# A Criterion of Optimization of a Modified Fundamental Solution for Two Dimensional Elastic Waves

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**Abstract**

*In this work we are concerned with the determination of an optimal choice for the multipole coefficients of the modified fundamental solution. The optimization criterion is the minimization of the kernel of the modified integral operator for two dimensional elastic waves. Interesting results are obtained for the special case of a circular boundaries.*

**Keywords:** *Multipole coefficients, Green's function, integral equations, linear elasticity.*

## 1 Introduction

The problem of scattering of waves (water, acoustic, elastic and electromagnetic waves) in a domain containing an inhomogeinety (cavity, inclusion or others) is very often formulated in terms of a boundary value problem. The solution to such problem can be sought using different methods (finite difference, finite element and so on). In the case of infinite domain, the boundary integral equation method seems to be more appropriate for solving this type of problems. This method reduces the solving of the problem to an integral equation on the internal boundary of the domain. However, a problem of uniqueness of the solution of the boundary integral equation appears. This anomaly is related to the method of the resolution used rather than to the physical nature of the problem. Some methods, to overcome this anomaly, were proposed (see [1] for a detailled discussion of the proposed solutions)

Indeed, when using the method of integral representations, the two problems; exterior problem (which has a unique solution) and the interior one (which has no unique solution for a certain specter of values of the frequency of waves) are represented by two integral equations with adjoint kernels, and therefore they will have the same number of solutions [2] which presents a contardiction. To recover the uniqueness of the solution of the interior problem, Jones [3] and Ursell [4] developed a technique in acoustics. In 1986, the second author [1] has developed this technique (called a modified Green's function technique) in the case of elastic waves by adding to the fundamental solution a set of functions called multipole. Physically talk, this technique is based on the injection of absorbent points or small circles inside the domain, to transform the phenomenon of stationary waves (interior problem) to a phenomenon of diverging progressive waves (exterior problem). This modification involves the complex coefficients called multipoles coefficients and which should satisfy a large condition (2.20).

In the case of acoustic waves, a method of determination of an optimal choice of those coefficients was elaborated by Roach and Kleinman [5]. This method is based on the minimization of the norm of the modified integral operator. In [6] Argyropoulos, Kiriaki and Roach determined another optimal choice for these coefficients by minimizing the norm of the kernel of the modified integral operator for the case of three dimensional elastic waves. The minimization of the norm of this integral operator or of the norm of its kernel is related to the convergence of the iterative method used for the resolution of that modified integral equation, namely the method of successive approximations. We have established in [7] an optimal choice for the multipole coefficients in the case of two dimensional elastic waves by minimizing the norm of the modified integral operator. In present paper, we propose to find another optimal choice based this time on the minimization of the norm of the kernel of the modified integral operator (where the Green's function is modified with simple or cross multipole coefficients).

# 2    Formulation of the problem

Consider a domain $D \subset IR^2$ which is homogeneous, elastic and isotropic, unbounded externally and bounded internally by $\partial D$. We denote $D_- = IR^2/(D \cup \partial D)$; $P, Q$ points of $D$; $p, q$ points of $\partial D$; and $P_-, Q_-$ points of $D_-$

The boundary problem is thus formulated as follows :
i) The field equation in $D$ :

$$\frac{1}{k^2} \, grad \, (div \, u \, (P)) - \frac{1}{K^2} \, rot \, (rot \, u \, (P)) + u \, (P) = 0 \quad P \in D \qquad (2.1)$$

ii) The boundary condition on $\partial D$ :

$$Tu(p) = f(p) \quad (\text{ Neumann condition}) \qquad p \in \partial D$$

or

$$u(p) = g(p) (\text{ Dirichlet condition}) \qquad p \in \partial D \qquad (2.2)$$

iii) The radiation conditions:

When the domain is unbounded, conditions on the behaviour of the solution at infinity must be imposed. These conditions ensure uniqueness of the solution. Physically talk, the solution must represent a phenomenon of diverging progressive waves. These conditions are called the radiation conditions and are expressed as follows [16] :

$$\lim_{r_P \longrightarrow +\infty} u'(P) = 0 \quad \text{and} \quad \lim_{r_P \longrightarrow +\infty} u''(P) = 0 \qquad (2.3)$$

$$\lim_{r_P \longrightarrow +\infty} (r_p)^{\frac{1}{2}} \left\{ \frac{\partial u'(P)}{\partial r_P} - i\,k\,u'(P) \right\} = 0$$

$$\lim_{r_P \longrightarrow +\infty} (r_p)^{\frac{1}{2}} \left\{ \frac{\partial u''(P)}{\partial r_P} - i\,k\,u''(P) \right\} = 0$$

where $u'(P) = -\frac{1}{k^2} grad\,(divu\,(P))$, and

$$u''(P) = -\frac{1}{K^2} rot\,(rot\,u\,(P)) \qquad (2.4)$$

with $k^2 = \frac{\rho\,\omega^2}{\lambda+2\mu}$ and $K^2 = \frac{\rho\,\omega^2}{\mu}$, and $\rho$ is the density. $\lambda, \mu$ are the Lamé constants and $\omega^2$ is the frequency of the waves.

$T$ is the traction operator which acts on the function $u(p)$ with respect to the point on the boundary. To avoid any ambiguity, we write $T_p$ or $T_q$ when the operator acts on a function of two points. In the case of an isotropic domain, $T$ takes the form :

$$Tu(p) = \lambda.\widehat{n}(p)\,div\,u(p) + 2\mu\,\frac{\partial u(p)}{\partial n_P} + \mu.\widehat{n}(p) \times rot\,u(p) \qquad (2.5)$$

where $\widehat{n}(p)$ is the normal in $p$ directed in the exterior of $D_-$.

The boundary problem as defined possesses a unique solution for all values of the frequency $\omega^2$ [16]

To obtain an integral representation of the solution of our problem, a fundamental singular solution is necessary. This solution, known as the Green tensor

is a singular function of two points, denoted $G_0\left(P,Q\right)$ , such that $G_0\left(P,Q\right)$ satisfies the equation (2.1) except when $P=Q$, and the radiation conditions (2.3). In the case of an isotropic and homogenous domain, this tensor is expressed as follows [17]:

$$G_0\left(P,Q\right) = \frac{i}{4\mu}\left\{\Psi.I + \frac{1}{K^2}grad\left(grad\left(\Psi - \Phi\right)\right)\right\} \qquad (2.6)$$

where $I$ is the unit tensor, $\Psi = H_0^1\left(K.R\right)$ and $\Phi = H_0^1\left(k.R\right)$, $H_0^1\left(\ .\right)$are the function of Hankel of zero order and first type and $R$ is the distance between the two points $P$ and $Q$ .

Having defined the fundamental solution, integral representations to solve our problem can now be obtained. There are two methods for obtaining integral representations, the direct method and the indirect method. The direct method is based on the Green's formula, where the unknown has a physical sense which is either displacement or traction. The indirect method is based on the concept of single layer or double layer potential , in this case the unknown does not have a clear physical sense. We will begin by the indirect method. For the Dirichlet problem, the solution is represented by a double layer potential :

$$u\left(P\right) = \int_{\partial D} T_q G_0\left(P,q\right).W\left(q\right).ds_q = \left(DW\right)\left(P\right), \quad P \in D \qquad (2.7)$$

where $W\left(q\right)$ is a density function defined on $\partial D$. This representation satisfies the equation (2.1) and the radiation conditions (2.3). If we apply the Dirichlet condition (2.2) taking into account the properties of double layer potential [16], then we obtain the following integral equation :

$$\frac{1}{2}W\left(p\right) + \int_{\partial D} T_q G_0\left(q,p\right).W\left(q\right).ds_q$$

$$= \frac{1}{2}W\left(p\right) + \left(\overline{K_0^*}W\right)\left(p\right) = g\left(p\right), \quad p \in \partial D \qquad (2.8)$$

where $K_0$ is the integral operator defined by :

$$\left(K_0 W\right)\left(p\right) = \int_{\partial D} T_p G_0\left(p,q\right).W\left(q\right).ds_q, \quad p \in \partial D \qquad (2.9)$$

For the Neumann problem, the solution is represented by a single layer potential :

$$u\left(P\right) = \int_{\partial D} G_0\left(P,q\right).W\left(q\right).ds_q = \left(SW\right)\left(P\right) \qquad P \in D \qquad (2.10)$$

where $W(q)$ is an unknown density function defined on $\partial D$. This representation satisfies the equation (2.1) and the radiation conditions (2.3). If we apply the Neumann condition (2.2) taking into account the properties of single layer potential [16], we obtain the following integral equation :

$$-\frac{1}{2}W(p) + \int_{\partial D} T_p G_0(p,q) . W(q) . ds_q$$

$$= -\frac{1}{2}W(p) + (K_0 W)(p) = f(p) \quad p \in \partial D \qquad (2.11)$$

For the direct method, the application of the Green's formula to the sought solution $u$ and to the Green's tensor $G_0$ in the domain limited internally by $\partial D$ and limited externally by a large circle of radius $r_\infty$, leads to the following relations:

$$u(P) = \int_{\partial D} [u(q) . T_q G_0(q,P) - Tu(q) . G_0(q,P)] . ds_q, \quad P \in D \qquad (2.12)$$

and

$$\frac{1}{2}u(p) = \int_{\partial D} [u(q) . T_q G_0(q,p) - Tu(q) . G_0(q,p)] . ds_q, \quad p \in \partial D. \qquad (2.13)$$

Where it was taken into account that $u$ and $G_0$ satisfy the equation (2.1) and the radiation conditions (2.3).

