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ALGORITHMS FOR FINDING AND PROVING BALANCED Q^2 IDENTITIES

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Abstract

This paper has three major parts: (1) A search algorithm is presented for finding mod 2 identities involving certain sums of products of the the familiar quintuple product in one variable, called balanced Q^2 identities. The method involves row reduction of large matrices whose columns contain the mod series coefficients of the Q^2 terms. The fact that the search is finite is based on an invariant conjecture. These mod 2 identities are lifted to integer equations by forcing the appropriate signs. All such tentative Q^2 identities can be written as T^2 identities, where T is the familiar triple product of Jacobi. (2) A proof algorithm is presented, where the truth of a tentative T^2 identity is ascertained by expressing that identity as a linear combination of several (true) identities obtained by correctly choosing the values in a fundamental six parameter formula. (3) A massive implementation of both the search algorithm and the proof algorithm is discussed, including data, statistics, pitfalls, and an elaboration of some of the new identities found, including several infinite parametric families of four term Q^2 identities.

–Dedicated to our friend Ron Graham

1. Introduction

The present paper is the culmination of a 25 year investigation of identities involving infinite series and products. During this period we[1] have written ten papers about our findings and the elementary methods we use. We briefly outline some of the highlights of this work.

In 1980 we began a systematic search [5] for Ramanujan pairs, which we define to be pairs of infinite increasing sequences of positive integers, $\{a_i\}$, $\{b_j\}$, satisfying

[1]the second two authors and the late Irving Gerst.

$$\prod_{i=1}^{\infty} \frac{1}{1-x^{a_i}} = \sum_{j=1}^{\infty} \frac{b_j}{(1-x)\cdots(1-x^j)}.$$

This is essentially the definition given by George Andrews [1], whose purpose was to generalize the two well-known Rogers-Ramanujan identities, which are Ramanujan pairs with $a_i \equiv 1,4 \pmod 5$, $b_j = j^2$ and with $a_i \equiv 2,3 \pmod 5$, $b_j = j(j+1)$. Using a computer search we found two infinite classes of Ramanujan pairs, both known to Euler, and eight sporadic pairs, including the two identities of Rogers and Ramanujan. On examination we discovered that two of these eight identities had simple power series expansions modulo 2 [5, p. 732, Table 1]:

$$\prod_{\substack{n\equiv\pm1,\pm2,\pm5,\\ \pm6,\pm8,\pm9\\ \text{mod } 20}} \frac{1}{1-x^n} \equiv \sum_{n=1}^{\infty}(x^{n(5n-3)/2}+x^{n(5n+3)/2}) \pmod 2 \tag{1.1}$$

and

$$\prod_{\substack{n\equiv\pm2,\pm3,\pm4,\\ \pm5,\pm6,\pm7\\ \text{mod } 20}} \frac{1}{1-x^n} \equiv \sum_{n=1}^{\infty}(x^{n(5n-1)/2}+x^{n(5n+1)/2}) \pmod 2. \tag{1.2}$$

Since the left sides of these two congruences are generating functions of the number of partitions whose "modular" parts come from their indexing sets, we immediately obtain parity results for such partitions. Thus, while no one has found a simple description of the parity of the partition function $p(n)$, parity results can easily be obtained for such restricted partition functions. (See also [6], [8], and [10].)

We next began a search for identities similar to the two above. The search was done in both directions. One way, we looked at mod 2 series expansions whose form was like those on the right sides of (1.1) and (1.2) and attempted to factor them into recognizable products of linear factors whose index set consisted of residues modulo some integer m (called *modular-part products*). In the other direction, we computed the mod 2 series expansion of various modular-part products and identified those for which the exponents in the series were quadratically spaced, as in (1.1).

