Int. J. Open Problems Comp. Math., Vol. 2, No. 1, March 2009

Ramanujan's Entries and Solutions of Incomplete Elliptic Integrals in Terms of Hypergeometric Functions

M. I. Qureshi and M. P. Chaudhary

Department of Applied Sciences and Humanities
Faculty of Engineering and Technology
Jamia Millia Islamia (A Central University)
New Delhi-110025, India
e-mail: mpchaudhary_2000@yahoo.com

Abstract

The main object of this note is to obtain some necessary conditions (in terms of truncated Gaussian hypergeometric forms) associated with four results of entry 7 of chapter 17 of second note book of Ramanujan. Unsolved - incomplete elliptic integrals of first and second kinds are also solved and generalized in terms of hypergeometric functions.

Keywords: *Wavelet Transform, Image Despeckling, Translation invariant, Bayesian estimation, and SAR images.*

2000 AMS Classifications: 33-Special Functions.

1 Introduction and Preliminary Results

(i) Pochhammer Symbol - If a is neither zero nor negative integer then

$$(a)_r = \frac{\Gamma(a+r)}{\Gamma(a)} = \begin{cases} 1 & ; \quad r = 0 \\ a(a+1)\cdots(a+r-1) & ; \quad r = 1,2,3,\cdots \end{cases} \tag{1.1}$$

where the notation (Γ) stands for Gamma function.

(ii) Gauss ordinary hypergeometric function ${}_2F_1$ is defined as

$$ {}_2F_1\left[\begin{matrix} a,b\,; \\ c\ \ \ ; \end{matrix} z\right] = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r\, r!} z^r $$

where $|\,z\,| < 1$ and $c \neq 0, -1, -2, -3\ldots$ (1.2)

(iii) Truncated ordinary hypergeometric function [21] is defined as

$$ {}_2F_1\left[\begin{matrix} a,b\,; \\ c\ \ \ ; \end{matrix} z\right] \text{to } m \text{ terms} = {}_2F_1\left[\begin{matrix} a,b\,; \\ c\ \ \ ; \end{matrix} z\right]_{m-1} = \sum_{r=0}^{m-1} \frac{(a)_r (b)_r}{(c)_r\, r!} z^r \qquad (1.3) $$

when $m \to \infty$, then (1.3) reduces to (1.2).

(f) Complete Elliptic Integral of First and Second Kind

$$ \mathrm{K}(\sqrt{x}\,) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{(1 - x\sin^2\theta)}} = \frac{\pi}{2}\ {}_2F_1\left[\begin{matrix} 1/2, 1/2\,; \\ 1\qquad\quad ; \end{matrix} x\right] $$

$$ \mathrm{E}(\sqrt{x}\,) = \int_0^{\pi/2} \sqrt{(1 - x\sin^2\theta)}\,\mathrm{d}\theta = \frac{\pi}{2}\ {}_2F_1\left[\begin{matrix} 1/2, -1/2\,; \\ 1\qquad\quad ; \end{matrix} x\right] $$

where $|\,x\,| < 1$ (1.4)

(v). Kampé de Fériet’s ordinary double hypergeometric function [22; 23]:

(In the modified notation of *Srivastava and Panda* [24; 25; 26])

$$ F^{A:B;D}_{E:G;H}\left[\begin{matrix} (a_A):(b_B);(d_D); \\ (e_E):(g_G);(h_H); \end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} \frac{[(a_A)]_{m+n}[(b_B)]_m[(d_D)]_n\, x^m y^n}{[(e_E)]_{m+n}[(g_G)]_m[(h_H)]_n\, m!\,n!} \qquad (1.10) $$

where above all notations have their standard usual meanings. For convergence of above double power series, any one of the following conditions should be satisfied

Either, (i) $A+B < E+G+1$, $A+D < E+H+1$, then max $\{|x|, |y|\} < \infty$ (1.6)

