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On The Φ Class Operators

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Abstract

In this paper we characterize Hermitian operators defined on Hilbert space. Using this result we establish several new characterizations to (Φ) ***class operators. Further, we apply these results to investigate on the relation between this class and other usual classes of operators. Some applications are also given.***

Keywords: *Hermitian operator, Hyponormal operators,* (Φ) *class.*

1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle , \rangle$ and let $B(\mathcal{H})$ be the algebra of all bounded operators acting on $\mathcal{H}$. We denote by (Φ) the class of operators satisfying the following equality

$$T^*[T^*, T]T = [T^*, T], \quad [T^*, T] = T^*T - TT^*.$$

This class was introduced by F. Ming[2].

It is clear that the class (Φ) contains the class of normal operators. For any operators $T \in B(\mathcal{H})$ set, as usual, $|T| = (T^*T)^{\frac{1}{2}}$ and $[T^*, T] = T^*T - TT^* = |T|^2 - |T^*|^2$ (the self commutant of T), and consider the following standard definitions: T is hyponormal if $|T^*|^2 \leq |T|^2$ (i.e., if $[T^*, T]$ is nonnegative or, equivalently, if $\| T^*x \| \leq \| Tx \|$ for every $x \in \mathcal{H}$), co-hyponormal, if T^* is hyponormal, normal if $T^*T = TT^*$. For the related topics and basic properties of class (Φ) operators, (see [1]).

Let $T \in B(\mathcal{H})$, in the following we will denote the kernel, the range, the spectrum, the convex hull of the spectrum and the numerical range of the operator T by $\ker T,\ ran\ T,\ \sigma(T),\ cov\sigma(T)$ and $W(T)$ respectively. The present paper is organized as follows. In Theorem 2.1 in section two we give a characterization of hermitian operators. Hence, by using this result, in section three we will give a characterization of the (Φ) class operators and we will investigate on the relation between this class and other usual classes of operators. Some applications are also given.

2 Preliminaries

In the next, we need to prove the following famous theorem concerning the decomposition of Hermitian operators.

Theorem 2.1 *If $D \in B(\mathcal{H})$ is a self-adjoint operator and $D = U|D|$ its polar decomposition, then $D = D^+ \oplus D^-$, where D^+ is a positif operator and D^- is a negative operator. Moreover, U is Hermitian, commutes with D and verify $U^2 = I$.*

To prove theorem 2.1 we need to show the following proposition.

Proposition 2.2 *Let $D \in B(\mathcal{H})$ be a Hermitian operator. If $\mathcal{H}^+ = \{x \in \mathcal{H} : (Dx, x) \geq 0\}$ and $\mathcal{H}^- = \{x \in \mathcal{H} : (Dx, x) \leq 0\}$ are two subsets of $\mathcal{H}$, then $\mathcal{H}^+$ and $\mathcal{H}^-$ are two closed linear subspaces of $\mathcal{H}$.*

The proof of the proposition follows from the following lemmas.

Lemma 2.3 *Let $D = U|D|$ be the polar decomposition of the Hermitian operator D. If V and S are the restrictions of U and D on $\overline{ran\ D}$ respectively, then V is a Hermitian isometry which commutes with S. Where $\overline{ran\ D}$ is the closure of ran D.*

Proof. Let $\mathcal{H} = \ker D \oplus \overline{ran\ D}$, since D is Hermitian, $\ker D$ reduces D orthogonally and if $x \in \ker D$, we get $Dx = |D|U^*x = 0$, therefore $U^*x \in \ker |U| = \ker U$. Thus

$$U^*(\ker D) = U^*(\ker U) \subseteq \ker D.$$

We know also that $\ker U$ reduces orthogonally U. Thus U and D can be written as follows:

$$U = \begin{bmatrix} 0 & 0 \\ 0 & V \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix},$$

Hence, S has the polar decomposition

$$S = V|S|, \tag{1}$$

where V is an isometry, i.e.,

$$V^*V = I_R, \tag{2}$$

where I_R denote the identity on $\overline{ran\ D}$. Since S is Hermitian, we get $S^2 = V|S|^2V^* = VS^2V^*$. Thus $S^2V = VS^2$ and so V commutes with $|S|$. From the relations (2.1), (2.2) and the commutativity, it follows that $|S| = |S|V^2$, since $|S|$ is injective, then

$$V^2 = I_R. \tag{3}$$

Now, from the injectivity of V^*, (2.1) and (2.2), we get that V is a Hermitian isometry which commutes with $|S|$, and hence commutes with S.