It often turned out that the mod 2 congruences could be lifted to actual equations over the integers when the signs in the linear factors were chosen judiciously. The following example [7, p. 301] is typical of this phenomenon: the congruence

$$\prod_{\substack{n\not\equiv5\\ \text{mod } 10}} (1-x^n) \equiv \sum_{n=0}^{\infty}(x^{n(n+1)}+x^{5n(n+1)+1}) \pmod 2 \tag{1.3}$$

lifts to the equation

$$\prod_{\substack{n\equiv0,\pm3\\ \text{mod } 10}} (1-x^n) \prod_{\substack{n\equiv\pm1,\pm2,\pm4\\ \text{mod } 10}} (1+x^n) = \sum_{n=0}^{\infty}(x^{n(n+1)}+x^{5n(n+1)+1}). \tag{1.4}$$

An observant reader will have noticed that the left side of (1.3) is an ordinary product rather than a reciprocal product like the one in (1.1). We explain how to transform a regular modular-part product into a reciprocal modular-part product mod 2 in [8], where parity matters are discussed at length. Using the methods of that paper, the product in (1.3) can be expressed as the reciprocal modular-part product

$$\prod_{\substack{n\not\equiv 5 \\ \text{mod } 10}} (1-x^n) \equiv \prod_{\substack{n\equiv\pm1,\pm3 \\ \text{mod } 10}} \frac{1}{1-x^n} \pmod 2. \tag{1.5}$$

Combining (1.3) and (1.5) gives parity results for the number of partitions whose parts are $\pm1, \pm3 \pmod{10}$.

Our investigations, past and present, have several themes:
(1) we are interested in equations expressing infinite products as infinte sums;
(2) many of the series have interesting partition interpretations;
(3) the series expansions are viewed mod 2;
(4) the products are modular-part products;
(5) the sums have quadratically spaced exponents;
(6) mod 2 congruences often lift to equations over the integers.

This work leads naturally to the one-variable triple product of Jacobi, which satisfies all five of the above requirements,

$$T(k,l) = \prod_{\substack{n\equiv 0,\pm(k-l) \\ \text{mod } 2k}} (1-x^n) = \sum_{-\infty}^{\infty} x^{ks^2+ls},$$

and the one-variable quintuple product

$$Q(m,k) = \prod_{\substack{n\equiv 0,\pm k \\ \text{mod } m}} (1-x^n) \prod_{\substack{n\equiv \pm(m-2k) \\ \text{mod } 2m}} (1-x^n) = \sum_{-\infty}^{\infty} x^{\frac{n(3n+1)m}{2}} \left(x^{-3nk} - x^{(3n+1)k}\right).$$

We found that many of the equations we wished to prove in various of our papers could be transformed into equations where each side consists of a sum of products of two T functions of the form $x^\alpha T(k_1,*)T(k_2,*)$, where the pair of values (k_1,k_2) is the same for each term in the equation. These *balanced T^2 identities*, as we called them, became more a central focus in what we were doing.

As a way of manufacturing T^2 identities, we found a general expansion formula [3, Th. 2] which under certain conditions expresses a power of x times the product of two T functions as a sum of powers of x times such products. Bruce Berndt pointed out that this expansion generalizes a formula for such a product by Schröter. The formula, expressed in Ramanujan's $f(a,b)$ notation can be found in [2, p. 73]. Berndt and Yesilyurt have recently employed this formula to provide elementary proofs of several of Ramanujan's "forty identities" [3].

In [11] we gave a fundamental 6-parameter formula, which generates balanced T^2 identities. This fundamental formula was used to give a new proof of the familiar Ramanujan identity

$H(x)G(x^{11}) - x^2G(x)H(x^{11}) = 1$, where $G(x) = \prod_{n=0}^{\infty}[(1-x^{5n+1})(1-x^{5n+4})]^{-1}$ and $H(x) = \prod_{n=0}^{\infty}[(1-x^{5n+2})(1-x^{5n+3})]^{-1}$, after we had first transformed the Ramanujan identity into a balanced T^2 equation, a general requirement to be able to use this formula in making a proof.

