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Partizan Geography on $K_n \times K_2$

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Abstract

We give the values, temperatures and a guide to play partizan geography on a $n \times 2$ board where Left can move along the rows and Right along the column. Clearly, Left has an advantage for $n > 2$, but has to play carefully to fully utilize this advantage.

1. Introduction

In this game, Left moves the token (O) to any empty cell on the same row and Right moves it to any empty cell in the same column (roles are reversed in the negative game) and no cell can be visited twice—visited cells are indicated by an X. Before reading the rest of the paper, the reader is invited to discover who wins, and how, in the following sum?

X	X	X			
			O		X

$-$

X	X	X			
			O	X	X

$-$

X		
	O	X

The game called *Kotzig's Nim* in [1] and *Modular Nim* in [2] consists of a directed cycle of length n with the vertices labeled 0 through $n-1$, a token placed initially on vertex 0, and a set of integers called the move set. There are two players, who alternate moves, where the last player to move wins; a move consists of moving the token from the vertex i on which it currently resides to vertex $i+m \bmod n$, where m is a member of the move set. However, the coin can only land on a vertex once, thus ensuring that the game is finite. Most of the known results concern themselves with move sets of small cardinality and consisting of small numbers (see p. 515 of [1], and [2]).

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Obviously, this game can be extended to more general directed graphs, the move set being indicated by directed edges, for clarity. This has become known as *Geography*, [3, 4], after the children's game of the same name. In [2], it was shown that deciding who wins in Geography is P-space complete. In [6, 5], the class of graphs was restricted to the cartesian product of two directed cycles.

All these versions are impartial, i.e. both players have the same moves. In this paper, we consider the partizan version played on the Cartesian product of K_n and K_2. Recall that the cartesian product of graphs G and H, $G \,\square\, H$, has vertex set $V(G) \times V(H)$ and there is an edge from (i, j) to (k, l) if i is adjacent to k and $j = l$ or if j is adjacent to l and $i = k$.

We will visualize this graph as a 2 by n rectangular board with rows labeled $0, 1$ from top to bottom and the columns are numbered $0, 1, \ldots, (n-1)$ from left to right.

Our starting board is

	0	1	...	$n-1$
1				
0	O			

In the partizan version, from this starting position of $(0, 0)$, Right can only move to $(0, 1)$ and Left to any position $(x, 0)$ where $x \in \{1, \ldots, n-1\}$. Equivalently, on our rectangular board, Left is allowed move horizontally within the same row to any open cell while Right is only allowed to to move up or down within the same column and no cell can be visited twice.

After Right's initial move on the opening board, Right will always have a response to a Left move because the cells on a column can be paired off. In other words, Right moving first moves to a game of value 0. After a Left move though, Right moves to a game whose value depends on the parity as well as then number of cells that can be paired off—Right, after playing the paired cells, would like to finish in the column in which Left doesn't have a move. However, if Right allows Left two consecutive moves before responding, Left has gained a move so there are threats that have to be taken into account when evaluating a position. The first few values are:

n	1	2	3	4	5	6	7
value of $K_n \,\square\, K_2$	-1	$*$	$\{\frac{1}{2} \mid 0\}$	$\{1* \mid 0\}$	$\{\frac{3}{2} \mid 1 \,\|\, 0\}$	$\{2* \mid 1 \,\|\, 0\}$	$\{\frac{5}{2} \mid 2 \,\|\, 1 \,\|\|\, 0\}$
(Mean, Temp.)		$(0, 0)$	$(\frac{1}{4}, \frac{1}{4})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{5}{8}, \frac{5}{8})$	$(\frac{3}{4}, \frac{3}{4})$	$(\frac{13}{16}, \frac{13}{16})$

This pattern continues. In Theorem 2 (F), we show that, with $k = \lfloor \frac{n-1}{2} \rfloor$, the starting position has Mean and Temperature $1 - \frac{1}{2^k}$ if n is even and $1 - \frac{3}{2^{k+1}}$ if n is odd. In fact, in Theorem 2, we give the strategy and a 'players-guide' to the game. In Theorem 7, we give the exact values and in Corollary 8, we give a more detailed interpretation of these values. The proof of Theorem 2 follows from Theorem 7 and Corollary 8.

This work is an extension of results found in [7]. The authors would like to acknowledge the use and usefulness of CGSuite [8] for this paper.

2. Basic Strategy

In order to more easily discuss positions in the game, we must first develop a simple notation to describe a board position. We adopt the convention that the token will always be on the bottom row. We use an ordered triple (x, y, z) to represent a board in which:

(1) There are x cells open in the bottom row that do not have top row positions available.

(2) There are y columns that have cells open in both the bottom and top rows.

(3) There are z cells open in the top row that do not have bottom row positions available from them.

(4) (i) if the token occupies one of the x cells, this board will be denoted by $B(x, y, z)$. Note: by definition then $x > 0$ and also as a consequence, only Left can move.