If we apply the Neumann condition (2.2) on the boundary to the equation (2.13), we obtain the following integral equation :

$$\frac{1}{2}u(p) - \int_{\partial D} u(q) . T_q G_0(q,p) . ds_q$$

$$= -\int_{\partial D} f(q) . G_0(q,p) . ds_q, \quad p \in \partial D \qquad (2.14)$$

Or in the form :

$$\frac{1}{2}u(p) - \left(\overline{K}_0^* u\right)(p) = -(S f)(p), \quad p \in \partial D \qquad (2.15)$$

The two integral equations (2.11) and (2.15) corresponding respectively to the integral representations (2.10) and (2.12) for our problem, are adjoint integral equations. So, by virtue of theorems of Fredholm [2] if one of the two equations has a unique solution it will be the same for the other. It follows then that only one of the two needs to be studied. But it is known and established (see for example [1] ) that these two equations have not unique

solutions for a specter of discrete values of the frequency $\omega^2$ wich presents a contardiction. This anomaly observed in the uniqueness of the solution of the integral equations above is due to the method of resolution selected rather than the physical nature of the problem.

Many methods have been proposed to overcome this anomaly. one of them is based on the modification of the function of Green $G_0$ by adding a series of functions called multipoles. The modified Green's tensor is given in [1] by :

$$G_1\left(P,Q\right) = G_0\left(P,Q\right) + \frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}[a_m^{\sigma l}.F_m^{\sigma l}(P)\otimes F_m^{\sigma l}(Q)] \qquad (2.16)$$

(modification with simple multipole coefficients)

$$G_1(p,q) = G_0(p,q) + \frac{i}{4\mu K^2}\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\sum_{\ell=1}^{2}[a_m^{\sigma\ell}.F_m^{\sigma\ell}(P)\otimes F_m^{\sigma\ell}(Q)$$

$$+(-1)^{\sigma+\ell}.b_m.F_m^{\sigma\ell}(P)\otimes F_m^{(3-\sigma)(3-\ell)}(Q)] \qquad (2.17)$$

(modification with simple and cross multipole coefficients)

where $a_m^{\sigma l}$ and $b_m$ are respectively, the simple and cross multipole coefficients,

$$F_m^{\sigma 1}(P) = grad\left(H_m^1\left(k.r_p\right).E_m^{\sigma}\left(\theta_p\right)\right)$$

$$F_m^{\sigma 2}(P) = rot\left(H_m^1\left(K.r_p\right).E_m^{\sigma}\left(\theta_p\right)\ \widehat{e}_3\right)$$

$(r_p,\theta_p)$ are the polar coordinates of the point, $H_m^1(\ .)$ is the fonction of Hankel of order $m$ and type 1 and

$$E_m^{\sigma}(\theta) = \sqrt{\varepsilon_m}.\left\{\begin{array}{ll}\cos(m\theta_p) & \sigma = 1 \\ \sin(m\theta_p) & \sigma = 2\end{array}\right. \text{ with } \varepsilon_m = \left\{\begin{array}{ll}1 & m = 0 \\ 2 & m > 0\end{array}\right.,$$

$\otimes$ design the tensorial product and $e_3$ is the unit vector in the direction of $Z$.

The modified integral equations corresponding to (2.11) and (2.15) becomes then respectively :

$$-\frac{1}{2}W\left(p\right) + \left(K_1W\right)\left(p\right) = f\left(p\right), \quad p \in \partial D \qquad (2.18)$$

$$\frac{1}{2}u\left(p\right) - \left(\overline{K}_1^*u\right)\left(p\right) = -\left(S_1\ f\right)\left(p\right), \quad p \in \partial D \qquad (2.19)$$

where $K_1$ is the modified intégral operator defined in the same way as $K_0$ but with $G_0$ replaced by $G_1$.

In order to ensure the uniqueness of the solutions of integral equation (2.18) and (2.19), a sufficient but not necessary condition has been established in [1]. This condition is related to the multipole coefficients and it is expressed in the following form :

$$\bar{b}_m \cdot \left( a_m^{\sigma 1} + \frac{1}{2} \right) + b_m \cdot \left( \bar{a}_m^{\sigma 2} + \frac{1}{2} \right) = 0 \qquad (2.20)$$

and

$$|a_m^{\sigma l} + \frac{1}{2}|^2 + |b_m|^2 - \frac{1}{4} < 0, \quad \forall m = 0 : \infty \quad \text{and} \quad \forall \sigma, l = 1 : 2$$

Since the condition (2.20) is a large condition, we propose to determine an optimal choice for these coefficients.

# 3 Main results

## 3.1 General case

In the following paragraph, we determine the expressions of the multipole coefficients which minimizes the norm of the kernel of $K_1$, i.e., we minimize the norm of the modified Green's function $G_1$.

### 3.1.1 Theorem

If the kernel of the modified integral operator $K_1$, namely the function of Green $G_1$ is defined by (2.16) then the quantity

$$\int_{r_p = A} \|G_1\|_{L_2(\partial D)}^2 \cdot ds_p \qquad \forall A \succeq \max r_q \ , \qquad q \in \partial D \qquad (3.1)$$

is bounded if the simple multipole coefficients are selected as follows :

$$a_m^{\sigma l} = -\frac{< \widehat{F}_m^{\sigma \varrho} , F_m^{\sigma l} >}{\|F_m^{\sigma l}\|_{L_2(\partial D)}^2} \qquad (3.2)$$

where $\widehat{F}_m^{\sigma \varrho} = Re \left( F_m^{\sigma l} \right)$

**Proof :**

we have :

$$G_1\left(P,Q\right) = G_0\left(P,Q\right) + \frac{i}{4\mu K^2} \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} [a_m^{\sigma l}.F_m^{\sigma l}(P) \otimes F_m^{\sigma l}(Q)]$$

$$= \frac{i}{4\mu K^2} \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} [F_m^{\sigma l}(P) \otimes \widehat{F}_m^{\sigma \varrho}(Q) + a_m^{\sigma l}.F_m^{\sigma l}(P) \otimes F_m^{\sigma l}(Q)] \qquad (3.3)$$

we set :

$$f_m^{\sigma l}(Q) = \left[\widehat{F}_m^{\sigma \varrho}(Q) + a_m^{\sigma l}.F_m^{\sigma l}(Q)\right] \qquad (3.4)$$

So the modified Green's function $G_1$ is written in the form :

$$G_1\left(P,Q\right) \; = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} [F_m^{\sigma l}(P) \otimes f_m^{\sigma \varrho}(Q)] \qquad (3.5)$$

so :  $\int_{r_p=A} \|G_1\|_{L_2(\partial D)}^2 \; . \; ds_p = \int_{r_p=A} \int_{\partial D} G_1\left(P,q\right) \; : \; \overline{G}_1\left(q,P\right) \; . \; ds_p \; . ds_q$

$$= \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} \sum_{n=0}^{\infty} \sum_{\nu=1}^{2} \sum_{k=1}^{2} \int_{r_p=A} F_m^{\sigma l}(P).\overline{F}_n^{\nu k}(P).ds_p \; . \; \int_{\partial D} f_m^{\sigma l}(q).\overline{f}_n^{\nu k}(q).ds_q$$

Using inner product relations of the functions $\left\{F_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ on the circle of radius $A$, we obtain :

$$\int_{r_p=A} \|G_1\|_{L_2(\partial D)}^2 \; . \; ds_p = 2 \; \pi \; A \; . \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \left(|a|^2 + |b|^2\right) . \int_{\partial D} \left|f_m^{\sigma 1}(q)\right|^2 .ds_q$$

$$- \left(a \; \overline{d} + \overline{c} \; b\right) . \, (-1)^{\sigma} . \int_{\partial D} f_m^{\sigma 1}(q).\overline{f}_m^{(3-\sigma)2}(q).ds_q$$

$$- \left(\overline{a} \; d + c \; \overline{b}\right) . \, (-1)^{\sigma} . \int_{\partial D} \overline{f}_m^{\sigma 1}(q).f_m^{(3-\sigma)2}(q).ds_q$$

$$+ \left(|c|^2 + |d|^2\right) . \int_{\partial D} \left|f_m^{\sigma 2}(q)\right|^2 .ds_q$$

with

$$a = k \; . \; H_m^{'}\left(k \; r\right) \quad , \quad b = k \; . \; \frac{m}{k \; r} H_m\left(k \; r\right)$$

$$c = K \; . \; H_m^{'}\left(K \; r\right) \quad , \quad d = K \; . \; \frac{m}{K \; r} H_m\left(K \; r\right)$$

so :

