},\frac{(\mathcal{D}^j_{b,\lambda+3}f(z))'}{(\mathcal{D}^j_{b,\lambda+2}f(z))'};z\right):z\in\mathbb{U}\right\}\subset\Omega,\quad (2.15)$$

then $\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'}\prec q(z)$.

Proof. Define the analytic function p in $\mathbb{U}$ by:

$$p(z)=\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'}.\qquad (2.16)$$

Then, by using (2.3), we get

$$\frac{(\mathcal{D}^j_{b,\lambda+2}f(z))'}{(\mathcal{D}^j_{b,\lambda+1}f(z))'}=\frac{1}{\lambda+2}\Big\{(\lambda+1)p(z)+1+\frac{zp'(z)}{p(z)}\Big\}.\qquad (2.17)$$

Differentiating logarithmically (2.17), and further computations show that

$$\frac{(D^j_{b,\lambda+3}f(z))'}{(D^j_{b,\lambda+2}f(z))'}=\frac{1}{\lambda+3}\left\{(\lambda+1)p(z)+2+\frac{zp'(z)}{p(z)}\right.$$
$$\left.+\frac{[(\lambda+1)p(z)+1]\frac{zp'(z)}{p(z)}+\frac{z^2p''(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2}{(\lambda+1)p(z)+1+\frac{zp'(z)}{p(z)}}\right\}.\qquad (2.18)$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r,\ v=\frac{1}{\lambda+2}\Big((\lambda+1)r+1+\frac{s}{r}\Big),$$
$$w=\frac{1}{\lambda+3}\left((\lambda+1)r+2+\frac{s}{r}+\frac{[(\lambda+1)r+1]\frac{s}{r}+\frac{t}{r}-\left(\frac{s}{r}\right)^2}{(\lambda+1)r+1+\frac{s}{r}}\right).\qquad (2.19)$$

Let

$$\begin{aligned} &\Psi(r,s,t;z)\\ =\ &\phi(u,v,w;z)\\ =\ &\phi\Bigg(r, \frac{1}{\lambda+2}\Big((\lambda+1)r+1+\frac{s}{r}\Big),\\ &\frac{1}{\lambda+3}\Bigg((\lambda+1)r+2+\frac{s}{r}+\frac{[(\lambda+1)r+1]\frac{s}{r}+\frac{t}{r}-\left(\frac{s}{r}\right)^2}{(\lambda+1)r+1+\frac{s}{r}}\Bigg);z\Bigg). \end{aligned} \tag{2.20}$$

Using (2.16), (2.17) and (2.18), from (2.20), it follows that

$$\begin{aligned} &\psi(p(z), zp'(z), z^2p''(z); z)\\ &=\phi\left(\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+2}f(z))'}{(\mathcal{D}^j_{b,\lambda+1}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+3}f(z))'}{(\mathcal{D}^j_{b,\lambda+2}f(z))'};z\right). \end{aligned} \tag{2.21}$$

Hence (2.15) implies $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$. The proof is complete if it can be shown that the admissibility condition for $\phi \in \Phi_{n,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.3.

For this purpose, note that

$$\begin{aligned} \frac{s}{r} &= (\lambda+2)v - [(\lambda+1)u+1],\\ \frac{t}{r} &= [(\lambda+3)w-(\lambda+2)v-1](\lambda+2)v - [(\lambda+2)v-2\frac{s}{r}]\frac{s}{r}, \end{aligned}$$

and thus

$$\frac{t}{s}+1 = \frac{[(\lambda+3)w-(\lambda+2)v-1](\lambda+2)v}{(\lambda+2)v-[(\lambda+1)u+1]} - [2(\lambda+1)u+1-(\lambda+2)v].$$

Hence $\psi \in \Psi[\Omega, q]$ and by Theorem 1.5, $p(z) \prec q(z)$ or $\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'} \prec q(z)$.

In the case $\Omega \neq \mathbb{C}$ is a simply connected domain with $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω, the class $\Phi_{n,1}[h(\mathbb{U}), q]$ is written as $\Phi_{n,1}[h, q]$. The following result is an immediate consequence of Theorem 2.13.