(ii) if the current position is in the y cells, this board will be denoted by $A(x, y, z)$. Note: by definition $y > 0$ and Right can move.

For example, the notation $B(3, 0, 0)$ represents each of the following two $K_2 \square K_4$ board positions:

	0	1	2	3
1	X	X	X	X
0	O	X		

	0	1	2	3
1	X	X	X	X
0		X	O	

which are of course, isomorphic positions. Also The following two $K_2 \square K_4$ boards, which are isomorphic, would be represented by the notation $A(2, 2, 0)$:

	0	1	2	3
1	X	X		
0				O

	0	1	2	3
1	X		X	
0		O		

Using this notation, it is now easy to describe the options of the games.

Lemma 1 *Given a position (x, y, z):*

$$B(x, y, z) = \left\{ B(x-1, y, z)\, ,\, A(x-1, y, z) \,\middle|\, \right\} \tag{1}$$

$$A(x, y, z) = \left\{ B(x, y-1, z+1)\, ,\, A(x, y-1, z+1) \,\middle|\, B(z+1, y-1, x) \right\} \tag{2}$$

provided the options exist.

Theorem 2 (Strategy) *The value of $A(x, y, z)$ is:*

(A) *If* $z \geq y-2$ *and* $x > y-2$, *then*

$$A(x, y, z) = \{y + x - 2 \,|\, y + z - 1\} = y + m + \{x - 2 - m \,|\, z - 1 - m\}$$

where $m = min\{x, z\}$ *and Left's best move is to* $B(x, y-1, z+1)$*;*

(B) *If* $z \geq y-2$ *and* $x \leq y-2$, *then*

$$A(x, y, z) = \{y + x - 2 \,|\, x + z\} = y + x - 2 + \{0 \,|\, z - y + 2\}$$

and Left's moves are incomparable;

(C) *If* $z < y-2$ *and* $x > y-2$, *then* $A(x, y, z) = y + z - 1 + H$ *where* H *has mean and temperature* $\frac{x-y+1}{2}$ *and Left's moves are incomparable.*

(D) *If* $z < y-2$ *and* $x \leq y-2$, *then Left's best move is to* $A(x, y-1, z+1)$ *and*

(i) *if* $2x < y+z$ *then* $A(x, y, z) = x + z + H$ *where, putting* $k = \lfloor \frac{y-z-1}{2} \rfloor$, H *has mean and temperature* $1 - \frac{1}{2^k}$ *if* $y-z$ *is even; and mean and temperature* $1 - \frac{3}{2^{k+1}}$ *if* $y-z$ *is odd;*

(ii) *if* $2x \geq y+z$, *then* $A(x, y, z) = x + z + H$ *where* H *has mean and temperature* $1 - \frac{1}{2^{y-x-1}}$.

The value of $B(x, y, z)$ *is:*

(E) *If* $z < y$, *then* $B(x, y, z) = x + z - 1$*;*

(F) *If* $z \geq y$, *then* $B(x, y, z) = y + x - 1$.

In parts (A) and (B) of the following theorem, the game value reduces to numbers for some values of x and z.

In our opening example, the game in our new notation is $A(3, 2, 1) - A(3, 1, 2) - A(1, 1, 1)$ which has value $\{3 \,|\, 2\} - 2 * -\frac{1}{2} = \{\frac{1}{2} * \,|\, -\frac{1}{2}*\}$. The game is a first player win and both should move in $A(3, 2, 1)$, Left to $B(3, 2, 2)$ and Right to $B(3, 0, 3)$.

As a general heuristic, a player should play in the game with highest temperature. In $G = A(3, 12, 4) + A(7, 12, 1) - A(9, 15, 6)$, G is fuzzy with 0, i.e. a first player win, G has mean $\frac{29}{32}$ and a temperature of $\frac{15}{16}$. The individual summands have temperatures $\frac{7}{8}$, $\frac{15}{16}$ and $\frac{29}{32}$ respectively. The only winning move for Right is to $A(3, 12, 4) + 8 - A(9, 15, 6)$ which is less than 0 and has a mean of $-\frac{1}{32}$. Left is less constrained, he has winning moves in both $A(7, 12, 1)$ and $-A(9, 15, 6)$ but playing in $-A(9, 15, 6)$ results in a strictly greater game than playing in $A(7, 12, 1)$.

3. Evaluation of Positions

In order to evaluate the game, we will first look at some basic positions.

Lemma 3 *For $x > 0$, $B(x, 0, z) = x - 1$.*

Proof. If $x = 1$, then $B(x, 0, z) = B(1, 0, z) = \{\,|\,\} = 0$.