Or, (ii) $A+B = E + G + 1$, $A + D = E + H + 1$ and $A > E$

then $$|x|^{\frac{1}{(A-E)}} + |y|^{\frac{1}{(A-E)}} < 1 \tag{1.7}$$

Or, (iii) If $A+B = E+G+1$, $A+D = E + H + 1$ and $A \leq E$

then max$\{|x|, |y|\} < 1$ (1.8)

(vi) Reduction formula for Clausenian hypergeometric function ${}_3F_2$

$${}_3F_2\left[\begin{matrix} a,b,1 & ; \\ c,2 & ; \end{matrix} z\right] = \frac{(c-1)}{(a-1)(b-1)z}\left\{{}_2F_1\left[\begin{matrix} a-1,b-1 & ; \\ c-1 & ; \end{matrix} z\right]-1\right\} \tag{1.9}$$

The equation (1.14) can be derived easily from the following method

$${}_2F_1\left[\begin{matrix} a-1,b-1 & ; \\ c-1 & ; \end{matrix} z\right] = \sum_{m=0}^{\infty} \frac{(a-1)_m (b-1)_m z^m}{(c-1)_m m!}$$

$$= 1 + \sum_{m=1}^{\infty} \frac{(a-1)_m (b-1)_m z^m}{(c-1)_m m!} = 1 + \sum_{m=0}^{\infty} \frac{(a-1)_{m+1} (b-1)_{m+1} z^{m+1}}{(c-1)_{m+1} (m+1)!}$$

$$= 1 + \frac{(a-1)(b-1)z}{(c-1)} \sum_{m=0}^{\infty} \frac{(a)_m (b)_m z^m}{(c)_m (2)_m}$$ □

2 Generalized Form of Incomplete Elliptic Integrals of First and Second Kind

The following reduction formula is familiar in the literature

(a) $$\mathrm{I}_n = \int \sin^n \theta d\theta = \frac{-\sin^{n-1}\theta\cos\theta}{n} + \frac{n-1}{n}\mathrm{I}_{n-2}\,; \quad n \geq 2 \qquad (2.1)$$

where $\mathrm{I}_0 = \theta$, $\mathrm{I}_1 = -\cos\theta$

By successive application of (2.1), we can find

(b) $$\mathrm{I}_{2m} = \int \sin^{2m}\theta d\theta$$

$$= \left(\frac{-(1/2)_m \sin\theta\cos\theta}{m!}\sum_{r=0}^{m-1}\frac{(1)_r(\sin^2\theta)^r}{(3/2)_r}\right) + \left(\frac{(1/2)_m\theta}{m!}\right);\ \text{where } m \geq 0$$

(2.2)

Here $I_0 = \theta$ and empty sum $\sum_{r=0}^{-1}$ is treated as zero. The integral (2.2) can be verified for $m = 0, 1, 2, 3, 4, 5...$

In this note, the following incomplete elliptic integral of first kind, will play an important role, to decide the necessary conditions.

$$\mathrm{F}(\sqrt{x},\alpha) = \int_0^\alpha \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} = \alpha\, {}_2F_1\left[\begin{matrix}1/2, 1/2\,; \\ 1\quad ;\end{matrix}\ x\right] - \frac{x\sin\alpha\cos\alpha}{4}\times$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{matrix}3/2, 3/2 : 1; 1, 1\ ; \\ 2, 2\ : -; 3/2\ ;\end{matrix}\quad x, x\sin^2\alpha\right] \qquad (2.3)$$

where $\max\{|x|, |x\sin^2\alpha|\} < 1$

Proof of (2.3):

$$\mathrm{F}(\sqrt{x},\alpha)=\int_0^{\alpha}\frac{\mathrm{d}\theta}{\sqrt{1-x\sin^2\theta}}=\int_0^{\alpha}(1-x\sin^2\theta)^{-1/2}\mathrm{d}\theta$$

$$=\int_0^{\alpha}{}_1F_0\left[\begin{matrix}1/2; \\ -\ ;\end{matrix}x\sin^2\theta\right]\mathrm{d}\theta=\int_0^{\alpha}\left(\sum_{m=0}^{\infty}\frac{(1/2)_m\,x^m\sin^{2m}\theta}{m!}\right)\mathrm{d}\theta$$