In [12, (1.4)] we took a combinatorial approach to proving a balanced T^2 identity by first writing each term as a double sum, viz.,

$$x^{\alpha}T(k_1,l_1)T(k_2,l_2) = \sum_{(i,j)\in Z^2} x^{k_1i^2+k_2j^2+l_1+l_2j+\alpha}.$$

In this form the T^2 identity asserts that a power of x in some double sum on the left side of the identity, generated at a point (i,j) in the indexing plane Z^2 for that sum, must be paired with the same power of x in some double sum on the right side of the identity, generated at a point (i',j') in the indexing plane Z^2 for that sum.

What we found from this rewriting were three rather remarkable results, the first two of which remain unproved: (1) The powers of x in a particular double sum on the left, which appear in a particular double sum on the right, are generated at index points (i,j) which lie on *affine lattices* in the indexing plane of the double sum on the left; (2) The quadratic exponents in the double sums of a T^2 identity form a family having a certain invariant; (3) A second proof of the 6-parameter formula generating balanced T^2 identities can be given, using the lattices and affine maps between them [13].

The first of these three results does not play a direct part in the algorithms of this paper. However, the second result is a conjecture on which the search algorithm is based and the Fundamental Formula in (3) is the heart of the proving algorithm.

In 1991 we began the study of balanced Q^2 identities with the publication of the identity

(1.6) $$Q(8,3)Q(56,7) + x^3Q(8,1)Q(56,21) = Q(8,2)Q(56,14)$$

in [9, (16)]. Since that time we have published two other trinomial identities, viz.,

(1.7) $$Q(14,3)Q(70,5) + x^3Q(14,1)Q(70,25) = Q(14,5)Q(70,15)$$

in [11, (4.8)] and

(1.8) $$Q(7,3)Q(35,5) + x^3Q(7,2)Q(35,15) = Q(7,1)Q(35,10)$$

in [12, (3.1)].

Proofs of (1.6) and (1.7) were given in [11, Theorem 3] and [11, Theorem 4] respectively, while a proof of (1.8) appeared in [12, Theorem 4]. These three proofs used the proof method involving the fundamental T^2 formula discussed in Section 5 of this paper, although in these proofs, we gave only results essential to the proofs, omitting a description of the underlying computational techniques that were used to produce those results. Using the Search Algorithm presented in the present paper, we found two other three-term identities: equations (8.2) and (8.3).

Initially we knew only a few balanced Q^2 identities and, because of the complicated nature of the general term of the series for Q, we thought there might not be many of them, especially with a large number of terms. This is no longer our opinion, since now we have a systematic way of finding them, a way that has produced nearly 100,000 such identities, some with a huge number of terms. In fact, we now know parametric families with four terms in them (Sec. 8), although as yet we don't know if there are more than just the five trinomials referred to above.

It is our purpose in this paper to give an account of both the search method we use to find balanced Q^2 identities and the proof method we use to prove them. In Sections 4 and 6 respectively, we illustrate the use of these two methods by finding and proving a minimal spanning set of Q^2 identities that, like (1.7), are balanced at (14, 70).

In Section 7 we describe a huge computer search for Q^2 identities balanced at (m_1, m_2), where $5 \leq m_1 \leq 100,\ m_1 \leq m_2 \leq 1000$, and give some statistics about various aspects of the search. We present information about what we found and how well our methods actually worked. In particular these data confirm our experience that Q^2 identities most often occur when m_1 divides m_2. The paper concludes with a presentation of some parametric families of four-term identities.

We should point out that we think the methods given in this paper may well be more important than actual results. In fact, the value of a particular Q^2 identity may be quite a bit smaller when it is one of a hundred thousand such identities than when it is only one among five such identities. Put differently, a seashell found on the top of a moutaitn is more valuable than one found among thousands on a beach.

In summary, our interest has been to show how we used mod 2 power series to find identities and how we used our proof method to show these identities are true.