Using induction,

$$B(x, 0, z) = \big\{B(x-1, 0, z)\,\big|\ \big\} = \big\{(x-1) - 1\,\big|\ \big\} = \big\{x - 2\,\big|\ \big\} = x - 1.$$

□

Lemma 4 *For $x > 0$, $A(x, 1, z) = \{x - 1\,|\,z\}$.*

Proof. From Lemma 1 $A(x, 1, z) = \big\{B(x, 0, z+1)\,\big|\,B(z+1, 0, x)\big\}$. From Lemma 3, we know that $B(x, 0, z) = x - 1$ and $B(z + 1, 0, x) = z$. Thus, $A(x, 1, z) = \{x - 1\,|\,z\}$. □

Since $\{x - 1|z\} \le x$ this proves

Corollary 5 *$A(x, 1, z) \le x$.*

Lemma 6 *For $x > 0$, $B(x, 1, z) = x$.*

Proof. $B(x, 1, z) = \big\{B(x-1, 1, z), A(x-1, 1, z)\,\big|\ \big\}$. From Lemma 4, we know that $A(x - 1, 1, z) = \{x - 2\,|\,z\} \le x - 1$. By induction, $B(x - 1, 1, z) = x - 1$. Therefore, $B(x, 1, z) = \{x - 1\,|\ \} = x$.

□

In order to go any further, we need to introduce some new notation to describe the unusual values that we see in this game.

Notation 1 *Let a and c be integers and g any game, then*

$$\Big\langle a+i, \langle g\rangle\,\Big|\,c+i\Big\rangle_{i=0}^{k} = \Bigg\{a, \Big\{a+1, \Big\{\ldots\{a+k, g\,|\,c+k\}\,\Big|\,\ldots\Big\}\,\Big|\,c+1\Big\}\,\Bigg|\,c\Bigg\},$$

$$\Big\langle a, \langle g\rangle\,\Big|\,c+i\Big\rangle_{i=0}^{k} = \Bigg\{a, \Big\{a, \Big\{\ldots\{a, g\,|\,c+k\}\,\Big|\,\ldots\Big\}\,\Big|\,c+1\Big\}\,\Bigg|\,c\Bigg\},$$

$$\Big\langle \langle g\rangle\,\Big|\,c+i\Big\rangle_{i=0}^{k} = \Bigg\{\Big\{\Big\{\ldots\{g\,|\,c+k\}\,\Big|\,\ldots\Big\}\,\Big|\,c+1\Big\}\,\Bigg|\,c\Bigg\},$$

and similarly, if the index i is not present on the right side, then

$$\left\langle a+i, \langle g\rangle \,\middle|\, c\right\rangle_{i=0}^{k} = \left\{a, \left\{a+1, \left\{\ldots \{a+k, g \,|\, c\} \,\middle|\, \ldots\right\} \,\middle|\, c\right\} \,\middle|\, c\right\}.$$

Typical games positions in the usual and in the new notation are:

$$\begin{aligned}
A(17,17,9) &= \{26, \{27, \{28, \{29 \,|\, 25\} \,|\, 25\} \,|\, 25\} \,|\, 25\} \\
&= \langle 26+i, \langle 29 \,|\, 25\rangle \,|\, 25\,\rangle_{i=0}^{2} \\
A(5,10,3) &= \{\{\{\{10 \,|\, 11\} \,|\, 10\} \,|\, 9\} \,|\, 8\} \\
&= \langle\langle 10 \,|\, 11\rangle \,|\, 8+i\,\rangle_{i=0}^{2} \\
A(12,15,3) &= \{\{\{17, \{18, \{19, \{20 \,|\, 17\} \,|\, 17\} \,|\, 17\} \,|\, 17\} \,|\, 16\} \,|\, 15\} \\
&= \langle\,\langle\,\langle 17+i, \langle 20 \,|\, 17\rangle \,|\, 17\,\rangle_{i=0}^{2}\,\rangle \,|\, 15+j\,\rangle_{j=0}^{1}.
\end{aligned}$$

In the last case $g = \{17, \{18, \{19, \{20 \,|\, 17\} \,|\, 17\} \,|\, 17\} \,|\, 17\} = \langle 17+i, \langle 20 \,|\, 17\rangle \,|\, 17\,\rangle_{i=0}^{2}$.

We are now ready to find the value of a game.