$$\int_{r_p=A} \|G_1\|^2_{L_2(\partial D)} \, . \, ds_p = 2 \, \pi \, A \, . \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \int_{\partial D} \left| a \, f_m^{\sigma 1}(q) \right|^2 . ds_q + \int_{\partial D} \left| b \, f_m^{\sigma 1}(q) \right|^2 . ds_q$$

$$- \int_{\partial D} \left( a \, f_m^{\sigma 1}(q) \right) . \left( \overline{(-1)^{\sigma} . d . f_m^{(3-\sigma)2}(q)} \right) . ds_q$$

$$- \int_{\partial D} \left( b \, f_m^{\sigma 1}(q) \right) . \left( \overline{(-1)^{\sigma} . c . f_m^{(3-\sigma)2}(q)} \right) . ds_q$$

$$- \int_{\partial D} \left( \overline{a \, f_m^{\sigma 1}(q)} \right) . \left( (-1)^{\sigma} . d . f_m^{(3-\sigma)2}(q) \right) . ds_q$$

$$- \int_{\partial D} \left( \overline{b \, f_m^{\sigma 1}(q)} \right) . \left( (-1)^{\sigma} . c . f_m^{(3-\sigma)2}(q) \right) . ds_q$$

$$+ \int_{\partial D} \left( \left| (-1)^{\sigma} . c . f_m^{(3-\sigma)2}(q) \right|^2 \right) . ds_q + \int_{\partial D} \left( \left| (-1)^{\sigma} . d . f_m^{(3-\sigma)2}(q) \right|^2 \right) . ds_q$$

$$= 2 \, \pi \, A \, . \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \int_{\partial D} \left( \left| a \, f_m^{\sigma 1}(q) - (-1)^{\sigma} . d . f_m^{(3-\sigma)2}(q) \right|^2 \right) . ds_q$$

$$+ \int_{\partial D} \left( \left| b \, f_m^{\sigma 1}(q) - (-1)^{\sigma} . c . f_m^{(3-\sigma)2}(q) \right|^2 \right) . ds_q \qquad (3.6)$$

On the other hand, and from the Minkowski inequality, we have :

$$\int_{\partial D} \left( \left| a \, f_m^{\sigma 1}(q) - (-1)^{\sigma} . d . f_m^{(3-\sigma)2}(q) \right|^2 \right) . ds_q$$

$$\preceq \left[ \left[ \int_{\partial D} \left( \left| a \, f_m^{\sigma 1}(q) \right|^2 \right) . ds_q \right]^{\frac{1}{2}} + \left[ \int_{\partial D} \left( \left| (-1)^{\sigma} . d . f_m^{(3-\sigma)2}(q) \right|^2 \right) . ds_q \right]^{\frac{1}{2}} \right]^2$$

and $\int_{\partial D} \left( \left| b \, f_m^{\sigma 1}(q) - (-1)^{\sigma} . c . f_m^{(3-\sigma)2}(q) \right|^2 \right) . ds_q$

$$\preceq \left[ \left[ \int_{\partial D} \left( \left| b \, f_m^{\sigma 1}(q) \right|^2 \right) . ds_q \right]^{\frac{1}{2}} + \left[ \int_{\partial D} \left( \left| (-1)^{\sigma} . c . f_m^{(3-\sigma)2}(q) \right|^2 \right) . ds_q \right]^{\frac{1}{2}} \right]^2$$

so :

$$\int_{r_p=A} \|G_1\|^2_{L_2(\partial D)} \, . \, ds_p \leq 2 \, \pi \, A \, . \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \left( |a|^2 + |b|^2 \right) . \int_{\partial D} \left| f_m^{\sigma 1}(q) \right|^2 . ds_q$$

$$+ \left( |c|^2 + |d|^2 \right) \cdot \int_{\partial D} \left| f_m^{(3-\sigma)2}(q) \right|^2 . ds_q$$

$$+2 \cdot \left( |a| \cdot |d| + |b| \cdot |c| \right) \cdot \left[ \int_{\partial D} \left| f_m^{\sigma 1}(q) \right|^2 . ds_q \cdot \int_{\partial D} \left| f_m^{(3-\sigma)2}(q) \right|^2 . ds_q \right]$$

Also, the term $\int_{\partial D} \left| f_m^{\sigma l}(q) \right|^2 . ds_q$ can be rearranged as follows:

$$\int_{\partial D} \left| f_m^{\sigma l}(q) \right|^2 . ds_q = \int_{\partial D} \left| \widehat{F}_m^{\sigma \varrho}(Q) + a_m^{\sigma l} . F_m^{\sigma l}(Q) \right|^2 . ds_q$$

$$= \int_{\partial D} \left( \widehat{F}_m^{\sigma \varrho}(Q) + a_m^{\sigma l} . F_m^{\sigma l}(Q) \right) \cdot \left( \overline{\widehat{F}_m^{\sigma \varrho}}(Q) + \overline{a}_m^{\sigma l} . \overline{F}_m^{\sigma l}(Q) \right) ds_q$$

$$= \int_{\partial D} \left[ \begin{array}{c} \widehat{F}_m^{\sigma \varrho}(Q) . \overline{\widehat{F}_m^{\sigma \varrho}}(Q) + \overline{a}_m^{\sigma l} . \widehat{F}_m^{\sigma \varrho}(Q) . \overline{F}_m^{\sigma l}(Q) + a_m^{\sigma l} . F_m^{\sigma l}(Q) . \overline{\widehat{F}_m^{\sigma \varrho}}(Q) \\ + \left| a_m^{\sigma l} \right|^2 . F_m^{\sigma l}(Q) . \overline{F}_m^{\sigma l}(Q) \end{array} \right] ds_q$$

$$= \left[ \begin{array}{c} \left\| \widehat{F}_m^{\sigma \varrho} \right\|_{L_2(\partial D)}^2 + \overline{a}_m^{\sigma l} . < \widehat{F}_m^{\sigma \varrho} , F_m^{\sigma l} > + a_m^{\sigma l} . < F_m^{\sigma l} , \widehat{F}_m^{\sigma \varrho} > \\ + \left| a_m^{\sigma l} \right|^2 . \left\| F_m^{\sigma l} \right\|_{L_2(\partial D)}^2 \end{array} \right]$$

$$= \left[ \left\| \widehat{F}_m^{\sigma \varrho} \right\|_{L_2(\partial D)}^2 - \frac{\left| < \widehat{F}_m^{\sigma \varrho} , F_m^{\sigma l} > \right|^2}{\left\| F_m^{\sigma l} \right\|_{L_2(\partial D)}^4} + \left\| F_m^{\sigma l} \right\|_{L_2(\partial D)}^2 \cdot \left| a_m^{\sigma l} + \frac{< \widehat{F}_m^{\sigma \varrho} , F_m^{\sigma l} >}{\left\| F_m^{\sigma l} \right\|_{L_2(\partial D)}^2} \right|^2 \right]$$

from this effect, it is clear that the quantity (3.1) will be bounded if the value $\left| a_m^{\sigma l} + \frac{< \widehat{F}_m^{\sigma \varrho} , F_m^{\sigma l} >}{\left\| F_m^{\sigma l} \right\|_{L_2(\partial D)}^2} \right|^2$ is equal to zero. So we have : $a_m^{\sigma l} = -\frac{< \widehat{F}_m^{\sigma \varrho} , F_m^{\sigma l} >}{\left\| F_m^{\sigma l} \right\|_{L_2(\partial D)}^2}$

This completes the demonstration of the theorem.

In what follows we will show that the optimal choice defined by (3.2), satisfies the large condition (2.20).

Remind that, in the case of simple multipole coefficients the expression of the large condition (2.20) is in the form :

$$\left| a_m^{\sigma l} + \frac{1}{2} \right| < \frac{1}{2} \quad \text{or} \quad \left| 2 . a_m^{\sigma l} + 1 \right| < 1$$

### 3.1.2 Lemma

If the simple multipole coefficients are defined by (3.2), then the condition (2.20) is satisfied.

**Proof :**

$$\left| 2 \, . a_m^{\sigma l} + 1 \right| = \left| -2 \frac{< \widehat{F}_m^{\sigma \varrho} \, , \, F_m^{\sigma l} >}{\| F_m^{\sigma l} \|_{L_2(\partial D)}^2} + 1 \right|$$

$$= \left| \frac{< F_m^{\sigma l} \, , \, F_m^{\sigma l} > \, - \, 2 \, < \widehat{F}_m^{\sigma \varrho} \, , \, F_m^{\sigma l} >}{\| F_m^{\sigma l} \|_{L_2(\partial D)}^2} \right|$$

$$= \left| \frac{- \, < \overline{F}_m^{\sigma l} \, , \, F_m^{\sigma l} >}{\| F_m^{\sigma l} \|_{L_2(\partial D)}^2} \right| = \left| \frac{< \overline{F}_m^{\sigma l} \, , \, F_m^{\sigma l} >}{\| F_m^{\sigma l} \|_{L_2(\partial D)}^2} \right|$$

we set $F_m^{\sigma l} = u \, + \, i \, v$, so :

$$\left| 2 \, . a_m^{\sigma l} + 1 \right| = \left| \frac{< u \, , \, u > \, - \, < v \, , \, v > \, - \, 2 \, i \, < u \, , \, v >}{< u \, , \, u > \, + \, < v \, , \, v >} \right|$$

$$= \frac{\left[ (< u \, , \, u > \, - \, < v \, , \, v >)^2 \, + \, 4 \, < u \, , \, v > \, < u \, , \, v > \right]^{\frac{1}{2}}}{< u \, , \, u > \, + \, < v \, , \, v >}$$

$$= \frac{\left[ (< u \, , \, u > \, - \, < v \, , \, v >)^2 \, + \, 4 \left( \begin{array}{c} < u \, , \, v >^2 \\ - \, < u \, , \, u > \, < v \, , \, v > \end{array} \right)^2 \right]^{\frac{1}{2}}}{< u \, , \, u > \, + \, < v \, , \, v >} \tag{3.7}$$

It is clear, that if we have :

$$< u \, , \, v >^2 \, - \, < u \, , \, u > \, < v \, , \, v > \, < 0$$

Then, the quantity (3.7) is strictly less than 1.