Lemma 2.4 *let $D = U|D|$ be the polar decomposition of the Hermitian operator D and $P = I - U^*U$, then the following statements are satisfied*

1. *P is an orthogonal projection on $\ker D$ and $Q = I - P$ is an orthogonal projection on $\overline{ran\ D}$;*
2. *$Q(\mathcal{H}^+) = \mathcal{H}^+ \cap \overline{ran\ D}$ and $Q(\mathcal{H}^-) = \mathcal{H}^- \cap \overline{ran\ D}$.*

Proof. On $\mathcal{H} = \ker D \oplus \overline{ran\ D}$, it is clear that P is an orthogonal projection on $\ker D$ and so $Q = I - P$ is an orthogonal projection on $\overline{ran\ D}$. We first remark that $Q(\mathcal{H}^+) \subseteq \overline{ran\ D}$, Let $x \in \mathcal{H}^+$, if $y = Qx$, then $y - x \in \ker D$ and so there exists $z \in \ker D$ such that $y = x + z$, consequently

$$(Dx, x) = (Dy, y) \geq 0.$$

Hence $y \in \mathcal{H}^+$. Thus $Q(\mathcal{H}^+) \subseteq \mathcal{H}^+ \cap \overline{ran\ D}$. To prove the reverse inclusion, let $x \in \mathcal{H}^+ \cap \overline{ran\ D}$, then $x \in \mathcal{H}^+$ and $Qx = x$, so $x \in Q(\mathcal{H}^+)$. Hence $Q(\mathcal{H}^+) = \mathcal{H}^+ \cap \overline{ran\ D}$. Analogously, we prove that $Q(\mathcal{H}^-) = \mathcal{H}^- \cap \overline{ran\ D}$.

Lemma 2.5 *$\langle Dx, x\rangle = \langle |D|x, x\rangle$, for all $x \in \mathcal{H}^+$.*

Proof. From lemma 2.2 , we obtain, for all $x \in Q(\mathcal{H}^+)$ that

$$\langle S(x + Vx), (x + Vx)\rangle = 2[\langle Sx, x\rangle + \langle |S|x, x\rangle].$$

Hence by lemma 2.3, $\langle S(x + Vx), (x + Vx)\rangle \geq 0$. Thus, for all $x \in Q(\mathcal{H}^+)$, $x + Vx \in Q(\mathcal{H}^+)$. Since S and $|S|$ are injective, then $x + Vx \neq 0$, consequently if $x \neq 0$, then $Vx = x$ for all $x \in Q(\mathcal{H}^+)$.
Moreover, for all $x \in \mathcal{H}^+$, let $x = z + Qz$ where $z \in \ker D$, it follows that

$$\begin{aligned}\langle Dx, x\rangle = \langle DQx, Qx\rangle &= \langle SQx, Qx\rangle \\ &= \langle |S|Qx, Qx\rangle \\ &= \langle |D|x, x\rangle.\end{aligned}$$

In the next, we give the proof of the Proposition 2.2.

Proof. Let us prove that $\mathcal{H}^+$ is a linear subspace. We have for all $x \in \mathcal{H}^+$ and all $\lambda \in \mathbb{C}$,

$$\langle D\lambda x, x\rangle = |\lambda|\langle Dx, x\rangle.$$

Thus $\lambda x \in \mathcal{H}^+$. For $x, y \in \mathcal{H}^+$, we obtain

$$\langle D(x+y), (x+y)\rangle = \langle Dx, x\rangle + 2Re\langle Dx, y\rangle + \langle Dy, y\rangle.$$

We distinguish two cases:

1. If $Re\langle Dx, y\rangle \geq 0$, then $\langle D(x+y), (x+y)\rangle \geq 0$. Thus $x + y \in \mathcal{H}^+$.

2. If $Re\langle Dx, y\rangle \leq 0$. Apply lemma 2.4 and the Schwartz's inequality, we obtain

$$|Re\langle Dx, y\rangle|^2 \leq |\langle Dx, y\rangle|^2 \leq \langle |D|x, x)(|D|y, y\rangle, \text{ for all } x, y \in \mathcal{H}^+, \quad (4)$$

or

$$\langle Dx, x\rangle\langle Dy, y\rangle \leq \frac{1}{2}\big(\langle Dx, x\rangle^2 + \langle Dy, y\rangle^2\big). \quad (5)$$