Theorem 7 *The values of a general position in Partizan Geography played on $K_n \,\square\, K_2$ are as follows.*

The value of $A(x,y,z)$ is:

(A) If $z \geq y-2$ and $x > y-2$, then

$$A(x,y,z) = \{y+x-2 \,|\, y+z-1\} = \begin{cases} y+x-1 & \text{if } z \geq x+1 \\ y+x-\frac{3}{2} & \text{if } z = x \\ y+x-2* & \text{if } z = x-1 \\ \{y+x-2 \,|\, y+z-1\} & \text{if } z < x-1. \end{cases}$$

(B) If $z \geq y-2$ and $x \leq y-2$, then

$$A(x,y,z) = \{y+x-2 \,|\, x+z\} = \begin{cases} y+x-1 & \text{if } z > y-1 \\ y+x-\frac{3}{2} & \text{if } z = y-1 \\ y+x-2* & \text{if } z = y-2 \end{cases}$$

(C) If $z < y-2$ and $x > y-2$, then

$$A(x,y,z) = \left\langle x+z+i, \langle y+x-k-2 \,|\, y+z-1\rangle \,\middle|\, y+z-1\right\rangle_{i=0}^{k-1},$$

where $k = \frac{y-z-2}{2}$ if $y-z$ is even and $k = \frac{y-z-1}{2}$ if $y-z$ is odd;

(D) If $z < y-2$ and $x \leq y-2$, then

(i) if $2x < y+z$ then

$$A(x,y,z) = \Big\langle \langle x+z+k+g\rangle \Big| x+z+i \Big\rangle_{i=0}^{k-1}$$

where $k = \frac{y-z-2}{2}$, $g = *$ if $(y-z)$ is even and $k = \frac{y-z-1}{2}$, $g = -\frac{1}{2}$ if $(y-z)$ is odd;

(ii) if $2x \geq y+z$, then

$$A(x,y,z) = \Bigg\langle \Big\langle \Big\langle y+z-1+i, \langle 2x-k-1 \,|\, y+z-1\rangle \Big| y+z-1 \Big\rangle_{i=0}^{k-1} \Big\rangle \Bigg| x+z+j \Bigg\rangle_{j=0}^{y-x-2}$$

where $k = \frac{2x-y-z}{2}$ if $(y+z)$ is even and $k = \frac{2x-y-z+1}{2}$ if $(y+z)$ is odd.

The value of $B(x,y,z)$ is:

(E) If $z < y$, then

$$B(x,y,z) = x+z-1;$$

(F) If $z \geq y$, then

$$B(x,y,z) = y+x-1.$$

Proof. Since evaluating $A(x,y,z)$ is more complicated, we present the arguments for $B(x,y,z)$ first.

Let $G = B(x,y,z)$. Then $G = \Big\{ B(x-1,y,z), A(x-1,y,z) \Big| \Big\}$.

Case (E): Let $y \leq z$. In $y+x-1-B(x,y,z)$, Left playing second only plays the integers and never in the $B(x,y,z)$ component. Right, therefore only has $x+y-1$ moves and so loses going first. Left moving first either has no move or plays in $y+x-1$ to $y+(x-1)-1$, and Right replies moving to $y+(x-1)-1-B(x-1,y,z)$ which is a second player win by induction.

Case (F): Let $y > z$. In $z+x-1-B(x,y,z)$, Left moving first can only play in $z+x-1$ and Right responds to $z+(x-1)-1-B(x-1,y,z)$ which is 0 by induction. Right moving first could move to $z+x-1-B(x-1,y,z)$ but Left responds to $z+(x-1)-1-B(x-1,y,z) = 0$.

Suppose, therefore, that Right moves to $z+x-1-A(x-1,y,z)$, Left responds to $z+x-1-B(z+1,y-1,x-1)$. If $y-1 > x-1$ then

$$z+x-1-B(z+1,y-1,x-1) = z+x-1-(z+1+x-2) = 0$$

and Left has won. If $y-1 \leq x-1$ then in $z+x-1-B(z+1,y-1,x-1)$ Left never plays in the $B(z+1,y-1,x-1)$ component and so Right has $y-1+z+1-1 = z+y-1$ moves and $z+y-1 \leq z+x-1$ and so again Left has won.

Let $G = A(x, y, z)$. The options from G are given as the following:

$$G = \big\{B(x, y-1, z+1), A(x, y-1, z+1) \,\big|\, B(z+1, y-1, x)\big\}.$$

Case (A): Assume $z \geq y-2$ and $x > y-2$. By induction, $B(x, y-1, z+1) = y+x-2$ since $z+1 \geq y-1$, and $B(z+1, y-1, x) = y+z-1$ since $x \geq y-1$, giving

$$G = \big\{y+x-2, A(x, y-1, z+1) \,\big|\, y+z-1\big\}.$$

Since $z+1 \geq (y-1)-2$ and $x > (y-1)-2$, then $A(x, y-1, z+1)$ still falls under case (A). By induction, then, this is

$$A(x, y-1, z+1) = \{y+x-3 \,|\, y+z-1\}$$

which gives us

$$G = \big\{y+x-2, \{y+x-3 \,|\, y+z-1\} \,\big|\, y+z-1\big\}.$$

Since $\{y+x-3 \,|\, y+z-1\} \leq y+x-2$, the option to $A(x, y-1, z+1)$ is dominated. Thus, for case (A),

$$G = \{y+x-2 \,|\, y+z-1\}.$$

Case (B): Assume $z \geq y-2$ and $x \leq y-2$. By induction, $B(x, y-1, z+1) = y+x-2$ since $z+1 \geq y-1$, and $B(z+1, y-1, x) = x+z$ since $x < y-1$, giving

$$G = \big\{y+x-2, A(x, y-1, z+1) \,\big|\, x+z\big\}.$$

We have that $z+1 \geq (y-1)-2$ but it is possible for *(i)* $x = y-2 > (y-1)-2$ or *(ii)* $x \leq (y-1)-2$, then $A(x, y-1, z+1)$ may fall under either case (A) or (B). By induction, we will evaluate each case.