But from the Schwartz inequality, we have :

$$< u \, , \, v >^2 \, \preceq \, < u \, , \, u > \, < v \, , \, v >$$

$$\Longleftrightarrow \, < u \, , \, v >^2 \, - \, < u \, , \, u > \, < v \, , \, v > \, \preceq 0$$

Moreover, if the two functions $u = Re\left( F_m^{\sigma l} \right) = \, \widehat{F}_m^{\sigma \varrho}$ and $v = Im\left( F_m^{\sigma l} \right)$ are linearly independent, then the Schwartz inequality becomes strict

For this, we will study the case where $l = 1$ :

we have : $u = \nabla\left(J_m\left(k\ r\right).\ E_m^\sigma\left(\theta\right)\right)$    and    $v = \nabla\left(Y_m\left(k\ r\right).\ E_m^\sigma\left(\theta\right)\right)$

$$u = k\ J_m^{'}\left(k\ r\right).\ E_m^\sigma\left(\theta\right)\ .\ \overrightarrow{r}\ +\ \frac{m}{r}\ J_m\left(k\ r\right).\ \left(-1\right)^\sigma\ E_m^{(3-\sigma)}\left(\theta\right)\ .\ \overrightarrow{\theta}$$

$$v = k\ Y_m^{'}\left(k\ r\right).\ E_m^\sigma\left(\theta\right)\ .\ \overrightarrow{r}\ +\ \frac{m}{r}\ Y_m\left(k\ r\right).\ \left(-1\right)^\sigma\ E_m^{(3-\sigma)}\left(\theta\right)\ .\ \overrightarrow{\theta}$$

the functions $u$ and $v$ will be linearly independent if and only if :

$$\frac{k\ J_m^{'}\left(k\ r\right).\ E_m^\sigma\left(\theta\right)}{k\ Y_m^{'}\left(k\ r\right).\ E_m^\sigma\left(\theta\right)}\ \neq\ \frac{\frac{m}{r}\ J_m\left(k\ r\right).\ \left(-1\right)^\sigma\ E_m^{(3-\sigma)}\left(\theta\right)}{\frac{m}{r}\ Y_m\left(k\ r\right).\ \left(-1\right)^\sigma\ E_m^{(3-\sigma)}\left(\theta\right)}$$

In other words, $u$ and $v$ will be linearly independent if and only if :

$$\frac{J_m^{'}\left(k\ r\right)}{Y_m^{'}\left(k\ r\right)}\ \neq\ \frac{J_m\left(k\ r\right)}{Y_m\left(k\ r\right)}$$

Calculate :

$$J_m^{'}\left(k\ r\right).Y_m\left(k\ r\right)-\ J_m\left(k\ r\right).Y_m^{'}\left(k\ r\right)=\ ?$$

we have :

$$J_m^{'}\left(X\right) = -\ J_{m+1}\left(X\right) + \frac{m}{X}\ J_m\left(X\right)$$

and

$$Y_m^{'}\left(X\right) = -\ Y_{m+1}\left(X\right) + \frac{m}{X}\ Y_m\left(X\right)$$

so :

$$J_m^{'}\left(k\ r\right).Y_m\left(k\ r\right)-\ J_m\left(k\ r\right).Y_m^{'}\left(k\ r\right)$$

$$=\ \left[-\ J_{m+1}\left(k\ r\right) + \frac{m}{k\ r}\ J_m\left(k\ r\right)\right].\ Y_m\left(k\ r\right)$$

$$-\ J_m\left(k\ r\right).\left[-\ Y_{m+1}\left(k\ r\right) + \frac{m}{k\ r}\ Y_m\left(k\ r\right)\right]$$

$$=\ \left[-\ J_{m+1}\left(k\ r\right).\ Y_m\left(k\ r\right) + \frac{m}{k\ r}\ J_m\left(k\ r\right).\ Y_m\left(k\ r\right)\right]$$

$$+\ \left[\ J_m\left(k\ r\right).Y_{m+1}\left(k\ r\right) - \frac{m}{k\ r}\ J_m\left(k\ r\right).Y_m\left(k\ r\right)\right]$$

$$=-\ \left[\ J_{m+1}\left(k\ r\right).\ Y_m\left(k\ r\right) - J_m\left(k\ r\right).Y_{m+1}\left(k\ r\right)\right] = -\frac{2}{\pi\ k\ r}\ \neq 0$$

so $u$ and $v$ are linearly independent
then :

$$< u\ ,\ v >^2\ -\ \ < u\ ,\ u >\ \ < v\ ,\ v >\ \ < 0$$

so : $\left|2\ .a_m^{\sigma l} + 1\right| < 1$

### 3.1.3 Theorem

If the kernel of the modified integral operator $K_1$, namely the function of Green $G_1$ is defined by (2.16) then the quantity

$$\int_{r_p=A} \|G_1\|^2_{L_2(\partial D)} \cdot ds_p \qquad \forall \, A \succeq \max \, r_q \, , \qquad q \in \partial D$$

is minimized if the simple multiple coefficients are selected as follows :

$$a_m^{\sigma l} = \frac{\left[\alpha_m^{(3-\sigma)(3-l)} \cdot A_m^{(3-\sigma)(3-l)}\right] \cdot \left[g_m^{\sigma l}\right] - \left[\overline{\beta}_m^{\sigma l} \cdot \overline{B}_m^{\sigma l}\right] \cdot \left[g_m^{(3-\sigma)(3-l)}\right]}{\triangle_m^{\sigma}} \qquad (3.8)$$

where :

$$\alpha_m^{\sigma l} = \left\|F_m^{\sigma l}\right\|^2_A \qquad , \qquad \beta_m^{\sigma l} = \left\langle F_m^{\sigma l} \, , \, F_m^{(3-\sigma)(3-l)}\right\rangle_A$$

$$A_m^{\sigma l} = \left\|F_m^{\sigma l}\right\|^2_{\partial D} \qquad , \qquad B_m^{\sigma l} = \left\langle F_m^{\sigma l} \, , \, F_m^{(3-\sigma)(3-l)}\right\rangle_{\partial D}$$

$$\triangle_m^{\sigma} = \alpha_m^{\sigma 1} \cdot A_m^{\sigma 1} \cdot \alpha_m^{(3-\sigma)2} \cdot A_m^{(3-\sigma)2} - \beta_m^{\sigma 1} \cdot B_m^{\sigma 1} \cdot \overline{\beta}_m^{\sigma 1} \cdot \overline{B}_m^{\sigma 1}$$

and

$$g_m^{\sigma l} = - \left\langle \overline{\beta}_m^{\sigma l} \cdot \widehat{F}_m^{(3-\sigma)(3-l)} + \alpha_m^{\sigma l} \cdot \widehat{F}_m^{\sigma l} \, , \, F_m^{\sigma l}\right\rangle_{\partial D}$$

**Proof:**
**Step 1 :**
The modified Green's function $G_1$ written in the form (3.5) :

$$G_1(P,Q) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} \left[F_m^{\sigma l}(P) \otimes f_m^{\sigma \varrho}(Q)\right]$$

so :

$$\int_{r_p=A} \|G_1\|^2_{L_2(\partial D)} \cdot ds_p = \int_{r_p=A} \int_{\partial D} G_1(P,q) \; : \; \overline{G}_1(q,P) \cdot ds_p.ds_q$$