Since change the order of integration and summation, is permissible under suitable convergence condition [28, p.45], we get

$$\mathrm{F}(\sqrt{x},\alpha)=\sum_{m=0}^{\infty}\frac{(1/2)_m\,x^m}{m!}\int_0^{\alpha}\sin^{2m}\theta\mathrm{d}\theta$$

$$=\sum_{m=0}^{\infty}\frac{(1/2)_m\,x^m}{m!}\left\{\left(\frac{-(1/2)_m\sin\alpha\cos\alpha}{m!}\,{}_2F_1\left[\begin{matrix}1,1\ ; \\ 3/2\ ;\end{matrix}\ \sin^2\alpha\right]_{m-1}\right)+\left(\frac{(1/2)_m\alpha}{m!}\right)\right\} \tag{2.4}$$

$$=\alpha+\sum_{m=1}^{\infty}\frac{(1/2)_m\,x^m}{m!}\left[\left(-\sin\alpha\cos\alpha\frac{(1/2)_m}{m!}\sum_{r=0}^{m-1}\frac{r!(\sin^2\alpha)^r}{(3/2)_r}\right)+\left(\frac{(1/2)_m\alpha}{m!}\right)\right] \tag{2.5}$$

Replacing m by m+1 in equation (2.5) and expressing single and double power series in hypergeometric form, we get

$$\mathrm{F}(\sqrt{x},\alpha)=\alpha-\frac{x\sin\alpha\cos\alpha}{4}F^{2:1;2}_{2:0;1}\left[\begin{matrix}3/2,3/2:1\ ;1,1\ ; \\ 2,2\ :-;3/2\ ;\end{matrix}\ x,x\sin^2\alpha\right]+$$
$$+\frac{x\alpha}{4}\,{}_3F_2\left[\begin{matrix}1,3/2\ ,\ 3/2\ ; \\ 2\ ,2\ ;\end{matrix}\ x\right] \tag{2.6}$$

Now using (1.14) in equation (2.6), we get

$$\mathrm{F}(\sqrt{x},\alpha)=\alpha\,{}_2F_1\left[\begin{matrix}1/2,1/2; \\ 1 \quad ;\end{matrix} x\right]-\frac{x\sin\alpha\cos\alpha}{4}F^{2:1;2}_{2:0;1}\left[\begin{matrix}3/2,3/2:1\ ;1,1\ ; \\ 2,2 \quad :-;3/2\ ;\end{matrix}\ x, x\sin^2\alpha\right] \tag{2.7}$$

In view of the convergence conditions given in equations (1.2) and (1.13), we can say that right hand side of (2.7) is convergent. For $\alpha = \pi/2$, equation (2.7) will reduce to equation (1.9).

Using same technique, the equation (2.3) can be generalized in the following form

$$\int_0^\alpha {}_AF_B\left[\begin{matrix}a_1, a_2,\cdots a_A; \\ b_1, b_2,\cdots b_B;\end{matrix}\ x\sin^2\theta\right]\mathrm{d}\theta=\alpha\ {}_{A+1}F_{B+1}\left[\begin{matrix}1/2,\ (a_A); \\ 1,(b_B) \quad ;\end{matrix}\ x\right]-\frac{x\sin\alpha\cos\alpha\prod_{i=1}^{A}(a_i)}{2\prod_{j=1}^{B}(b_j)}\times$$

$$\times F^{A+1:1;2}_{B+2:0;1}\left[\begin{matrix}1+(a_A),3/2:1;1,1 \quad ; \\ 1+(b_B),2,2:-;3/2\ ;\end{matrix}\ x, x\sin^2\alpha\right] \tag{2.8}$$

Put A =1, B = 0, a_1 = ½ in equation (2.8), we get equation (2.3) or (2.7), which is the exact solution of incomplete elliptic integral of first kind and solution is not found in the literature [2;3;4;8;9;18;19].