(i) If $x = y-2 > (y-1)-2$, i.e. case (A), then $A(x, y-1, z+1) = \{y+x-3 | y+z-1\}$ which gives us

$$G = \big\{y+x-2, \{y+x-3 \,|\, y+z-1\} \,\big|\, x+z\big\}.$$

Since $\{y+x-3 \,|\, y+z-1\} \leq y+x-2$, this option is dominated and therefore $G = \{y+x-2 \,|\, x+z\}$.

(ii) If $x \leq (y-1)-2$, i.e case (B), then, by induction, $A(x, y-1, z+1) = \{y+x-3 \,|\, x+z+1\}$ which gives us

$$G = \big\{y+x-2, \{y+x-3 \,|\, x+z-1\} \,\big|\, x+z\big\}.$$

Since $\{y+x-3 \,|\, x+z+1\} \leq y+x-2$, this option is dominated or equal. Therefore, $G = \{y+x-2 \,|\, x+z\}$.

Therefore, the move to $A(x, y-1, z+1)$ is dominated and

$$G = \{y+x-2 \,|\, x+z\} = \begin{cases} y+x-1 & \text{if } z > y-1 \\ y+x-\frac{3}{2} & \text{if } z = y-1 \\ y+x-2* & \text{if } z = y-2 \end{cases}$$

Case (C): Assume $z < y-2$ and $x > y-2$. Then, by induction, $B(x, y-1, z+1) = x+z$ since $z+1 < y-1$, and $B(z+1, y-1, x) = y+(z+1)-1$ since $x \geq y-1$, giving

$$G = \{x+z, A(x, y-1, z+1) \mid y+z-1\}.$$

We know that $x > (y-1)-2$, but it is possible that for either *(i)* $z+1 < (y-1)-2$ or *(ii)* $z+1 \geq (y-1)-2$ if $z+1 = y-2$ or $z+1 = y-3$.

(i) If $z+1 = y-2$ or $z+1 = y-3$, i.e. $y-z$ equals 3 and 4, respectively, then $A(x, y-1, z+1)$ falls under case (A) and so, by induction, this is

$$A(x, y-1, z+1) = \{y+x-3 \mid y+z-1\},$$

and, since the two Left options are incomparable, giving

$$G = \{x+z, \{y+x-3 \mid y+z-1\} \mid y+z-1\}$$

which we can rewrite as

$$G = \Big\langle x+z+i, \langle y+x-k-2 \mid y+z-1\rangle \Big| y+z-1\Big\rangle_{i=0}^{k-1}$$

where $k = \frac{y-z-2}{2}$ when $y-z=4$ and $k = \frac{y-z-1}{2}$ when $y-z=3$.

(ii) If $z+1 < (y-1)-2$, then $A(x, y-1, z+1)$ still falls under case (C) and so, by induction, this is

$$A(x, y-1, z+1) = \Big\langle x+z+1+i, \langle y+x-k-3 \mid y+z-1\rangle \Big| y+z-1\Big\rangle_{i=0}^{k-1},$$

where $k = \frac{y-z-4}{2}$ if $y-z$ is even and $k = \frac{y-z-3}{2}$ if $y-z$ is odd. Let $w = k+1$, or $k = w-1$, so that we can recast this as

$$A(x, y-1, z+1) = \Big\langle x+z+1+i, \langle y+x-w-2 \mid y+z-1\rangle \Big| y+z-1\Big\rangle_{i=0}^{w-2},$$

where $w = \frac{y-z-2}{2}$ if $y-z$ is even and $w = \frac{y-z-1}{2}$ if $y-z$ is odd. Thus, since both Left options of G are incomparable, we obtain

$$G = \Big\{x+z, \Big\langle x+z+1+i, \langle y+x-w-2 \mid y+z-1\rangle \Big| y+z-1\Big\rangle_{i=0}^{w-2} \Big| y+z-1\Big\},$$

which, in our modified notation, becomes

$$G = \Big\langle x+z+i, \langle y+x-w-2 \mid y+z-1\rangle \Big| y+z-1\Big\rangle_{i=0}^{w-1}$$

with $w = \frac{y-z-2}{2}$ if $y-z$ is even and $w = \frac{y-z-1}{2}$ if $y-z$ is odd.