Using inner product relations of the functions $\left\{F_m^{\sigma l}\right\}_{m=0:\infty}^{\sigma,l=1:2}$ on the circle of radius $A$, we obtain :

$$\int_{r_p=A} \|G_1\|^2_{L_2(\partial D)} \cdot ds_p = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \left[\begin{array}{l} \|F_m^{\sigma 1}\|^2_A \cdot \langle f_m^{\sigma 1} \, , \, f_m^{\sigma 1}\rangle_{\partial D} \\ + \; \langle F_m^{\sigma 1} \, , \, F_m^{\sigma 2}\rangle_A \cdot \left\langle f_m^{\sigma 1} \, , \, f_m^{(3-\sigma)2}\right\rangle_{\partial D} \\ + \; \langle F_m^{\sigma 2} \, , \, F_m^{\sigma 1}\rangle_A \cdot \left\langle f_m^{(3-\sigma)2} \, , \, f_m^{\sigma 1}\right\rangle_{\partial D} \\ + \; \|F_m^{\sigma 2}\|^2_A \cdot \left\langle f_m^{(3-\sigma)2} \, , \, f_m^{(3-\sigma)2}\right\rangle_{\partial D} \end{array}\right] \qquad (3.9)$$

calculating the expressions $\langle f_m^{\sigma 1} , f_m^{\sigma 1} \rangle_{\partial D}$, $\langle f_m^{\sigma 1} , f_m^{(3-\sigma)2} \rangle_{\partial D}$
, $\langle f_m^{(3-\sigma)2} , f_m^{\sigma 1} \rangle_{\partial D}$ and $\langle f_m^{(3-\sigma)2} , f_m^{(3-\sigma)2} \rangle_{\partial D}$, and substituting in (3.9)
we obtain :
$\int_{r_p=A} \|G_1\|_{L_2(\partial D)}^2 \cdot ds_p$

$$= \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \alpha_m^{\sigma 1} \cdot \begin{pmatrix} \left\| \widehat{F}_m^{\sigma 1} \right\|_{\partial D}^2 \\ +\overline{a}_m^{\sigma 1} \left\langle \widehat{F}_m^{\sigma 1} , F_m^{\sigma 1} \right\rangle_{\partial D} \\ + a_m^{\sigma 1} \left\langle F_m^{\sigma 1} , \widehat{F}_m^{\sigma 1} \right\rangle_{\partial D} \\ +a_m^{\sigma 1} \cdot \overline{a}_m^{\sigma 1} \cdot A_m^{\sigma 1} \end{pmatrix}$$

$$+\beta_m^{\sigma 1} \cdot \begin{pmatrix} \left\langle \widehat{F}_m^{\sigma 1} , \widehat{F}_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ +\overline{a}_m^{(3-\sigma)2} \left\langle \widehat{F}_m^{\sigma 1} , F_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ + a_m^{\sigma 1} \left\langle F_m^{\sigma 1} , \widehat{F}_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ +a_m^{\sigma 1} \cdot \overline{a}_m^{(3-\sigma)2} \cdot B_m^{\sigma 1} \end{pmatrix}$$

$$+\overline{\beta}_m^{\sigma 1} \cdot \begin{pmatrix} \left\langle \widehat{F}_m^{(3-\sigma)2} , \widehat{F}_m^{\sigma 1} \right\rangle_{\partial D} \\ +a_m^{(3-\sigma)2} \left\langle F_m^{(3-\sigma)2} , \widehat{F}_m^{\sigma 1} \right\rangle_{\partial D} \\ + \overline{a}_m^{\sigma 1} \left\langle \widehat{F}_m^{(3-\sigma)2} , F_m^{\sigma 1} \right\rangle_{\partial D} \\ +a_m^{(3-\sigma)2} \cdot \overline{a}_m^{\sigma 1} \cdot \overline{B}_m^{\sigma 1} \end{pmatrix}$$

$$+\alpha_m^{(3-\sigma)2} \cdot \begin{pmatrix} \left\| \widehat{F}_m^{(3-\sigma)2} \right\|_{\partial D}^2 \\ +\overline{a}_m^{(3-\sigma)2} \left\langle \widehat{F}_m^{(3-\sigma)2} , F_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ + a_m^{(3-\sigma)2} \left\langle F_m^{(3-\sigma)2} , \widehat{F}_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ +a_m^{(3-\sigma)2} \cdot \overline{a}_m^{(3-\sigma)2} \cdot A_m^{(3-\sigma)2} \end{pmatrix}$$

This is a standard problem of minimization , where the necessary condition
for the existence of a minimum is the vanishing of the gradient.

so, if we cancel the gradient with respect to the coefficients $\overline{a}_m^{\sigma 1}$ and $\overline{a}_m^{(3-\sigma)2}$,
we obtain the following relations :

$$\alpha_m^{\sigma 1} \cdot \begin{pmatrix} \left\langle \widehat{F}_m^{\sigma 1} , F_m^{\sigma 1} \right\rangle_{\partial D} \\ + a_m^{\sigma 1} \cdot A_m^{\sigma 1} \end{pmatrix} + \overline{\beta}_m^{\sigma 1} \cdot \begin{pmatrix} \left\langle \widehat{F}_m^{(3-\sigma)2} , F_m^{\sigma 1} \right\rangle_{\partial D} \\ + a_m^{(3-\sigma)2} \cdot \overline{B}_m^{\sigma 1} \end{pmatrix} = 0$$

$$\beta_m^{\sigma 1} \cdot \begin{pmatrix} \left\langle \widehat{F}_m^{\sigma 1} , F_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ + a_m^{\sigma 1} \cdot B_m^{\sigma 1} \end{pmatrix} + \alpha_m^{(3-\sigma)2} \cdot \begin{pmatrix} \left\langle \widehat{F}_m^{(3-\sigma)2} , F_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ + a_m^{(3-\sigma)2} \cdot A_m^{(3-\sigma)2} \end{pmatrix} = 0$$

Or in the following form :

$$\left\{ \begin{array}{l} (\alpha_m^{\sigma 1} \cdot A_m^{\sigma 1}) \cdot a_m^{\sigma 1} + \left(\overline{\beta}_m^{\sigma 1} \cdot \overline{B}_m^{\sigma 1}\right) \cdot a_m^{(3-\sigma)2} = g_m^{\sigma 1} \\ (\beta_m^{\sigma 1} \cdot B_m^{\sigma 1}) \cdot a_m^{\sigma 1} + \left(\alpha_m^{(3-\sigma)2} \cdot A_m^{(3-\sigma)2}\right) \cdot a_m^{(3-\sigma)2} = g_m^{(3-\sigma)2} \end{array} \right.$$

According to the Schwartz inequality and the linear independence of the functions $\left\{ F_m^{\sigma l} \right\}_{m\,=\,0\,:\,\infty}^{\sigma, l\,=\,1\,:\,2}$, the latter system admits the following non-zero determinant :

$$\triangle_m^\sigma = \alpha_m^{\sigma 1} \cdot A_m^{\sigma 1} \cdot \alpha_m^{(3-\sigma)2} \cdot A_m^{(3-\sigma)2} - \beta_m^{\sigma 1} \cdot B_m^{\sigma 1} \cdot \overline{\beta}_m^{\sigma 1} \cdot \overline{B}_m^{\sigma 1}$$

And therefore, the solutions of the latter system will be given by (3.8).

**Step 2 :**
We will show that the choice of multipole coefficients as defined in (3.8) verify the minimum of the quantity (3.1).

For this, we suppose that the quantity (3.1) is a function of variables $x_m^{\sigma 1}$, $y_m^{\sigma 1}$, $x_m^{(3-\sigma)2}$ and $y_m^{(3-\sigma)2}$

where:

$$a_m^{\sigma 1} = x_m^{\sigma 1} + i \cdot y_m^{\sigma 1} \quad \text{and} \quad a_m^{(3-\sigma)2} = x_m^{(3-\sigma)2} + i \cdot y_m^{(3-\sigma)2}.$$

It is clear that the optimal choice defined by (3.8), makes the gradient of the quantity (3.1) equal to zero. Moreover, our choice will verify the minimum of the quantity (3.1) if the Hessien of (3.1) is a semi defined positive matrix. But since the Hessien is a symmetric matrix, then it is sufficient that the main determinants are strictly positives.

After some manipulation, we obtain :

$$\left| H \left[ \int_{r_p=A} \|G_1\|_{L_2(\partial D)}^2 \cdot ds_p \right] \right|$$

$$= \left( 16 \, \alpha_m^{\sigma 1} \cdot A_m^{\sigma 1} \cdot \alpha_m^{(3-\sigma)2} \cdot A_m^{(3-\sigma)2} \right) \cdot [\triangle_m^\sigma]$$

$$+ 16 \left( Re^4 \left( \beta_m^{\sigma 1} \cdot B_m^{\sigma 1} \right) + Im^4 \left( \beta_m^{\sigma 1} \cdot B_m^{\sigma 1} \right) \right) > 0$$

$$\left| H_{11} \left[ \int_{r_p=A} \|G_1\|_{L_2(\partial D)}^2 \cdot ds_p \right] \right| = \left( 8 \, \alpha_m^{(3-\sigma)2} \cdot A_m^{(3-\sigma)2} \right) \cdot [\triangle_m^\sigma] > 0$$

$$\left| H_{22} \left[ \int_{r_p=A} \|G_1\|_{L_2(\partial D)}^2 \cdot ds_p \right] \right| = \left( 8 \, \alpha_m^{(3-\sigma)2} \cdot A_m^{(3-\sigma)2} \right) \cdot [\triangle_m^\sigma] > 0$$

and

$$\left| H_{33}\left[ \int_{r_p=A} \|G_1\|_{L_2(\partial D)}^2 \ . \ ds_p \right] \right| = \left( 8 \ \alpha_m^{\sigma 1} \ . \ A_m^{\sigma 1} \right).\left[ \triangle_m^\sigma \right] > 0$$

Hence the optimal choice as defined in (3.8) satisfies the minimum of the quantity (3.1)

This completes the proof of the theorem.