In equation (2.8), put $A = B = 0$, we get

$$\int_0^\alpha \mathrm{e}^{x\sin^2\theta}\mathrm{d}\theta\ =\ \alpha\ {}_1F_1\left[\begin{matrix}1/2\ ; \\ 1 \quad ;\end{matrix}\ x\right]-\frac{x\sin\alpha\cos\alpha}{2}\times$$

$$\times F^{1:1;2}_{2:0;1}\left[\begin{matrix}3/2:1;1,1 \quad ; \\ 2,2\ :-;3/2\ ;\end{matrix}\ x, x\sin^2\alpha\right] \tag{2.9}$$

In (2.8) put A =1, B = 0, a_1 = - ½, we get

$$\mathrm{E}(\sqrt{x}\,,\alpha)=\int_0^{\alpha}\sqrt{(1-x\sin^2\theta)}\,\mathrm{d}\theta=\alpha\;{}_2F_1\left[\begin{matrix}1/2,-1/2\,;\\1\qquad\;\;;\end{matrix}x\right]+\frac{x\sin\alpha\cos\alpha}{4}\times$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{matrix}1/2,3/2:1\;\;;1,1\,;\\2,2\qquad\;\;:-;3/2;\end{matrix}x,\,x\sin^2\alpha\right] \qquad (2.10)$$

Which is the incomplete elliptic integral of second kind and solution is not found in literature.

Abramowitz and Stegun [1], Byrd and Friedman [5; 6], Cayley [7], Heuman [11], Jahnke – Emde [12], Jahnke, Emde and Lösch [13], King [14], Lawden [15], Neville [16], Pearson [17], Roberts [20], and Thomson [27] etc, have calculated approximately the complete and incomplete elliptic integrals of first kind and second kind for different values of x and α, using Trapezoidal rule, Simpson-1/3 rule, Simpson-3/8 rule, , Boole's rule, Weddle's rule etc.

3 Some entries of Ramanujan

[3; 8; 19] are given below

Entry 7(ii): If $\tan\alpha=\sqrt{(1-x)}\tan\beta$

then $$\int_0^{\alpha}\frac{\mathrm{d}\theta}{\sqrt{(1-x\cos^2\theta)}}=\int_0^{\beta}\frac{\mathrm{d}\theta}{\sqrt{(1-x\sin^2\theta)}}$$

Entry 7(v): If $\cot\alpha\cot\beta=\sqrt{1-x}$

then $$\int_0^{\alpha}\frac{\mathrm{d}\theta}{\sqrt{(1-x\sin^2\theta)}}+\int_0^{\beta}\frac{\mathrm{d}\theta}{\sqrt{(1-x\sin^2\theta)}}=\frac{\pi}{2}\;{}_2F_1\left[\begin{matrix}1/2,1/2;\\1\qquad\;;\end{matrix}x\right]$$

Entry 7(vi): If $\cot\alpha\tan\left(\dfrac{\beta}{2}\right)=\sqrt{(1-x\sin^2\alpha)}$

then $$2\int_0^{\alpha}\frac{d\theta}{\sqrt{(1-x\sin^2\theta)}}=\int_0^{\beta}\frac{d\theta}{\sqrt{(1-x\sin^2\theta)}}$$

Entry 7(viii): If $\displaystyle\int_0^{\alpha}\frac{d\theta}{\sqrt{(1-x\sin^2\theta)}}+\int_0^{\beta}\frac{d\theta}{\sqrt{(1-x\sin^2\theta)}}=\int_0^{\gamma}\frac{d\theta}{\sqrt{(1-x\sin^2\theta)}}$

then

(a) $$\tan\left(\frac{\gamma}{2}\right)=\frac{\sin\alpha\cos\beta\sqrt{(1-x\sin^2\beta)}+\sin\beta\cos\alpha\sqrt{(1-x\sin^2\alpha)}}{\cos\alpha+\cos\beta}$$