Case (D): Assume $z < y-2$ and $x \leq y-2$. By induction, $B(x, y-1, z+1) = x+z$ since $z+1 < y-1$, and $B(z+1, y-1, x) = x+z$ since $x < y-1$, giving

$$G = \{x+z, A(x, y-1, z+1) \mid x+z\}.$$

It is possible that for either *(i)* $z+1<(y-1)-2$ (i.e. $2x<y+z$) or *(ii)* $x>(y-1)-2$ or for both to happen at once (i.e. $2x \geq y+z$).

(i): Assume $2x<y+z$. Then $A(x,y-1,z+1)$ is either in case (B) or case (D).

If $A(x,y-1,z+1)$ is in case (B), then $x \leq y-3$ and $z+1 \geq y-3$, i.e. $z=y-4$ or $z=y-3$. Suppose that $z=y-3$. Then, by induction,

$$G=\big\{x+z,\{y+x-3\,|\,x+z+1\}\,|\,x+z\big\}=\left\{z+x+\frac{1}{2}\,|\,x+z\right\}$$

which we can re-write as

$$G=\left\langle\left\langle x+\frac{z}{2}+\frac{y}{2}-1\right\rangle\,\middle|\,x+z+i\right\rangle_{i=0}^{\frac{y-3-z}{2}}$$

Similarly, if $z=y-4$ then

$$G=\{z+x+1*\,|\,x+z\}=\left\langle\left\langle x+\frac{z}{2}+\frac{y}{2}-1+*\right\rangle\,\middle|\,x+z+i\right\rangle_{i=0}^{\frac{y-4-z}{2}}$$

If $A(x,y-1,z+1)$ is in case (D), then $z<y-3$ and $x \leq y-3$. Since

$$G=\big\{B(x,y-1,z+1),A(x,y-1,z+1)\,\big|\,B(z+1,y-1,x)\big\}$$

and $z+1<y-1$ and $x<y-1$, by induction,

$$G=\begin{cases}\left\{x+z,\left\langle\left\langle x+\frac{z+1}{2}+\frac{y-1}{2}-1\right\rangle\,\middle|\,x+z+i\right\rangle_{i=0}^{\frac{y-1-3-z-1}{2}}\,\middle|\,x+z\right\} & \text{if } y-z \text{ is odd}\\ \left\{x+z,\left\langle\left\langle x+\frac{z+1}{2}+\frac{y-1}{2}-1+*\right\rangle\,\middle|\,x+z+i\right\rangle_{i=0}^{\frac{y-1-4-z-1}{2}}\,\middle|\,x+z\right\} & \text{if } y-z \text{ is even.}\end{cases}$$

For Left, the move to $B(x,y-1,z+1)=x+z$ is dominated, therefore

$$G=\begin{cases}\left\{\left\langle\left\langle x+\frac{z}{2}+\frac{y}{2}-1\right\rangle\,\middle|\,x+z+i\right\rangle_{i=0}^{\frac{y-3-z}{2}}\,\middle|\,x+z\right\} & \text{if } y-z \text{ is odd}\\ \left\{\left\langle\left\langle x+\frac{z}{2}+\frac{y}{2}-1+*\right\rangle\,\middle|\,x+z+i\right\rangle_{i=0}^{\frac{y-4-z}{2}}\,\middle|\,x+z\right\} & \text{if } y-z \text{ is even}\end{cases}$$

(ii): Assume $2x \geq y+z$. Then $A(x,y-1,z+1)$ is either in case (A), case (C) or case (D).

If it is in case (A), then $z \geq y-3$ and $x>y-3$. By induction,

$$G=\big\{x+z,\{y+x-3\,|\,y+z-1\}\,\big|\,x+z\big\}.$$

Since $y-3 \leq z < y-2$, then $z = y-3$. Likewise, since $y-3 < x \leq y-2$, then $x = y-2 = z+1$. Thus, $x+z = y+z-2$ and $y+x-3 = y+z-2 = 2x-1$ so that

$$G = \big\{y+z-2, \{y+z-2 \,|\, y+z-1\} \,\big|\, y+z-1\big\},$$

in which the left option to $y+z-2$ is dominated by $\{y+z-2 \,|\, y+z-1\}$. And since $y+z-2 = 2x-1$, G becomes

$$G = \big\{\{2x-1 \,|\, y+z-1\} \,\big|\, x+z\big\}.$$

We can rewrite this as

$$G = \left\langle \left\langle \left\langle y+z-1+i, \langle 2x-k-1 \,|\, y+z-1\rangle \,\Big|\, y+z-1 \right\rangle_{i=0}^{k-1} \right\rangle \,\Bigg|\, x+z+j \right\rangle_{j=0}^{y-x-2}$$

where $k = 0$ and $y-x-2 = 0$.