### 3.1.4   Theorem

If the kernel of the modified integral operator $K_1$, namely the function of Green $G_1$ is defined by (2.17) then the quantity

$$\int_{r_p=A} \|G_1\|_{L_2(\partial D)}^2 \ . \ ds_p \qquad \forall \ A \succeq \max \ r_p \ , \qquad q \in \partial D$$

is minimized if the simple and cross multipole coefficients are selected as follows:

$$a_m^{\sigma l} = \frac{\begin{array}{c} \left(\overline{B}_m^{\sigma l}\right)\left[M_{m, \ 1}^{\sigma l}\right] + \left(\overline{\beta}_m^{\sigma l}\right)\left[M_{m, \ 2}^{\sigma l}\right] \\ - \left(A_m^{(3-\sigma)(3-l)}\right)\left[N_{m, \ 1}^{\sigma l}\right] - \left(\alpha_m^{(3-\sigma)(3-l)}\right)\left[N_{m, \ 2}^{\sigma l}\right] \end{array}}{\triangle_m^\sigma}$$

$$(-1)^{\sigma+l} \ .b_m = \frac{\begin{array}{c} \left(B_m^{\sigma l}\right)\left[N_{m, \ 1}^{\sigma l}\right] + \left(\beta_m^{\sigma l}\right)\left[N_{m, \ 2}^{\sigma l}\right] \\ - \left(A_m^{\sigma l}\right)\left[M_{m, \ 1}^{\sigma l}\right] - \left(\alpha_m^{\sigma l}\right)\left[M_{m, \ 2}^{\sigma l}\right] \end{array}}{\triangle_m^\sigma} \qquad (3.10)$$

where :

$$M_{m, \ 1}^{\sigma l} = \left(\triangle_{m, \ A}^\sigma\right) \ . \ \left[B_m^{(3-\sigma)(3-l)}.g_m^{(3-\sigma)(3-l)} - A_m^{(3-\sigma)(3-l)}.h_m^{\sigma l}\right]$$

$$M_{m, \ 2}^{\sigma l} = \left(\triangle_{m, \ \partial D}^\sigma\right) \ . \ \left[\beta_m^{(3-\sigma)(3-l)}.g_m^{(3-\sigma)(3-l)} - \alpha_m^{(3-\sigma)(3-l)}.h_m^{\sigma l}\right]$$

$$N_{m, \ 1}^{\sigma l} = \left(\triangle_{m, \ A}^\sigma\right) \ . \ \left[B_m^{(3-\sigma)(3-l)}.h_m^{(3-\sigma)(3-l)} - A_m^{(3-\sigma)(3-l)}.g_m^{\sigma l}\right]$$

$$N_{m, \ 2}^{\sigma l} = \left(\triangle_{m, \ \partial D}^\sigma\right) \ . \ \left[\beta_m^{(3-\sigma)(3-l)}.h_m^{(3-\sigma)(3-l)} - \alpha_m^{(3-\sigma)(3-l)}.g_m^{\sigma l}\right]$$

$$h_m^{\sigma l} = - \ \left\langle \overline{\beta}_m^{\sigma l} \ . \ \widehat{F}_m^{(3-\sigma)(3-l)} + \alpha_m^{\sigma l} \ . \ \widehat{F}_m^{\sigma l}, \ F_m^{(3-\sigma)(3-l)} \right\rangle_{\partial D}$$

**Proof :**
**Step 1 :**
We have :

$$G_1\left(P,Q\right) = G_0\left(P,Q\right)$$

$$+ \frac{i}{4\mu K^2} \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} \left[ \begin{array}{c} a_m^{\sigma l}.F_m^{\sigma l}(P) \otimes F_m^{\sigma l}(Q) \\ + (-1)^{\sigma+l}.b_m.F_m^{\sigma l}(P) \otimes F_m^{(3-\sigma)(3-l)}(Q) \end{array} \right]$$

$$= \frac{i}{4\mu K^2} \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} \left[ \begin{array}{c} F_m^{\sigma l}(P) \otimes \widehat{F}_m^{\sigma\varrho}(Q) \\ + a_m^{\sigma l}.F_m^{\sigma l}(P) \otimes F_m^{\sigma l}(Q) \\ + (-1)^{\sigma+l}.b_m.F_m^{\sigma l}(P) \otimes F_m^{(3-\sigma)(3-l)}(Q) \end{array} \right] \qquad (3.11)$$

We pose :

$$f_m^{\sigma l}(Q) = \left[ \widehat{F}_m^{\sigma\varrho}(Q) + a_m^{\sigma l}.F_m^{\sigma l}(Q) + (-1)^{\sigma+l}.b_m.F_m^{(3-\sigma)(3-l)}(Q) \right] \qquad (3.12)$$

So the modified Green's function is written in the form :

$$G_1\left(P,Q\right) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} \left[ F_m^{\sigma l}(P) \otimes f_m^{\sigma\varrho}(Q) \right] \qquad (3.13)$$

hence :

$$\int_{r_p=A} \|G_1\|_{L_2(\partial D)}^2 \ . \ ds_p = \int_{r_p=A} \int_{\partial D} G_1\left(P,q\right) \ : \ \overline{G}_1\left(q,P\right) \ . \ ds_p.ds_q$$

$$= \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \sum_{\ell=1}^{2} \sum_{n=0}^{\infty} \sum_{\nu=1}^{2} \sum_{k=1}^{2} \int_{r_p=A} F_m^{\sigma l}(P).\overline{F}_n^{\nu k}(P).ds_p \ . \ \int_{\partial D} f_m^{\sigma l}(q).\overline{f}_n^{\nu k}(q).ds_q$$

Using the inner product relations of the functions $\left\{ F_m^{\sigma l} \right\}_{m=0:\infty}^{\sigma,l=1:2}$ on the circle of radius $A$, we obtain :

$$\int_{r_p=A} \|G_1\|_{L_2(\partial D)}^2 \ . \ ds_p$$

$$= \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \left[ \begin{array}{c} \|F_m^{\sigma 1}\|_A^2 \ . \left\langle f_m^{\sigma 1} \ , \ f_m^{\sigma 1} \right\rangle_{\partial D} \\ + \ \left\langle F_m^{\sigma 1} \ , \ F_m^{\sigma 2} \right\rangle_A \ . \left\langle f_m^{\sigma 1} \ , \ f_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ + \ \left\langle F_m^{\sigma 2} \ , \ F_m^{\sigma 1} \right\rangle_A \ . \left\langle f_m^{(3-\sigma)2} \ , \ f_m^{\sigma 1} \right\rangle_{\partial D} \\ + \ \|F_m^{\sigma 2}\|_A^2 \ . \left\langle f_m^{(3-\sigma)2} \ , \ f_m^{(3-\sigma)2} \right\rangle_{\partial D} \end{array} \right] \qquad (3.14)$$

calculating the expressions $\left\langle f_m^{\sigma 1} \ , \ f_m^{\sigma 1} \right\rangle_{\partial D}$, $\left\langle f_m^{\sigma 1} \ , \ f_m^{(3-\sigma)2} \right\rangle_{\partial D}$ , $\left\langle f_m^{(3-\sigma)2} \ , \ f_m^{\sigma 1} \right\rangle_{\partial D}$ and $\left\langle f_m^{(3-\sigma)2} \ , \ f_m^{(3-\sigma)2} \right\rangle_{\partial D}$, and substituting in (3.14) we obtain :

$$\int_{r_p=A} \|G_1\|_{L_2(\partial D)}^2 \ . \ ds_p$$

$$= \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \alpha_m^{\sigma 1} \ . \ \begin{pmatrix} \left\|\widehat{F}_m^{\sigma 1}\right\|_{\partial D}^2 + \overline{a}_m^{\sigma 1} \left\langle \widehat{F}_m^{\sigma 1} \ , \ F_m^{\sigma 1} \right\rangle_{\partial D} \\ - (-1)^\sigma \overline{b}_m \left\langle \widehat{F}_m^{\sigma 1} \ , \ F_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ + a_m^{\sigma 1} \left\langle F_m^{\sigma 1} \ , \ \widehat{F}_m^{\sigma 1} \right\rangle_{\partial D} + a_m^{\sigma 1} . \ \overline{a}_m^{\sigma 1} . \ A_m^{\sigma 1} \\ - (-1)^\sigma a_m^{\sigma 1} \ \overline{b}_m \ B_m^{\sigma 1} - (-1)^\sigma b_m \left\langle F_m^{(3-\sigma)2} \ , \ \widehat{F}_m^{\sigma 1} \right\rangle_{\partial D} \\ - (-1)^\sigma \overline{a}_m^{\sigma 1} \ b_m \ \overline{B}_m^{\sigma 1} + b_m \ \overline{b}_m \ A_m^{(3-\sigma)2} \end{pmatrix}$$