Hence from 2.4 and 2.4, it follows that

$$|Re\langle Dx, y\rangle| \leq \frac{1}{2}\big(\langle Dx, x\rangle + \langle Dy, y\rangle\big). \quad (6)$$

Thus

$$-2Re\langle Dx, y\rangle \leq \langle Dx, x\rangle + \langle Dy, y\rangle.$$

Consequently, $x + y \in \mathcal{H}^+$ and $\mathcal{H}^+$ is a linear subspace of $\mathcal{H}$.

Analogously, we prove that $\mathcal{H}^-$ is a linear subspace of $\mathcal{H}$, (take $(-D)$). Now, we have to prove that $\mathcal{H}^+$ and $\mathcal{H}^-$ are closed for the topology of $\mathcal{H}$, for this, we have to show the following lemma:

Lemma 2.6 *The function $f : \mathcal{H} \to \mathbb{R}$ defined by $f(x) = \langle Dx, x\rangle$ is strongly continuous on $\mathcal{H}$.*

Proof. Note that, since D is bounded that

$$|f(x)| = |\langle Dx, x\rangle| \leq \| D \| \| x \|^2 \text{ for all } x \in \mathcal{H}. \quad (7)$$

Hence f is strongly continued at $x = 0$, let $\{y_n\}$ be a sequence in $\mathcal{H}$ defined by $y_n = x_n - x$, where $\{x_n\}$ is a sequence in $\mathcal{H}$ which converges strongly to x, by simple computation, we obtain

$$|f(x_n) - f(x)| = |f(y_n) + 2Re\langle Dx_n, x\rangle - f(x)| \leq |f(y_n)| + 2|Re\langle Dx_n, x\rangle - f(x)|. \quad (8)$$

By letting $n \to \infty$, $\langle Dx_n, x\rangle$ converges to $\langle Dx, x\rangle = f(x)$, and so $Re\langle Dx_n, x\rangle$ converges to $f(x)$. Thus by 2.7 and 2.8 it follows that f is strongly continuous on $\mathcal{H}$. Since $\mathcal{H}^+ = f^{-1}([0,\infty))$ and $\mathcal{H}^- = f^{-1}((-\infty, 0])$, by lemma 2.6, we deduces that $\mathcal{H}^+$ and $\mathcal{H}^-$ are closed linear subspaces in $\mathcal{H}$.

In the next, we give the proof of the Theorem 2.1.

Proof. To establish that $\overline{ran\ D} = Q(\mathcal{H}^+) \oplus Q(\mathcal{H}^-)$, we need to claim that $\mathcal{H}^+ \cap \mathcal{H}^- = \ker D$. Let us suppose the contrary, then there exists a vector z such that $z \in \mathcal{H}^+ \cap \mathcal{H}^-$ and $z \notin \ker D$. Set $z = a + Qz$, where $a \in \ker D$ and $Qz \in \overline{ran\ D}$. Since $Qz \neq 0$ and $\langle Dz, z\rangle = 0$ for all $z \in \mathcal{H}^+ \cap \mathcal{H}^-$, the following equalities

$$\langle Dz, z\rangle = \langle SQz, Qz\rangle = \langle |S|Qz, Qz\rangle = \| \, |S|^{\frac{1}{2}}Qz \, \| = 0, \ \ z \in \mathcal{H}^+$$

implies that $|S|Qz = 0$ and so $Qz = 0$, this contradicts the assumptions. Hence $\mathcal{H}^+ \cap \mathcal{H}^- \subseteq \ker D$. The reverse inclusion is trivial. Therefore $\mathcal{H}^+ \cap \mathcal{H}^- = \ker D$.