If it is in case (C), then $z < y-3$ and $x > y-3$. Since $z+1 < y-1$ and $x \geq y-1$, by induction,

$$G = \left\{x+z, \left\langle x+z+1+i, \langle y+x-k-3 \,|\, y+z-1\rangle \,\Big|\, y+z-1 \right\rangle_{i=0}^{k-1} \,\Bigg|\, x+z \right\},$$

where $k = \frac{y-z-4}{2}$ if $y-z$ is even and $k = \frac{y-z-3}{2}$ if $y-z$ is odd. Since $y-3 < x \leq y-2$, then $x = y-2$. Thus, $x+z = y+z-2$ and $y+x-k-3 = 2x-k-1$ and so

$$G = \left\{y+z-2, \left\langle y+z-1+i, \langle 2x-k-1 \,|\, y+z-1\rangle \,\Big|\, y+z-1 \right\rangle_{i=0}^{k-1} \,\Bigg|\, x+z \right\},$$

where $k = \frac{y-z-4}{2} = \frac{2x-y-z}{2}$ if $y-z$ is even and $k = \frac{y-z-3}{2} = \frac{2x-y-z+1}{2}$ if $y-z$ is odd. The left option to $y+z-2$ is dominated since the other option is at least $(y+z-1)*$. Thus, we have

$$G = \left\{ \left\langle y+z-1+i, \langle 2x-k-1 \,|\, y+z-1\rangle \,\Big|\, y+z-1 \right\rangle_{i=0}^{k-1} \,\Bigg|\, x+z \right\},$$

which we can rewrite as

$$G = \left\langle \left\langle y+z-1+i, \langle 2x-k-1 \,|\, y+z-1\rangle \,\Big|\, y+z-1 \right\rangle_{i=0}^{k-1} \,\Bigg|\, x+z \right\rangle_{j=0}^{y-x-2}$$

in which $k = \frac{2x-y-z}{2}$ if $y-z$ is even and $k = \frac{2x-y-z+1}{2}$ if $y-z$ is odd and $y-x-2 = 0$.

If it is in case (D), then $z < y-3$ and $x \leq y-3$. By induction,

$$G = \left\{x+z, \left\langle \left\langle \left\langle y+z-1+i, \langle 2x-k-1 \,|\, y+z-1\rangle \,\Big|\, y+z-1 \right\rangle_{i=0}^{k-1} \right\rangle \,\Bigg|\, x+z+1+j \right\rangle_{j=0}^{y-x-3} \,\Bigg|\, x+z \right\}$$

where $k = \frac{2x-y-z}{2}$ if $(y+z)$ is even and $k = \frac{2x-y-z+1}{2}$ if $(y+z)$ is odd.

Note, that in all cases , the move to the move to $B(x, y-1, z+1)$, is dominated. $\square$

Corollary 8 *(C) If $z < y-2$ and $x > y-2$, then*

$$A(x,y,z) = y+z-1+\Big\langle x-y+1+i, \langle x-z+1-k-2 \,|\, 0\rangle \,\Big|\, 0\Big\rangle_{i=0}^{k-1},$$

where $k = \frac{y-z-2}{2}$ if $y-z$ is even and $k = \frac{y-z-1}{2}$ if $y-z$ is odd where the second term has mean and temperature $\frac{x-y+1}{2}$.

(D) If $z < y-2$ and $x \le y-2$, then

(i) if $2x < y+z$ then

$$A(x,y,z) = x+z+\Big\langle \langle k+g\rangle \,\Big|\, i\Big\rangle_{i=0}^{k-1}$$

where $k = \frac{y-z-2}{2}$, $g = *$ if $(y-z)$ is even where the second term has mean and temperature are $1-\frac{1}{2^k}$; and $k = \frac{y-z-1}{2}$, $g = -\frac{1}{2}$ if $(y-z)$ is odd where the second term has mean and temperature are $1-\frac{3}{2^{k+1}}$;

(ii) if $2x \ge y+z$, then

$$A(x,y,z) = x+z+\Bigg\langle\Big\langle\Big\langle y-x-1+i, \langle x-z-k-1 \,|\, y-x-1\rangle \,\Big|\, y-x-1\Big\rangle_{i=0}^{k-1}\Big\rangle \,\Bigg|\, j\Bigg\rangle_{j=0}^{y-x-2}$$

where $k = \frac{2x-y-z}{2}$ if $(y+z)$ is even and $k = \frac{2x-y-z+1}{2}$ if $(y+z)$ is odd, where the second term has mean and temperature $1-\frac{1}{2^{y-x-1}}$.

Proof. The values are obtained by the number translation principle.