$$+ \beta_m^{\sigma 1} \ . \ \begin{pmatrix} \left\langle \widehat{F}_m^{\sigma 1} \ , \ \widehat{F}_m^{(3-\sigma)2} \right\rangle_{\partial D} + \overline{a}_m^{(3-\sigma)2} \left\langle \widehat{F}_m^{\sigma 1} \ , \ F_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ - (-1)^\sigma \overline{b}_m \left\langle \widehat{F}_m^{\sigma 1} \ , \ F_m^{\sigma 1} \right\rangle_{\partial D} \\ + a_m^{\sigma 1} \left\langle F_m^{\sigma 1} \ , \ \widehat{F}_m^{(3-\sigma)2} \right\rangle_{\partial D} + a_m^{\sigma 1} . \ \overline{a}_m^{(3-\sigma)2} . \ B_m^{\sigma 1} \\ - (-1)^\sigma a_m^{\sigma 1} \ \overline{b}_m \ A_m^{\sigma 1} - (-1)^\sigma b_m \left\langle F_m^{(3-\sigma)2} \ , \ \widehat{F}_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ - (-1)^\sigma \overline{a}_m^{(3-\sigma)2} \ b_m \ A_m^{(3-\sigma)2} + b_m \ \overline{b}_m \ \overline{B}_m^{\sigma 1} \end{pmatrix}$$

$$+ \overline{\beta}_m^{\sigma 1} \ . \ \begin{pmatrix} \left\langle \widehat{F}_m^{(3-\sigma)2} \ , \ \widehat{F}_m^{\sigma 1} \right\rangle_{\partial D} + a_m^{(3-\sigma)2} \left\langle F_m^{(3-\sigma)2} \ , \ \widehat{F}_m^{\sigma 1} \right\rangle_{\partial D} \\ - (-1)^\sigma \overline{b}_m \left\langle \widehat{F}_m^{(3-\sigma)2} \ , \ F_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ + \overline{a}_m^{\sigma 1} \left\langle \widehat{F}_m^{(3-\sigma)2} \ , \ F_m^{\sigma 1} \right\rangle_{\partial D} + a_m^{(3-\sigma)2} . \ \overline{a}_m^{\sigma 1} . \ \overline{B}_m^{\sigma 1} \\ - (-1)^\sigma a_m^{(3-\sigma)2} \ \overline{b}_m \ A_m^{(3-\sigma)2} - (-1)^\sigma b_m \left\langle F_m^{\sigma 1} \ , \ \widehat{F}_m^{\sigma 1} \right\rangle_{\partial D} \\ - (-1)^\sigma \overline{a}_m^{\sigma 1} \ b_m \ A_m^{\sigma 1} + b_m \ \overline{b}_m \ B_m^{\sigma 1} \end{pmatrix}$$

$$+ \alpha_m^{(3-\sigma)2} \ . \ \begin{pmatrix} \left\|\widehat{F}_m^{(3-\sigma)2}\right\|_{\partial D}^2 \\ + \overline{a}_m^{(3-\sigma)2} \left\langle \widehat{F}_m^{(3-\sigma)2} \ , \ F_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ - (-1)^\sigma \overline{b}_m \left\langle \widehat{F}_m^{(3-\sigma)2} \ , \ F_m^{\sigma 1} \right\rangle_{\partial D} \\ + a_m^{(3-\sigma)2} \left\langle F_m^{(3-\sigma)2} \ , \ \widehat{F}_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ + a_m^{(3-\sigma)2} . \ \overline{a}_m^{(3-\sigma)2} . \ A_m^{(3-\sigma)2} \\ - (-1)^\sigma a_m^{(3-\sigma)2} \ \overline{b}_m \ \overline{B}_m^{\sigma 1} \\ - (-1)^\sigma b_m \left\langle F_m^{\sigma 1} \ , \ \widehat{F}_m^{(3-\sigma)2} \right\rangle_{\partial D} \\ - (-1)^\sigma \overline{a}_m^{(3-\sigma)2} \ b_m \ B_m^{\sigma 1} + b_m \ \overline{b}_m \ A_m^{\sigma 1} \end{pmatrix} \qquad (3.15)$$

This is a standard problem of minimization , where the necessary condition for the existence of a minimum is the vanishing of the gradient.

so, if we cancel the gradient with respect to the coefficients $\overline{a}_m^{\sigma 1}$ and $\overline{a}_m^{(3-\sigma)2}$ and $b_m$, we obtain the following relations :

$$\left(\alpha_m^{\sigma 1} . A_m^{\sigma 1}\right) . a_m^{\sigma 1} + \left(\overline{\beta}_m^{\sigma 1} . \overline{B}_m^{\sigma 1}\right) . a_m^{(3-\sigma)2} - (-1)^\sigma \left(\alpha_m^{\sigma 1}\overline{B}_m^{\sigma 1} + A_m^{\sigma 1}\overline{\beta}_m^{\sigma 1}\right) b_m = g_m^{\sigma 1}$$

$$(\beta_m^{\sigma 1} . B_m^{\sigma 1}) . a_m^{\sigma 1} + \left(\alpha_m^{(3-\sigma)2} . A_m^{(3-\sigma)2}\right) . a_m^{(3-\sigma)2}$$
$$- (-1)^\sigma \left(\alpha_m^{(3-\sigma)2}B_m^{\sigma 1} + A_m^{(3-\sigma)2}\beta_m^{\sigma 1}\right) b_m = g_m^{(3-\sigma)2}$$

$$- (-1)^\sigma \left(\begin{matrix} B_m^{\sigma 1}\alpha_m^{\sigma 1} \\ + A_m^{\sigma 1}\beta_m^{\sigma 1} \end{matrix}\right) . a_m^{\sigma 1} - (-1)^\sigma \left(\begin{matrix} A_m^{(3-\sigma)2}\ \overline{\beta}_m^{\sigma 1} \\ + \overline{B}_m^{\sigma 1}\alpha_m^{(3-\sigma)2} \end{matrix}\right) . a_m^{(3-\sigma)2}$$
$$+ \left(\begin{matrix} A_m^{(3-\sigma)2}\alpha_m^{\sigma 1} + \overline{B}_m^{\sigma 1}\beta_m^{\sigma 1} \\ + B_m^{\sigma 1}\ \overline{\beta}_m^{\sigma 1} + A_m^{\sigma 1}\alpha_m^{(3-\sigma)2} \end{matrix}\right) b_m = - (-1)^\sigma \ \left(h_m^{\sigma 1} + h_m^{(3-\sigma)2}\right)$$

According to the Schwartz inequality and the linear independence of the functions $\left\{F_m^{\sigma l}\right\}_{m\ =\ 0\ :\ \infty}^{\sigma, l\ =\ 1\ :\ 2}$, the latter system admits the following non-zero determinant

$$\triangle_m^\sigma = \left[\begin{matrix} A_m^{\sigma 1} . \alpha_m^{(3-\sigma)2} \\ + \alpha_m^{\sigma 1} . A_m^{(3-\sigma)2} \\ - \left(\begin{matrix} B_m^{\sigma 1} . \overline{\beta}_m^{\sigma l} \\ + \beta_m^{\sigma 1}\ \overline{B}_m^{\sigma l} \end{matrix}\right) \end{matrix}\right] . \left(\begin{matrix} A_m^{\sigma 1} . A_m^{(3-\sigma)2} \\ - |B_m^{\sigma 1}|^2 \end{matrix}\right) . \left(\begin{matrix} \alpha_m^{\sigma 1} . \alpha_m^{(3-\sigma)2} \\ - |\beta_m^{\sigma 1}|^2 \end{matrix}\right)$$

And therefore, the solutions of the latter system will be given by (3.10).

**Step 2 :**

We will show that the choice of multipole coefficients as defined in (3.10) verify the minimum of the quantity (3.14).

For this, we suppose that the quantity (3.14) is a function of variables $x_m^{\sigma 1}$, $y_m^{\sigma 1}, x_m$ , $x_m^{(3-\sigma)2}$ , $y_m^{(3-\sigma)2}$ and $y_m$

where :

$$a_m^{\sigma 1} = x_m^{\sigma 1} + i . y_m^{\sigma 1}, \quad a_m^{(3-\sigma)2} = x_m^{(3-\sigma)2} + i . y_m^{(3-\sigma)2} \quad \text{and} \quad b_m = x_m + i . y_m.$$

It is clear that the optimal choice defined by (3.10), makes the gradient of the quantity (3.14) equal to zero, and in the same way of the demonstration of theorem 3.1.3, we proof that the main determinants of the Hessien of the quantity (3.14) are strictly positives

Hence the optimal choice as defined in (3.10) verifies the minimum of the quantity (3.14)

This completes the proof of the theorem.