Since $\langle Dx, x\rangle$ is real for all $x \in \mathcal{H}$, then it is either positive or negative, this implies that $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$ and so

$$Q(\mathcal{H}) = \overline{ran\ D} = Q(\mathcal{H}^+) \cup Q(\mathcal{H}^-).$$

The subspace $Q(\mathcal{H})$ is generated by $Q(\mathcal{H}^+)$ and $Q(\mathcal{H}^+)$ and so

$$\overline{ran\ D} = Q(\mathcal{H}^+) + Q(\mathcal{H}^-).$$

Now, from lemma 2.5 and the result above, it follows that $Q(\mathcal{H}^+) \cap Q(\mathcal{H}^-) = \{0\}$. In the other hand, let z be a nonzero vector in $\overline{ran\ D}$, if $z \in (Q(\mathcal{H}^+))^\perp$ and $z \notin Q(\mathcal{H}^-)$, then $\langle Dz, z\rangle > 0$ or equivalently $z \in (Q(\mathcal{H}^+))^\perp \cap Q(\mathcal{H}^+) = \{0\}$, this is absurd. Hence $(Q(\mathcal{H}^+))^\perp \subseteq Q(\mathcal{H}^-)$. Therefore $\overline{ran\ U^*} = Q(\mathcal{H}^+) \oplus Q(\mathcal{H}^-)$. Consequently, $\mathcal{H}$ can be represented as follows

$$\mathcal{H} = \ker D \oplus Q(\mathcal{H}^+) \oplus Q(\mathcal{H}^-).$$

Since S is the restriction of D on $\overline{ran\ D}$, we have $Sx = |S|x$ for all $x \in Q(\mathcal{H}^+)$. Hence

$$\langle S(Sx), Sx\rangle = \langle S|S|x, |S|x\rangle = \langle |S|Sx, |S|x\rangle = \| \, |S|^{\frac{3}{2}}x \, \|^2 \quad \text{for all } x \in Q(\mathcal{H}^+).$$

Thus, $Q(\mathcal{H}^+)$ reduces orthogonally S, this restriction denoted by D^+ is positive. Analogously, the restriction $(-S)$ to $Q(\mathcal{H}^-)$ denoted by D^- is negative. Finally D can be represented with respect to the decomposition $\mathcal{H} = \ker D \oplus Q(\mathcal{H}^+) \oplus Q(\mathcal{H}^-)$ as

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & D^+ & 0 \\ 0 & 0 & D^- \end{pmatrix},$$

where D^+ is a positive operator and D^- is a negative operator. Hence the proof is complete.

Remark 2.7 *If D is injective, then $\mathcal{H}^+ \cap \mathcal{H}^- = \{0\}$, where $Q(\mathcal{H}^+) = \mathcal{H}^+$ and $Q(\mathcal{H}^-) = \mathcal{H}^-$.*

3 Main Results

In this section we give a characterization of a subset of the (Φ) class operators when we suppose that $D = [T^*, T]$ is injective.

Proposition 3.1 *If $T \in \Phi$ and D is injective, then $T = A \oplus B$, where A is hyponormal completely nonnormal and B is co-hyponormal completely nonnormal.*

Proof. First let us prove that $\mathcal{H}^+$ and $\mathcal{H}^-$ are invariant for T. In fact, in view of the remark 2.7 and the following equality

$$\langle Dx, x\rangle = \langle T^*DTx, x\rangle = \langle DTx, Tx\rangle \text{ for all } x \in \mathcal{H},$$

we obtain either $Tx \in \mathcal{H}^+$ if $x \in \mathcal{H}^+$ or $Tx \in \mathcal{H}^-$ if $x \in \mathcal{H}^-$. In the next, we show that the restriction $A = T|_{\mathcal{H}^+}$ is hyponormal (resp. $B = T|_{\mathcal{H}^-}$ is co-hyponormal). Let P be the orthogonal projection on $\mathcal{H}^+$, since $\mathcal{H}^+$ is reduisant for T, then $[A^*, A] = PT^*TP - TPPT^* = P[T^*, T]P$. Hence A is hyponormal and B is co-hyponormal. In the next, we give an illustrative example for the above result.

Example 1. *Let $\{e_n : n \in \mathbb{Z}^2\}$ be an orthonormal system of a complex Hilbert space $\mathcal{H}$ and T be the bilateral shift with weights $(\gamma_n)_{n\in\mathbb{Z}}$ where $|\gamma_n| \neq 0$* for all $n \in \mathbb{Z}$ *and defined by $Te_n = \gamma_n e_n$* for all $n \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers.