In case (C), recall that $G = \Big\langle x-y+1+i, \langle x-z+1-k-2 \,|\, 0\rangle \,\Big|\, 0\Big\rangle_{i=0}^{k-1}$ is a game of the form $\{8, \{9, \{10 \,|\, 0\} \,|\, 0\} \,|\, 0\}$. By induction, the game $H = \Big\langle x-y+1+i, \langle x-z+1-k-2 \,|\, 0\rangle \,\Big|\, 0\Big\rangle_{i=1}^{k-1}$ has temperature $\frac{x-y+2}{2}$. At temperature $t = \frac{x-y+1}{2}$, the values in the expansion of $G_t = \{x-y+1+i-t, H_t - t \,|\, t\}$ are, in sequence from the left, $x-y+1+i-i(\frac{x-y+1}{2})$ for the Left 'options' and $(2-i)\frac{x-y+1}{2}$ for the Right. No interior game has frozen since $x-y+1+i-i(\frac{x-y+1}{2}) > (2-i)\frac{x-y+1}{2}$, and the first term in G_t is $x-y+1-(\frac{x-y+1}{2})$ which is larger than all other terms in $H_t - t$. Therefore, the option to $H_t - t$ is dominated and $G_t = \{x-y+1+i-t \,|\, t\} = \frac{x-y+1}{2}*$ for $t = \frac{x-y+1}{2}$. Therefore the mean and temperatures are both $\frac{x-y+1}{2}$.

In case (D)(i), recall that the game $G = \Big\langle \langle k+g\rangle \,\Big|\, i\Big\rangle_{i=0}^{k-1}$ is a game of the form $\{3* \,|\, 2 \,||\, 1 \,|||\, 0\}$ or $\{\frac{5}{2}* \,|\, 2 \,||\, 1 \,|||\, 0\}$. Moreover,

$$\Big\langle \langle k+g\rangle \,\Big|\, i\Big\rangle_{i=0}^{k-1} = \Big\{\Big\langle \langle k+g\rangle \,\Big|\, i\Big\rangle_{i=1}^{k-1} \,\Big|\, 0\Big\} = \Big\{\Big\langle 1+\langle k+g-1\rangle \,\Big|\, j\Big\rangle_{j=0}^{k-2} \,\Big|\, 0\Big\}$$

If $y-z$ is even then the game is of the first form, $G = \left\{\left\langle 1 + \langle k+g-1\rangle \,\middle|\, j\right\rangle_{i=0}^{k-2} \,\middle|\, 0\right\}$ and by induction when $t = 1 - \frac{1}{2^k}$ then

$$G_t = \left\{1 + \left(1 - \frac{1}{2^k}\right) - t \,\middle|\, t\right\} = \left\{1 \,\middle|\, 1 - \frac{1}{2^k}\right\}$$

and for $s > 0$

$$G_{t+s} = \left\{1 - s \,\middle|\, 1 - \frac{1}{2^k} + s\right\}$$

which freezes when $t + s = 1 - \frac{1}{2^{k+1}}$.

Similarly, it can be shown that when $y-z$ is odd then the mean and temperature are $1 - \frac{3}{2^{k+1}}$.

In case (D)(ii), recall that the game

$$G = \left\langle \left\langle \left\langle y-x-1+i, \langle x-z-k-1 \,|\, y-x-1\rangle \,\middle|\, y-x-1\right\rangle_{i=0}^{k-1} \right\rangle \,\middle|\, j \right\rangle_{j=0}^{y-x-2}$$

is a game of the form $\{\{\{2,\{3,\{3\,|\,2\}|2\}\,|\,2\}|1\}\,|\,0\}$. Moreover,

$$\left\langle \left\langle \left\langle y-x-1+i, \langle x-z-k-1 \,|\, y-x-1\rangle \,\middle|\, y-x-1\right\rangle_{i=0}^{k-1} \right\rangle \middle| j \right\rangle_{j=0}^{y-x-2} =$$
$$\left\{ \left\langle \left\langle \left\langle y-x-1+i, \langle x-z-k-1 \,|\, y-x-1\rangle \,\middle|\, y-x-1\right\rangle_{i=0}^{k-1} \right\rangle \,\middle|\, j \right\rangle_{j=1}^{y-x-2} \,\middle|\, 0 \right\},$$

so that $G = \{H \,|\, 0\}$ where by induction, H has mean and temperature $1 - \frac{1}{2^{y-x-2}}$. Therefore, with $t = 1 - \frac{1}{2^{y-x-2}}$ and $s > 0$,

$$\begin{aligned} G_{t+s} &= \{H_t - (t+s) \,|\, t+s\} \\ &= \left\{1 - \frac{1}{2^{y-x-2}} - \left(1 - \frac{1}{2^{y-x-2}} + s\right) \,\middle|\, 1 - \frac{1}{2^{y-x-2}} + s\right\} \\ &= \left\{1 - s \,\middle|\, 1 - \frac{1}{2^{y-x-2}} + s\right\} \end{aligned}$$

so therefore the mean and temperature of G are $1 - \frac{1}{2^{y-x-1}}$. □

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