## 3.2   Case of a circular boundaries

In our article published recently [20], we applied the results found in this paper for the case of circular boundaries. When the boundary is a circle of radius $a$, the simple and cross multipole coefficients are given by the following relatively simple expressions :

$$a_m^{11} = \frac{(\bar{c}_m.\hat{c}_m - \hat{a}_m^1.a_m^2)}{\Delta_{m,\,a}} \tag{3.16}$$

$$a_m^{21} = \frac{(\bar{c}_m.\hat{c}_m - \hat{a}_m^1.a_m^2)}{\Delta_{m,\,a}} \tag{3.17}$$

$$a_m^{22} = \frac{\left(c_m.\hat{d}_m - a_m^1.\hat{a}_m^2\right)}{\Delta_{m,\,a}} \tag{3.18}$$

$$a_m^{12} = \frac{\left(c_m.\hat{d}_m - a_m^1.\hat{a}_m^2\right)}{\Delta_{m,\,a}} \tag{3.19}$$

$$b_m = \frac{\left(a_m^2.\hat{d}_m - \hat{a}_m^2.\bar{c}_m\right)}{\Delta_m} \tag{3.20}$$

$$b_m = \frac{(\hat{c}_m.a_m^1 -\ c_m.\hat{a}_m^1)}{\Delta_{m,\,a}} \tag{3.21}$$

where [see **7**] :

$$a_m^1 = 2\pi a k^2 \left[\left|H_m^{'}(ka)\right|^2 + \frac{m^2}{(ka)^2}\left|H_m(ka)\right|^2\right]$$

$$a_m^2 = 2\pi a K^2 \left[\left|H_m^{'}(Ka)\right|^2 + \frac{m^2}{(Ka)^2}\left|H_m(Ka)\right|^2\right]$$

$$c_m = 2\pi a k K \left[\frac{m}{Ka}H_m^{'}(ka).\overline{H}_m(Ka) + \frac{m}{ka}H_m(ka).\overline{H}_m^{'}(Ka)\right]$$

$$\hat{a}_m^1 = 2\pi a k^2 \left[J_m^{'}(ka).\overline{H}_m^{'}(ka) + \frac{m^2}{(ka)^2}J_m(ka).\overline{H}_m(ka)\right]$$

$$\widehat{a}_m^2 = 2\pi a K^2 \left[ J_m'(Ka).\overline{H}_m'(Ka) + \frac{m^2}{(Ka)^2} J_m(Ka).\overline{H}_m(Ka) \right]$$

$$\widehat{c}_m = 2\pi a k K \left[ \frac{m}{Ka} J_m'(ka).\overline{H}_m(Ka) + \frac{m}{ka} J_m(ka).\overline{H}_m'(Ka) \right]$$

$$\widehat{d}_m = 2\pi a k K \left[ \frac{m}{ka} J_m'(Ka).\overline{H}_m(ka) + \frac{m}{Ka} J_m(Ka).\overline{H}_m'(Ka) \right]$$

$J_m(.)$ is the Bessel's function of order $m$ and type 1

### 3.2.1 Lemma

If the boundary of the domain $\partial D$ is a circle of radius $a$, then the two expressions (3.20) and (3.21) found for the cross multipole coefficients $b_m$ are equal.

**Proof.**

To simplify the expressions, introduce the following ratings:

$$x' = H_m'(ka) \quad , \quad x = \frac{m}{ka} H_m(ka)$$

$$y' = H_m'(Ka) \quad , \quad y = \frac{m}{Ka} H_m(Ka)$$

$$X_1 = k^2 \left[ 2\mu H_m''(ka) - \lambda H_m(ka) \right]$$

$$X_2 = \frac{2\mu m}{a} \left[ k H_m'(ka) - \frac{H_m(ka)}{a} \right]$$

$$Y_1 = \mu K^2 \left[ 2 H_m''(Ka) + H_m(Ka) \right]$$

$$Y_2 = \frac{2\mu m}{a} \left[ K H_m'(Ka) - \frac{H_m(Ka)}{a} \right]$$

then, the expressions $a_m^1$, $a_m^2$, $c_m$, $\widehat{a}_m^1$, $\widehat{a}_m^2$, $\widehat{c}_m$, $\widehat{d}_m$, $\alpha_m^1$, $\alpha_m^2$ and $\beta_m$ are written as:

$$a_m^1 = 2\pi a k^2 \left[ \left| x' \right|^2 + |x|^2 \right] \quad \text{and} \quad a_m^2 = 2\pi a K^2 \left[ \left| y' \right|^2 + |y|^2 \right]$$

$$c_m = 2\pi a k K \left[ x'.\overline{y} + x.\overline{y}' \right]$$

$$\widehat{a}_m^1 = 2\pi a k^2 \left[ \widehat{x}'.\overline{x}' + \widehat{x}.\overline{x} \right] \quad \text{and} \quad \widehat{a}_m^2 = 2\pi a K^2 \left[ \widehat{y}'.\overline{y}' + \widehat{y}.\overline{y} \right]$$

$$\widehat{c}_m = 2\pi akK \left[ \widehat{x}^{'} . \overline{y} + \widehat{x} . \overline{y}^{'} \right]$$

$$\widehat{d}_m = 2\pi akK \left[ \overline{x}^{'} . \widehat{y} + \overline{x} . \widehat{y}^{'} \right]$$

$$\alpha_m^1 = 2\pi a \left[ |X_1|^2 + |X_2|^2 \right] \quad \text{and} \quad \alpha_m^2 = 2\pi a \left[ |Y_1|^2 + |Y_2|^2 \right]$$

$$\beta_m = 2\pi a \left[ X_1 . \overline{Y}_2 + X_2 . \overline{Y}_1 \right]$$

where the expressions of $x$, $x^{'}$, $y$, $y^{'}$, $X_1$, $X_2$, $Y_1$ and $Y_2$ are the same for $x$, $x^{'}$, $y$, $y^{'}$, $X_1$, $X_2$, $Y_1$ and $Y_2$ but with the function of Hankel $H_m$ replaced by the function of Bessel $J_m$.

inserting these expressions in (3.20) and (3.21) we obtain :

$$\Delta_{m,\,a} = a_m^1 . a_m^2 - |c_m|^2 = (2\pi akK)^2 \left| x^{'} . y^{'} - x.y \right|^2$$

$$a_m^2 . \widehat{d}_m - \widehat{a}_m^2 . \overline{c}_m = - (2\pi a)^2 kK^3 \overline{(x^{'} . y^{'} - x.y)} \left( y.\widehat{y}^{'} - y^{'} . \widehat{y} \right) \qquad (3.22)$$

$$a_m^1 . \widehat{c}_m - \widehat{a}_m^1 . c_m = - (2\pi a)^2 Kk^3 \overline{(x^{'} . y^{'} - x.y)} \left( x.\widehat{x}^{'} - x^{'} . \widehat{x} \right) \qquad (3.23)$$

Using the Wronskien [9] of the functions of Bessel and Hankel, we obtain :

$$y.\widehat{y}^{'} - y^{'} . \widehat{y} = - \frac{im\pi}{(Ka)^2} \quad \text{and} \quad x.\widehat{x}^{'} - x^{'} . \widehat{x} = - \frac{im\pi}{(ka)^2}$$

inserting these expressions in (3.22) and (3.23) we obtain :

$$a_m^2 . \widehat{d}_m - \widehat{a}_m^2 . \overline{c}_m = -4im\pi^3 kK \overline{(x^{'} . y^{'} - x.y)}$$

$$a_m^1 . \widehat{c}_m - \widehat{a}_m^1 . c_m = -4im\pi^3 kK \overline{(x^{'} . y^{'} - x.y)}.$$

So the expression $(3.22)\,, (3.23)$ are equal. This complete the Proof.

The main result in the case of the circle, is the obtening of the exact Green's function for the Dirichlet problem :

$$G_1^D (P,\,Q) = G_{ex}^D (P,\,Q)$$

which makes the norm of the modified integral operator equal to zero :

$$G_1^D (P,Q) = G_{ex}^D (P,Q) \implies \|K_1^P\| = 0.$$

This result delete the obligation to verify the large condition (2.20), and ensures the convergence of the method of successive approximation for all values of the frequency of the waves.

Also, when the boundary is a slightly distorted circle defined in polar coordinates by :

$$r = a + \varepsilon\varphi(\theta), \qquad 0 \le \theta \le 2\pi$$

where $a$ is the radius of the circle not distorted and $\varphi$ and $\frac{\partial\varphi}{\partial\theta}$ are two bounded functions.

Then, the expressions of the multipole coefficients is relative to that found in the case of circle, and we have :

$$G_1(p,\ q) = G_1^{circle}(p,q) + O(\varepsilon) \quad \implies \quad \|K_1\| = O(\varepsilon).$$

# 4 Conclusion

In this paper, we have presented a new criterion of optimization of the modified fundamental solution for two dimensional elastic waves. We based on the work of E.ARGYROPOULOS and K.KIRIAKI [19]. In the first part of this paper we have presented the formulation of the boundary problem using the method of integral representations based on the single and double layer potential technique. The second part of our work consider the general case, and we have calculated the explicit expressions of the multipole coefficients for the following cases :

1- Modification of the fundamental solution with simple multipole coefficients

2- Modification of the fundamental solution with simple and cross multipole coefficients

In the third Part, we present some applications of these results for the special case where the boundary is a circle or a slightly distorted circle.

# 5 Open problems

The modified Green's function techniques which use the multipole coefficients has many open problems which deserve to be treated. In this way we can mentioned the following:

1- verification of the large condition (2.20) for the general case where the boundary takes any form.

2- Consider the cases of other simple geometric forms, such as square, rectangle, triangle, ellipse, ...

3- Treat the same subject by changing the criterion of optimality and consider for example the minimization of the condition number of the integral operator associated with our boundary problem, (in the case of three dimensions, see [12] for acoustic waves and [13] for elastic waves).

4- Establish the numerical applications for the results obtained in this paper (some numerical applications given in [14] and [15]).

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