(b) $$\gamma=\tan^{-1}\left(\tan\alpha\sqrt{(1-x\sin^2\beta)}\right)+\tan^{-1}\left(\tan\beta\sqrt{(1-x\sin^2\alpha)}\right)$$

(c) $$\cot\alpha\cot\beta=\frac{\cos\gamma}{\sin\alpha\sin\beta}+\sqrt{(1-x\sin^2\gamma)}$$

(d) $$\frac{\sqrt{x}}{2}=\frac{\sqrt{\{\sin(s)\sin(s-\alpha)\sin(s-\beta)\sin(s-\gamma)\}}}{\sin\alpha\sin\beta\sin\gamma}\text{; where } s=\frac{\alpha+\beta+\gamma}{2}$$

If x is independent of α, β, γ and other parameters then we have the following necessary conditions provided that associated power series is convergent.

If $\displaystyle\sum_{m=0}^{\infty}a_m x^m+\sum_{m=0}^{\infty}b_m x^m=\sum_{m=0}^{\infty}c_m x^m$ then $a_m+b_m=c_m$

and if $\displaystyle p\sum_{m=0}^{\infty}a_m x^m=\sum_{m=0}^{\infty}c_m x^m$ then $pa_m=c_m$; where p is a finite number

B.C.Berndt [3, pp 104 -113] and R.Y.Denis *et al*. [8, pp 115 – 117] have derived and verified the above entries.

4 Necessary Conditions

If x is independent of α, β, γ and other parameters then we have the following necessary conditions provided that associated integrals are convergent.

If $\displaystyle\int_0^{\alpha} \frac{d\theta}{\sqrt{(1-x\cos^2\theta)}} = \int_0^{\beta} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}}$ (4.1)

then

$$\sin\alpha\cos\alpha\ {}_2F_1\left[\begin{matrix}1,1\ ; \\ 3/2;\end{matrix}\ \cos^2\alpha\right]_{m-1} + \alpha = -\sin\beta\cos\beta\ {}_2F_1\left[\begin{matrix}1,1\ ; \\ 3/2;\end{matrix}\ \sin^2\beta\right]_{m-1} + \beta$$

where m = 0, 1, 2, 3…. (4.2)

If $\displaystyle\int_0^{\alpha} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} + \int_0^{\beta} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} = \frac{\pi}{2}\ {}_2F_1\left[\begin{matrix}1/2,1/2; \\ 1\quad ;\end{matrix}\ x\right]$ (4.3)

then

$$-\sin\alpha\cos\alpha\ {}_2F_1\left[\begin{matrix}1,1\ ; \\ 3/2\ ;\end{matrix}\ \sin^2\alpha\right]_{m-1} - \sin\beta\cos\beta\ {}_2F_1\left[\begin{matrix}1,1\ ; \\ 3/2\ ;\end{matrix}\ \sin^2\beta\right]_{m-1} + \alpha + \beta = \frac{\pi}{2}$$

where m = 0, 1, 2, 3… (4.4)

If $\displaystyle p\int_0^{\alpha} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} = \int_0^{\beta} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}},$ (4.5)

then

$$-p\sin\alpha\cos\alpha\ {}_2F_1\left[\begin{matrix}1,1 & ; \\ 3/2; & \end{matrix}\ \sin^2\alpha\right]_{m-1} + p\alpha = -\sin\beta\cos\beta\ {}_2F_1\left[\begin{matrix}1,1 & ; \\ 3/2; & \end{matrix}\ \sin^2\beta\right]_{m-1} + \beta$$

where m = 0, 1,2,3,…. (4.6)