Theorem 2.14 *Let $\phi \in \Phi_{n,1}[h(\mathbb{U}), q]$ with $q(0) = 1$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+2}f(z))'}{(\mathcal{D}^j_{b,\lambda+1}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+3}f(z))'}{(\mathcal{D}^j_{b,\lambda+2}f(z))'};z\right) \prec h(z), \tag{2.22}$$

then $\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'} \prec q(z)$.

In the particular case $q(z) = 1 + Mz$, $M > 0$, the class of admissible functions $\Phi_{n,1}[\Omega, q]$, is simply denoted by $\Phi_{n,1}[\Omega, M]$.

Definition 2.15 *Let Ω be a set in $\mathbb{C}$ and $M>0$. The class of admissible functions $\Phi_{n,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ such that*

$$\phi\Bigg(1+Me^{i\theta}, 1+\frac{(\lambda+1)(1+Me^{i\theta})+K}{(\lambda+2)(1+Me^{i\theta})}Me^{i\theta}, 1+\frac{K+(\lambda+1)(1+Me^{i\theta})}{(\lambda+3)(1+Me^{i\theta})}$$
$$+\frac{(M+e^{-i\theta})\big[[(\lambda+1)(1+Me^{i\theta})+1]KM+Le^{-i\theta}\big]-K^2M^2}{(\lambda+3)(M+e^{-i\theta})\big[(M+e^{-i\theta})[(1+Me^{i\theta})+1]+KM\big]};z\Bigg) \notin \Omega, \tag{2.23}$$

whenever $K \geq nM$, $\Re[Le^{-i\theta}] \geq (n-1)K$, $z \in \mathbb{U}$ and $\theta \in \mathbb{R}$.

Corollary 2.16 *Let $\phi \in \Phi_{n,1}[\Omega, M]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+2}f(z))'}{(\mathcal{D}^j_{b,\lambda+1}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+3}f(z))'}{(\mathcal{D}^j_{b,\lambda+2}f(z))'};z\right) \in \Omega,$$

then $\left|\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'} - 1\right| < M.$

In the special case $\Omega = q(\mathbb{U}) = \{w : |w-1| < M\}$, the class $\Phi_{n,1}[\Omega, M]$ is defined by $\Phi_{n,1}[M]$, and Corollary 2.16 takes the following form:

Corollary 2.17 *Let $\phi \in \Phi_{n,1}[M]$. If $f \in \mathcal{A}$ satisfies*

$$\left|\phi\left((\frac{\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+2}f(z))'}{(\mathcal{D}^j_{b,\lambda+1}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+3}f(z))'}{(\mathcal{D}^j_{b,\lambda+2}f(z))'};z\right) - 1\right| < M,$$

then $\left|\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'} - 1\right| < M.$

Corollary 2.18 *Let $M>0$, and $f \in \mathcal{A}$ satisfies*

$$\left|\frac{(\mathcal{D}^j_{b,\lambda+2}f(z))'}{(\mathcal{D}^j_{b,\lambda+1}f(z))'} - \frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'}\right| < \frac{M^2}{(\lambda+1)(1+M)},$$

then $\left|\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'} - 1\right| < M.$

Proof. This follows from Corollary 2.16 by taking $\phi(u,v,w;z) = v-u$ and $\Omega = h(\mathbb{U})$ where $h(z) = \frac{M^2}{(1+\lambda)(1+M)}z$, $M>0$. To use Corollary 2.16, we need to show that $\phi \in \Phi_{n,1}[M]$, that is the admissability condition (2.23) is satisfied. This follows since

$$\begin{aligned} |\phi(u, v, w; z)| &= \left|1 + \frac{K + (\lambda+1)(1+Me^{i\theta})}{(\lambda+2)(1+Me^{i\theta})} Me^{i\theta} - 1 - Me^{i\theta}\right| \\ &= \frac{M}{\lambda+2}\left|\frac{K - (1+Me^{i\theta})}{1+Me^{i\theta}}\right| \geq \frac{M}{\lambda+2}\left|\frac{K-(1+M)}{1+M}\right| \\ &\geq \frac{M}{\lambda+2}\left|\frac{1}{1+Me^{i\theta}} - 1\right| = \frac{M^2}{(\lambda+2)(1+M)}. \end{aligned}$$

$K \geq nM$, $K \neq 1 + M$, $z \in \mathbb{U}$ and $\theta \in \mathbb{R}$. Hence the result is easily deduced from Corollary 2.16.