By direct computation, we obtain $T^*e_n = \overline{\gamma_{n-1}}e_{n-1}$ for all $n \in \mathbb{Z}$, and

$$TT^*x = \sum_{n=-\infty}^{n=\infty} |\gamma_n|^2 x_n e_n, \;\; T^*Tx = \sum_{n=-\infty}^{n=\infty} |\gamma_{n-1}|^2 x_n e_n$$

Hence

$$[T^*, T]x = \sum_{n=-\infty}^{n=\infty} (|\gamma_n|^2 - |\gamma_{n-1}|^2) x_n e_n, \tag{9}$$

and

$$T^*[T^*, T]Tx = \sum_{n=-\infty}^{n=\infty} (|\gamma_{n+1}|^2 - |\gamma_n|^2)(|\gamma_n|^2) x_n e_n, \tag{10}$$

for all $x = \sum_{n=-\infty}^{n=\infty} x_n$ in $\mathcal{H}$ (i.e., $\sum_{n=-\infty}^{n=\infty} |x_n|^2 < \infty$).

Now $T \in \Phi$ is equivalent to

$$\sum_{n=-\infty}^{n=\infty} [(|\gamma_{n+1}|^2 - |\gamma_n|^2)(|\gamma_n|^2) - |\gamma_n|^2 + |\gamma_{n-1}|^2] x_n e_n = 0, \text{ for all } x \in \mathcal{H}.$$

Hence

$$(|\gamma_{n+1}|^2 - |\gamma_n|^2)(|\gamma_n|^2) - |\gamma_n|^2 + |\gamma_{n-1}|^2] = 0, \text{ for all } n \in \mathbb{Z}.$$

Or

$$|\gamma_n|^2 = |\gamma_0|^2 + \sum_{k=1}^{k=n-1} \Big(1 - \frac{|\gamma_{k-1}|^2}{|\gamma_k|^2}\Big), n \geq 1 \tag{11}$$

$$|\gamma_{-n}|^2 = |\gamma_0|^2 + \sum_{k=1}^{k=n-1} \Big(1 - \frac{|\gamma_{-k-1}|^2}{|\gamma_{-k}|^2}\Big), \ n \geq 1. \tag{12}$$

We remark that if n is sufficiently large, then $|\gamma_n| > 1$ and so the sums 3.3 and 3.4 are finite, hence we can choose two sequences $(|\alpha_n|)_{n\in\mathbb{Z}}$ and $(|\beta_n|)_{n\in\mathbb{Z}})$ such that $(|\alpha_n|)_{n\in\mathbb{Z}}$ is increasing and $(|\beta_n|)_{n\in\mathbb{Z}})$ is decreasing defined as follows

$$|\alpha_{-1}| = \frac{1}{2}, \ |\alpha_0| = \frac{3}{4}, \ |\beta_{-1}| = \frac{1}{2}, \ |\beta_0| = \frac{1}{3},$$

The others terms are computed from relations 3.3 and 3.4. The operators A and B are the bilateral shifts with weights $(\alpha_n)_{n\in\mathbb{Z}}$ and $(\beta_n)_{n\in\mathbb{Z}}$ respectively. From 3.2, it is easy to check that A is hyponormal in the (Φ) class, B is co-hyponormal in the (Φ) class and D is injective.

Theorem 3.2 *Let $T \in \Phi$. If* ker $[T^*, T]$ *is invariant by T^*, then $T = N \oplus A \oplus B$, where N is a normal operator, A is a pur hyponormal operator and B is a pur co-hyponormal operator.*

Proof. Let $D = [T^*, T]$, we first remark that $\ker D$ is invariant by T, in fact, if $x \ker D$, then

$$\langle DTx, Tx\rangle = \langle T^* DTx, x\rangle = \langle Dx, x\rangle = 0.$$

By applying theorem 2.1, $\ker D$ reduces orthogonally T, consequently we can write T as $T = N \oplus L$, where $N = T|_{\ker D}$ is normal and $L = T|_{\overline{ran\ U^*}}$ is in Φ and in which the commutator is injective. Therefore $L = A \oplus B$ by proposition 3.1. In the next, we will investigate on the relation between this class and other usual classes of operators.

Theorem 3.3 *If $T \in \Phi$ such that* ker $[T^*, T]$ *is invariant by T^*, then $r(T) = \|T\|$.*

Proof. Since ker $[T^*, T]$ is invariant by T^*, then $T = N \oplus A \oplus B$. In view of [4, Lemma 1], $\|T\| = \max(\|N\|, \|A\|, \|B\|)$.