If $$\int_0^{\alpha} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} + \int_0^{\beta} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} = \int_0^{\gamma} \frac{d\theta}{\sqrt{(1-x\sin^2\theta)}} \quad (4.7)$$

then

$$-\sin\alpha\cos\alpha\ {}_2F_1\left[\begin{matrix}1,1 & ; \\ 3/2 & ; \end{matrix}\ \sin^2\alpha\right]_{m-1} - \sin\beta\cos\beta\ {}_2F_1\left[\begin{matrix}1,1 & ; \\ 3/2 & ; \end{matrix}\sin^2\beta\right]_{m-1} + \alpha + \beta$$

$$= -\sin\gamma\cos\gamma\ {}_2F_1\left[\begin{matrix}1,1 & ; \\ 3/2 & ; \end{matrix}\ \sin^2\gamma\right]_{m-1} + \gamma$$, where, m = 0, 1, 2, 3…. (4.8)

<u>Derivation for Necessary Conditions</u>

Using (2.4) in (4.3), we get

$$\sum_{m=0}^{\infty} \frac{(1/2)_m\ x^m}{m!} \left\{ \left(\frac{-(1/2)_m \sin\alpha\cos\alpha}{m!}\ {}_2F_1\left[\begin{matrix}1,1 & ; \\ 3/2 & ; \end{matrix}\ \sin^2\alpha\right]_{m-1} \right) + \left(\frac{(1/2)_m\alpha}{m!} \right) \right\}$$
$$+ \sum_{m=0}^{\infty} \frac{(1/2)_m\ x^m}{m!} \left\{ \left(\frac{-(1/2)_m \sin\beta\cos\beta}{m!}\ {}_2F_1\left[\begin{matrix}1,1 & ; \\ 3/2 & ; \end{matrix}\ \sin^2\beta\right]_{m-1} \right) + \left(\frac{(1/2)_m\beta}{m!} \right) \right\}$$
$$= \sum_{m=0}^{\infty} \frac{(1/2)_m\ x^m}{m!} \left\{ \frac{\pi(1/2)_m}{2(m!)} \right\} \quad (4.9)$$

Since above three power series are convergent, then equate the coefficients of x^m on both sides and we get necessary condition (4.4). Similarly we can derive the conditions (4.6) and (4.8) for (4.5) and (4.7) respectively.

5 Open Problem(s)

In the study of space problems we need to calculate several distances / lengths mostly are elliptic paths / curves in nature, which involved incomplete elliptic integrals and their exact solutions, most of such solutions are still unknown.

In this research paper we have presented solutions of some incomplete elliptic integrals. Now we can search various elliptic paths in space and try to calculate their exact lengths (Not approximate, which we are knowing yet) with the help of solutions of incomplete integrals and use them in our space study.

Acknowledgement: The authors are thankful to Prof. R.Y. Denis (Emeritus Fellow, U.G.C., New Delhi) and Prof. M.A.Pathan (former Head of the department of Mathematics, A.M.U., Aligarh) for their valuable suggestions during preparation of this paper. Authors are also thankful to the referees for their valuable suggestions to improve the presentation of this paper.

The second author of this paper is grateful to Prof. G.E. Andrews (U.S.A.) for their sincere guidance and help in this difficult area of Mathematics during his visit to the Department of Mathematics, Pennsylvania State University, during June 2007.

References

[1] M. Abramowitz and I. A. Stegun; *Handbook of mathematical functions, with formulas, graphs and mathematical tables,* National Bureau of Standards, Washington D. C., 1964., Reprinted by Dover Publications, Inc., New York, 1972.

[2] G. E. Andrews and B. C. Berndt; *Ramanujan's lost notebook, Part I*, Springer, New York, 2005, MR 2135178 (2005m: 11001).

[3] B. C. Berndt; *Ramanujan's notebooks, Part III*, Springer Verlag, New York, 1991, MR1117903 (92j: 11024).

[4] B. C. Berndt; *Ramanujan's notebooks, Part V*, Springer Verlag, New York 1998, MR1486573 (99f: 11024).