3 Superordination and Sandwich Results

The dual problem of differential subordination, that is differential superordination of the multiplier transformation is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

Definition 3.1 *Let Ω be a set in $\mathbb{C}$, $q(z) \in \mathcal{H}[q(0), 1]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_n[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w, \zeta) \in \Omega$$

whenever

$$u = q(z),\ \ v = \frac{zq'(z) + m\lambda q(z)}{m(\lambda+1)}$$

$$\Re\left\{\frac{(\lambda+2)(w-u)}{v-u} - (2\lambda+3)\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$
$$(z \in \mathbb{U},\ \zeta \in \partial\mathbb{U},\ m \geq 1).$$

Theorem 3.2 *Let $\phi \in \Phi'_n[\Omega, q]$. If $f \in \mathcal{A}$, $(\mathcal{D}^j_{b,\lambda} f(z))' \in Q_1$ and*

$$\phi\big((\mathcal{D}^j_{b,\lambda} f(z))', (\mathcal{D}^j_{b,\lambda+1} f(z))', (\mathcal{D}^j_{b,\lambda+2} f(z))'; z\big)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi\big((\mathcal{D}^j_{b,\lambda} f(z))', (\mathcal{D}^j_{b,\lambda+1} f(z))', (\mathcal{D}^j_{b,\lambda+2} f(z))'; z\big) : z \in \mathbb{U}\right\} \tag{3.1}$$

implies $q(z) \prec (\mathcal{D}^j_{b,\lambda} f(z))'$.

Proof. Let p be defined by (2.2) and ψ by (2.7). Since $\phi \in \Phi'_n[\Omega, q]$, (2.8) and (3.1) yield

$$\Omega \subset \{\psi(p(z), p'(z), p''(z); z) : z \in \mathbb{U}\}.$$

From (2.6), the admissibility condition for $\phi \in \Phi'_n[\Omega, q]$ is univalent to the admissibility condition for ψ as given in Definition 1.4. Hence $\psi \in \Psi'_n[\Omega, q]$, and by Theorem 1.6, $q(z) \prec p(z)$ or $q(z) \prec (\mathcal{D}^j_{b,\lambda} f(z))'$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω, the the class $\Phi'_n[h(\mathbb{U}), q]$ is written as $\Phi'_n[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.2.

Theorem 3.3 *Let* $q(z) \in \mathcal{H}[q(0), 1]$, $h(z)$ *be analytic in* $\mathbb{U}$ *and* $\phi \in \Phi'_n[h, q]$. *If* $f \in \mathcal{A}$, $(\mathcal{D}^j_{b,\lambda} f(z))' \in Q_1$ *and*

$$\phi\big((\mathcal{D}^j_{b,\lambda} f(z))', (\mathcal{D}^j_{b,\lambda+1} f(z))', (\mathcal{D}^j_{b,\lambda+2} f(z))'; z\big)$$

is univalent in $\mathbb{U}$, *then*

$$h(z) \prec \phi\big((\mathcal{D}^j_{b,\lambda} f(z))', (\mathcal{D}^j_{b,\lambda+1} f(z))', (\mathcal{D}^j_{b,\lambda+2} f(z))'; z\big) \tag{3.2}$$

implies $q(z) \prec (\mathcal{D}^j_{b,\lambda} f(z))'$.

Theorems 3.2 and 3.3 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.2). The following theorem proves the existence of the best subordinant of (3.2) for an appropriate ϕ.