2. Preliminaries

Recall from [11, p. 83] that the series for the triple product $T(k, l)$ is given by

$$T(k,l) = T(k,l;x) \stackrel{\text{def}}{=} \sum_{-\infty}^{\infty} x^{ks^2+ls}. \tag{2.1}$$

Also, from [9, p. 780], we have the reduction formulas for $T(k, l)$:

$$T(k,-l) = T(k,l) \quad \text{and} \quad T(k,l) = x^{k-l}T(k,2k-l), \tag{2.2}$$

which allow us to put $T(k, l)$ into *reduced* form, $0 \leq l \leq k$. We say a T^2 identity is *balanced* at a pair of positive half-integers (k_1, k_2) if each of its terms has the form

$$x^{\alpha}\, T(k_1,l_1)\, T(k_2,l_2), \tag{2.3}$$

where $l_1 + k_1,\ l_2 + k_2,\ \alpha \in \mathbb{Z}$ and $\alpha \geq 0$ (See [11, p. 83]).

Next, recall from [12, p. 1284] that the series for the quintuple product is given by

$$Q(m,n) = Q(m,n;x) \overset{\text{def}}{=} \sum_{-\infty}^{\infty} x^{s(3s+1)m/2}\left(x^{-3sn} - x^{(3s+1)n}\right). \tag{2.4}$$

We also have reduction formulas for $Q(m,n)$ [4, (13) and (14)]: If $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$, then

$$Q(m,-n) = -x^{-n}Q(m,n) \tag{2.5}$$

and

$$Q(m,n) = x^{2m-3n}Q(m,n-m). \tag{2.6}$$

From [11, (4.2)], there is a simple formula connecting Q to T, viz.,

$$Q(m,n) = T\left(\frac{3m}{2}, \frac{m}{2} - 3n\right) - x^n T\left(\frac{3m}{2}, \frac{m}{2} + 3n\right). \tag{2.7}$$

Using (2.5) and (2.6), we can put $Q(m,n)$ into *reduced* form, $0 \le n \le \frac{m}{2}$. We say a Q^2 identity is *balanced* at a pair of positive integers (m_1, m_2) if each of its terms has the form

$$x^{\alpha}\, Q(m_1,n_1)\, Q(m_2,n_2), \tag{2.8}$$

where n_1, n_2, $\alpha \in \mathbb{Z}$ and $\alpha \ge 0$ (See [9, p. 785]).

In this paper we will not consider identities involving the three other T and Q functions [4, Definitions 1 and 3], except in 7(c). We will assume in general that the T's and Q's in an identity are in reduced form and that the largest possible power of x has been divided through an identity so the powers of x are non-negative and at least two terms have $\alpha = 0$.

We often omit the word *balanced* to avoid excessive use of this word and say a T^2 (or Q^2) identity is *reducible* if it can be expressed as the sum of two or more T^2 (or Q^2) identities. Otherwise, we say it is *irreducible*.

3. Searching

In this section we give an algorithm for finding Q^2 identities that are balanced at a particular (m_1, m_2). In order to make the search finite and efficient, we require that all terms of an identity have the same invariant value, called I_Q. The scheme of the search is to expand these Q^2 terms into a power series mod 2 up to a sufficiently large degree and then use Gaussian elimination to find a basis for the set of all mod 2 identities. We then attempt to lift each mod 2 identity to an actual identity with ± 1 coefficients.

(a) Invariant Conjecture.

We begin with some material that deals with T^2 identities. In [13, (1.4)], a term of a T^2 identity balanced at (k_1, k_2) was written as the double sum

$$x^\alpha T(k_1, l_1) T(k_2, l_2) = \sum_{(i,j)\in Z^2} x^{k_1 i^2 + k_2 j^2 + l_1 i + l_2 j + \alpha}. \tag{3.1}$$

For the exponent polynomial $k_1 x^2 + k_2 y^2 + l_1 x + l_2 y + \alpha$, define the *invariant* function I_T by

$$I_T = I_T(k_1, k_2, l_1, l_2, \alpha) \overset{\text{def}}{=} k_1 l_2^2 + k_2 l_1^2 - 4k_1 k_2 \alpha. \tag{3.2}$$

From the discussion in [13, Sec. 5], we derive the following conjectured necessary condition that the terms of a T^2 identity should satisfy.

Conjecture I_T: In a balanced, irreducible T^2 identity, the value I_T is the same for each term of the identity.