[5] P. F. Byrd and M. D. Friedman; *Handbook of elliptic integrals for engineers and physicists,* Springer, Verlag, Berlin, Heidelberg, 1954.

[6] P. F. Byrd and M. D. Friedman; *Handbook of elliptic integrals for engineers and scientists*, Springer Heidelberg, 1971.

[7] A. Cayley; *An elementary treatise on elliptic function*, Dover Publications, Inc., New York, 1956, 1996.

[8] R. Y. Denis, S. N. Singh and S. P. Singh; On certain elliptic integrals of Ramanujan, *J. Indian Math. Soc*., 37(3-4) (2006), 113-119.

[9] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi; *Higher transcendental function*, Vol. 2., McGraw-Hill Book Company, New York, Toronto and London, 1953.

[10] H. Exton; *Handbook of Hypergeometric Integrals, Theory, Applications, Tables and Computer Programs*, Halsted Press(Ellis Horwood Ltd., Chichester, U. K.), John Wiley and Sons, New York, Chichester, Sussex, England, 1978.

[11] C. Heuman; Tables of complete elliptic integrals, *J. Math. Phys.*, 20, 1941, 127-206.

[12] E. Jahnke and F. Emde; *Tables of functions with formulas and Curves,* 4th edition, Dover Publications, Inc., New York, 1945.

[13] E. Jahnke and F. Emde and F. Lösch; *Tables of higher functions, 6th edition,* McGraw-Hill, New York, 1960.

[14] L.V. King; *On the direct numerical calculation of elliptic functions and integrals*, Cambridge University Press, Cambridge, England, 1924.

[15] D. F. Lawden; *Elliptic functions and applications*, Springer- Verlag, Berlin, 1989.

[16] E. H. Neville; *Jacobian elliptic functions*, 2nd edition, Oxford University Press, London, England, 1951 (1st edition 1944).

[17] K. Pearson; *Tables of the complete and incomplete elliptic integrals*, Cambridge University Press, Cambridge, England, 1934.

[18] A. P. Prudnikov; Yu. A. Brychkovand and O. I. Marichev,; *Integrals and series* vol. 3.*, more special functions,* Nauka, Moscow, 1986. Translated from the Russian by G.G.Gould, Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo, Melbourne, 1990.

[19] E. H. Neville; *Notebook Volume II*, Tata Institute of Fundamental Research, Bombay, 1957.

[20] W. R. W. Roberts; *Elliptic and hyperelliptic integrals and allied theory*, Cambridge University Press, 1938.

[21] L. J. Slater; *Generalized hypergeometric functions*, Cambridge, University Press, Cambridge, 1966.

[22] H. M. Srivastava and Per W. Karlsson ; *Multiple gaussian hypergeometric series*, Halsted Press (Ellis Horwood Ltd., Chichester, Brisbane, U.K.) John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[23] H. M. Srivastava and H. L. Manocha; *A treatise on generating functions*, Halsted Press (Ellis Horwood Ltd., Chichester, Brisbane, U.K.) John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[24] H. M. Srivastava and R. Panda; An integral representation for the product of two Jacobi polynomials, *J.London Math. Soc*., (2), 12, 1976, 419-425.

[25] H. M. Srivastava and M. A. Pathan ; Some bilateral generating functions for the extended Jacobi polynomial-I*, Comment. Math. Uni. St. Paul*., 28(1), 1979, 23-30

[26] H. M. Srivastava and M. A. Pathan ; Some bilateral generating functions for the extended Jacobi polynomial-II*, Comment. Math. Uni. St. Paul.,* 29 (2), 1980, 105-114.

[28] L. M. Milne Thomson; Ten-figure table of the complete elliptic integrals, *Proc. London Math.Soc*. (2), 33, 1931.

[29] E. C. Titchmarsh ; T*he theory of functions*, Oxford University Press, Ely House, London, Second edition, 1939.