Also From [3], $r(T) = \max(r(N), r(A), r(B))$. Since $\sigma(T) = \sigma(N) \cup \sigma(A) \cup \sigma(B)$, it follows that $\|T\| = r(T)$.

Definition 3.4 *$T \in B(\mathcal{H})$ is called (G_1) class operator if*

$$\|(T - zI\|^{-1} = [dist\ (z, \sigma(T))]^{-1},\ \textit{for all } z \notin \sigma(T).$$

This class includes normal, subnormal and hyponormal operators [5].

Theorem 3.5 *If $T \in \Phi$ such that* ker $[T^*, T]$ *is invariant by T^*, then $T \in G_1$.*

To prove this theorem, we need the following lemma.

Lemma 3.6 *The class (G_1) operators contains the hyponormal operators and their adjoints.*

Proof. We known [6] that the hyponormal operators belongs to the class (G_1), we have to prove that (G_1) contains also their adjoints. In fact, let B be a co-hyponormal operator, then

$$\|(B^* - \lambda I)^{-1}\| = \|(B - \overline{\lambda} I)^{-1}\| = [dist\ (\lambda, \sigma(B^*))]^{-1},\ \text{for all } \lambda \notin \sigma(B^*).$$

Moreover, we have

$$\begin{aligned} dist\ (\lambda, \sigma(B^*)) &= dist\ (\lambda, \overline{\sigma(B)}) = \inf_{\mu \in \overline{\sigma(B)}} |\lambda - \mu| \\ &= \inf_{\overline{\mu} \in \sigma(B)} |\lambda - \mu| = \inf_{\overline{\mu} \in \sigma(B)} |\overline{\lambda} - \mu| \\ &= dist\ (\overline{\lambda}, \sigma(B)) \end{aligned}$$

Hence, set $\zeta = \overline{\lambda}$, it follows

$$\|(B - \zeta I)^{-1}\| = [dist\ (\zeta, \sigma(B))]^{-1},\ \text{for all } \zeta \notin \sigma(B).$$

This completes the proof. If $T \in \Phi$ such that $\ker D$ is invariant by T^*, then by theorem 3.2, T can be written as $T = N \oplus A \oplus B$. From the previous lemma, we have for all $\lambda \notin \sigma(T)$

$$\begin{aligned} \|(N - \lambda I)^{-1}\| &= [dist\ (\lambda, \sigma(N))]^{-1} \\ \|(A - \lambda I)^{-1}\| &= [dist\ (\lambda, \sigma(A))]^{-1} \\ \|(B - \lambda I)^{-1}\| &= [dist\ (\lambda, \sigma(B))]^{-1}. \end{aligned}$$

Also

$$\|(T - \lambda I)\| = \max(\|(N - \lambda I)\|, \|(A - \lambda I)\|, \|(B - \lambda I)\|).$$

Hence

$$\|(T-\lambda I)^{-1}\| = \max(\|(N-\lambda I)^{-1}\|, \|(A-\lambda I)^{-1}\|, \|(B-\lambda I)^{-1}\|).$$

Thus

$$dist\ (\lambda, \sigma(N)) = \inf_{\mu \in \sigma(N)} |\lambda - \mu| \geq \inf_{\mu \in \sigma(T)} |\lambda - \mu| = dist\ (\lambda, \sigma(N)).$$

It follows that

$$\|(N-\lambda I)^{-1}\| = [dist\ (\lambda, \sigma(N))]^{-1} \leq [dist\ (\lambda, \sigma(T))]^{-1}. \quad (13)$$

The inequality 3.5 is also verified by A and B, finally we deduces that

$$\|(T-\lambda I)^{-1}\| \leq [dist\ (\lambda, \sigma(T))]^{-1}, \text{ for all } \lambda \notin \sigma(T).$$

Since the reverse inequality is trivially verified for all bounded operators, then the proof is complete.

Corollary 3.7 *If* $T \in \Phi$ *such that* $\ker [T^*, T]$ *is invariant by* T^**, then* $cov\sigma(T) = \overline{W(T)}$.

Proof. In [5] it is shown that if $T \in G_1$, then $cov\sigma(T) = \overline{W(T)}$. Thus the result follows from the theorem 3.6.

4 Open Problem

The following problems are open till now.

1. What the class Φ contains exactly,

2. Is $T \in \Phi$ with real spectrum self-adjoint.

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