We can extend this conjecture to a similar one for Q^2 identities. Let the function I_Q be defined for a term $x^\alpha Q(m_1, n_1) Q(m_2, n_2)$ by

$$I_Q = I_Q(m_1, m_2, n_1, n_2, \alpha) \overset{\text{def}}{=} \frac{3}{8}\left[m_1(m_2 - 6n_2)^2 + m_2(m_1 - 6n_1)^2\right] - 9m_1 m_2 \alpha. \tag{3.3}$$

According to the Remarks in Sec. 2, m_1, m_2 are positive integers, the pair of integers (n_1, n_2) is in $\left[0, \frac{m_1}{2}\right] \times \left[0, \frac{m_2}{2}\right]$, and $\alpha \geq 0$. Then Conjecture I_T implies that the following conjecture is a necessary condition that the terms of a Q^2 identity should satisfy.

Conjecture I_Q: In a balanced, irreducible Q^2 identity, the value I_Q is the same for each term of the identity.

To see this, observe first that (2.7) gives the formula

$$\begin{aligned} x^\alpha Q(m_1, n_1) Q(m_2, n_2) = x^\alpha &\left[T\left(\frac{3m_1}{2}, \frac{m_1}{2} - 3n_1\right) - x^{n_1} T\left(\frac{3m_1}{2}, \frac{m_1}{2} + 3n_1\right)\right] \cdot \\ &\left[T\left(\frac{3m_2}{2}, \frac{m_2}{2} - 3n_2\right) - x^{n_2} T\left(\frac{3m_2}{2}, \frac{m_2}{2} + 3n_2\right)\right], \end{aligned} \tag{3.4}$$

which, when multiplied out, associates a sum of four T^2 terms with each Q^2 term. In this way, a Q^2 identity balanced at (m_1, m_2) is transformed into a T^2 identity balanced at $\left(\frac{3m_1}{2}, \frac{3m_2}{2}\right)$. The value I_T for the four T^2 terms is I_Q as stated in (3.3), which establishes that Conjecture I_Q is implied by Conjecture I_T. Conjecture I_Q is central to the Search Algorithm, because it makes the search for identities balanced at a given (m_1, m_2) finite.

(b) The Search Algorithm.

Step 1. [**Initialization**] Choose $m_1, m_2 \in \mathbb{Z}^+$, where $5 \leq m_1 \leq m_2$. Choose a large value for the parameter L. The value $L = 10,000$ is adequate for small values of m_1 and m_2.

Step 2. [**Construct families of triples**] For each pair of integers $(n_1, n_2) \in \left(0, \frac{m_1}{2}\right) \times \left(0, \frac{m_2}{2}\right)$, use (3.3) to compute the three values

$$I_0 = I_Q(m_1, m_2, n_1, n_2, 0), \quad \alpha = \left\lfloor \frac{I_0}{9m_1m_2} \right\rfloor, \quad \text{and} \quad I = I_0 - 9m_1m_2\alpha.$$

Include the triple (α, n_1, n_2) in a family C_I of previously found triples with the same value of I.

Step 3. [**Process each family** C_I] For each family C_I found in Step 2 that contains at least two triples with the same value of α, carry out the remaining Steps 4 – 12 of this algorithm.

Step 4. [**Construct a bit string for each member of** C_I] Find the set of integer-coefficient Maclaurin expansions of the Q^2 products (2.8) associated with each triple in C_I. Reduce each of these expansions mod 2 and make the first L coefficients into a bit string.

To carry out this step efficiently, first write the triple as the (mod 2) sum of four T^2 terms as in (3.4). Using (3.1), each of these bit strings can be built by adding a 1 mod 2 into an initialized string of L zeros at position $t = k_1i^2 + k_2j^2 + l_1i + l_2j + \alpha$ for each pair $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ for which $t < L$. Adding these four bit strings together mod 2 gives the desired bit string.

Step 5. [**Form the coefficient matrix**] Form the $L \times n$ matrix A that has the respective bit strings constructed in Step 4 as its columns. Here n is the number of triples in C_I.