Theorem 3.4 *Let the function* h *be analytic in* $\mathbb{U}$ *and* $\phi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$. *Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z) + (\lambda+1)q(z)}{\lambda+1}, \frac{z^2q''(z) + 2(\lambda+2)zq'(z) + (\lambda+1)(\lambda+2)q(z)}{(\lambda+2)(\lambda+1)}; z\right) = h(z)$$

has a solution $q \in Q_1$. *If* $\phi \in \Phi'_n[h, q]$, $f \in \mathcal{A}$, $(\mathcal{D}^j_{b,\lambda} f(z))' \in Q_1$ *and*

$$\phi\big((\mathcal{D}^j_{b,\lambda} f(z))', (\mathcal{D}^j_{b,\lambda+1} f(z))', (\mathcal{D}^j_{b,\lambda+2} f(z))'; z\big)$$

is univalent in $\mathbb{U}$, *then*

$$h(z) \prec \phi\big((\mathcal{D}^j_{b,\lambda} f(z))', (\mathcal{D}^j_{b,\lambda+1} f(z))', (\mathcal{D}^j_{b,\lambda+2} f(z))'; z\big)$$

implies $q(z) \prec (\mathcal{D}^j_{b,\lambda} f(z))'$, *and* $q(z)$ *is the best subordinant.*

Proof. The proof is similar to the proof of Theorem 2.6 and is omitted.

Combining Theorem 2.3 and 3.3, we obtain the following sandwich-type theorem.

Corollary 3.5 *Let $h_1(z)$ and $q_1(z)$ be analytic functions in $\mathbb{U}$, $h_2(z)$ be univalent in $\mathbb{U}$, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$, and $\phi \in \Phi_n[h_2, q_2] \cap \Phi'_n[h_1, q_1]$. If $f \in \mathcal{A}$, $(\mathcal{D}^j_{b,\lambda} f(z))' \in \mathcal{H}[q(0), 1] \cap Q_1$ and*

$$\phi\big((\mathcal{D}^j_{b,\lambda} f(z))', (\mathcal{D}^j_{b,\lambda+1} f(z))', (\mathcal{D}^j_{b,\lambda+2} f(z))'; z\big)$$

is univalent in $\mathbb{U}$, then

$$h_1(z) \prec \phi\big((\mathcal{D}^j_{b,\lambda} f(z))', (\mathcal{D}^j_{b,\lambda+1} f(z))', (\mathcal{D}^j_{b,\lambda+2} f(z))'; z\big) \prec h_2(z)$$

implies $q_1(z) \prec (\mathcal{D}^j_{b,\lambda} f(z))' \prec q_2(z)$.

Definition 3.6 *Let Ω be a set in $\mathbb{C}$, and $q(z) \in \mathcal{H}[q(0), 1]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{n,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w; \zeta) \in \Omega$$

whenever

$$\begin{aligned} &u = q(z), \;\; v = \frac{1}{\lambda+2}\Big((\lambda+1)q(z) + 1 + \frac{zq'(z)}{mq(z)}\Big), \qquad (q(z) \neq 0),\\ &\Re\left\{\frac{[(\lambda+3)w - (\lambda+2)v - 1](\lambda+2)v}{(\lambda+2)v - [(\lambda+1)r + 1]} - [2(\lambda+1)r + 1 - (\lambda+2)v]\right\}\\ &\leq \frac{1}{m}\,\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\}, \qquad (z \in \mathbb{U},\, \zeta \in \partial\mathbb{U},\, m \geq 1). \end{aligned}$$

Now we will give the dual result of Theorem 2.13 for differential superordination.

Theorem 3.7 *Let $\phi \in \Phi'_{n,1}[\Omega, q]$. If $f \in \mathcal{A}$, $\frac{(\mathcal{D}^j_{b,\lambda+1} f(z))'}{(\mathcal{D}^j_{b,\lambda} f(z))'} \in Q_1$ and*

$$\phi\left(\frac{(\mathcal{D}^j_{b,\lambda+1} f(z))'}{(\mathcal{D}^j_{b,\lambda} f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+2} f(z))'}{(\mathcal{D}^j_{b,\lambda+1} f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+3} f(z))'}{(\mathcal{D}^j_{b,\lambda+2} f(z))'}; z\right),$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{(\mathcal{D}^j_{b,\lambda+1} f(z))'}{(\mathcal{D}^j_{b,\lambda} f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+2} f(z))'}{(\mathcal{D}^j_{b,\lambda+1} f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+3} f(z))'}{(\mathcal{D}^j_{b,\lambda+2} f(z))'}; z\right) : z \in \mathbb{U}\right\}, \qquad (3.3)$$

implies $q(z) \prec \frac{(\mathcal{D}^j_{b,\lambda+1} f(z))'}{(\mathcal{D}^j_{b,\lambda} f(z))'}$.

Proof. Let p be defined by (2.16) and ψ by (2.20). Since $\phi \in \Phi'_n[\Omega, q]$, (2.21) and (3.3) yield

$$\Omega \subset \left\{\psi(p(z), p'(z), p''(z); z) : z \in \mathbb{U}\right\}.$$

From (2.19), the admissibility condition for $\phi \in \Phi'_{n,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.4. Hence $\psi \in \Psi'_n[\Omega, q]$, and by Theorem 1.6, $q(z) \prec p(z)$ or $q(z) \prec \frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'}$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω, the the class $\Phi'_{n,1}[h(\mathbb{U}), q]$ is written as $\Phi'_{n,1}[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.7.

Theorem 3.8 *Let* $q(z) \in \mathcal{H}[q(0), 1]$, $h(z)$ *be analytic in* $\mathbb{U}$ *and* $\phi \in \Phi'_{n,1}[h, q]$. *If* $f \in \mathcal{A}$, $\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'} \in Q_1$ *and*

$$\phi\left(\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+2}f(z))'}{(\mathcal{D}^j_{b,\lambda+1}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+3}f(z))'}{(\mathcal{D}^j_{b,\lambda+2}f(z))'}; z\right)$$

is univalent in $\mathbb{U}$, *then*

$$h(z) \prec \phi\left(\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+2}f(z))'}{(\mathcal{D}^j_{b,\lambda+1}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+3}f(z))'}{(\mathcal{D}^j_{b,\lambda+2}f(z))'}; z\right) \tag{3.4}$$

implies $q(z) \prec \frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'}$.

Combining Theorem 2.14 and 3.8, we obtain the following sandwich-type theorem.

Corollary 3.9 *Let* $h_1(z)$ *and* $q_1(z)$ *be analytic functions in* $\mathbb{U}$, $h_2(z)$ *be univalent in* $\mathbb{U}$, $q_2 \in Q_1$ *with* $q_1(0) = q_2(0) = 1$, *and* $\phi \in \Phi_{n,1}[h_2, q_2] \cap \Phi'_{n,1}[h_1, q_1]$. *If* $f \in \mathcal{A}$, $\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'} \in \mathcal{H}[q(0), 1] \cap Q_1$ *and*

$$\phi\left(\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+2}f(z))'}{(\mathcal{D}^j_{b,\lambda+1}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+3}f(z))'}{(\mathcal{D}^j_{b,\lambda+2}f(z))'}; z\right)$$

is univalent in $\mathbb{U}$, *then*

$$h_1(z) \prec \phi\left(\frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+2}f(z))'}{(\mathcal{D}^j_{b,\lambda+1}f(z))'}, \frac{(\mathcal{D}^j_{b,\lambda+3}f(z))'}{(\mathcal{D}^j_{b,\lambda+2}f(z))'}; z\right) \prec h_2(z)$$

implies $q_1(z) \prec \frac{(\mathcal{D}^j_{b,\lambda+1}f(z))'}{(\mathcal{D}^j_{b,\lambda}f(z))'} \prec q_2(z)$.

4 Open Problem

From the Definition 1.1 we have the following relation:

$$z(\mathcal{D}^{j}_{b,\lambda}f(z))' = (1+b)\mathcal{D}^{j}_{b+1,\lambda}f(z) - b\mathcal{D}^{j}_{b,\lambda}f(z).$$

One can use this relation and the same techniques to prove the earlier results to obtain a new set of results. Compare these results with the results given by [9].

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