Step 6. [**Row reduction**] Put matrix A into row-reduced, echelon form mod 2. Let B be the set of basis vectors of the null space of A. If B is empty, return to Step 4.

Step 7. [**Joint reduction**] Transform basis B in Step 6 into a basis $\widehat{B}$ of S via a sequence of bases $B_0 = B,\ B_1, \cdots, B_j,\ B_{j+1}, \cdots, \widehat{B}$. To derive B_{j+1} from B_j, $j \geq 0$, first re-order the vectors in B_j (getting the set $\bar{B}_j$) so that the number of 1's in successive vectors in $\bar{B}_j$ forms a non-decreasing sequence. Then compute the mod 2 sums of the first vector $\mathbf{v}_1$ in $\bar{B}_j$ with each vector that follows it, the second vector $\mathbf{v}_2$ in $\bar{B}_j$ with each vector that follows it, and so on, until a sum $\mathbf{v}_r + \mathbf{v}_{r+k}$, is found which has fewer 1's in it than the vector $\mathbf{v}_{r+k}$ itself. This sum then replaces $\mathbf{v}_{r+k}$ in $\bar{B}_j$ giving the next basis B_{j+1}. Eventually, when no such sum is found, the process ends, and the final set of vectors arrived at is the desired basis $\widehat{B}$.

Step 8. [**List the potential mod 2 identities**] For each vector $\mathbf{v} \in \widehat{B}$, collect into a set $T_{\mathbf{v}}$ the ith triples of the family C_I for each i for which the ith position in $\mathbf{v}$ is a 1. The mod 2 sum of the series expansion of $x^\alpha Q(m_1, n_1)Q(m_2, n_2)$ for each triple (α, n_1, n_2) in $T_{\mathbf{v}}$, truncated at degree L, is zero, and thus $T_{\mathbf{v}}$ represents a potential mod 2 identity in which the series are extended to infinity.

Step 9. [Eliminate linear identities] If the first components or second components of all the vectors of a $T_{\mathbf{v}}$ are the same when $m_1 < m_2$, then discard $T_{\mathbf{v}}$. If $m_1 = m_2$ eliminate all $T_{\mathbf{v}}$ where there is a value n such that all triples in $T_{\mathbf{v}}$ have n as either the first or second component.

Step 10. [Ignore non-primitive identities] If in a $T_{\mathbf{v}}$ surviving Step 9 the greatest common divisor d of m_1, m_2, and every n_1, n_2 ranging over all the triples of this particular $T_{\mathbf{v}}$ is greater than 1, then discard this $T_{\mathbf{v}}$.

Step 11. [Start series at degree 0] In each $T_{\mathbf{v}}$ surviving Step 10, subtract the smallest value of α in the triples from the α value in each triple.

Step 12. [Lift to $\mathbb{Z}$] Let N be a conveniently large integer. Process each $T_{\mathbf{v}}$ surviving Step 11 by the following steps:

(i) For each triple in $T_{\mathbf{v}}$, construct an N-dimensional vector of integers whose components are the first N coefficients of the Maclaurin expansion of the Q^2 term associated with this triple. Construct a matrix K with these vectors as its rows.

(ii) Assign a plus sign to row 1. Then repeat the following steps until all rows of K are zero:

> Examine the non-zero entries in row 1 of K until an entry k_{1c} is found whose absolute value equals the sum of the absolute values of the other entries in column c. If $k_{1c} > 0$, then add to (resp. subtract from) row 1 the rows of K that have a negative (resp. positive) entry in colum c. If $k_{1c} < 0$, then do the opposite. After a row is combined with row 1, zero out that row in K and attach a plus or minus sign to its row number if the row was respectively added to or subtracted from row 1.

(iii) Output the set of row numbers with attached signs as a Q^2 identity, the terms with the same sign being being placed on their respective side of an equals sign.

(c) Comments on the Search Algorithm.

Step 1. We state that $m_1 \geq 5$, because there are no values n_1 when $m_1 = 1, 2$ and only one value, $n_1 = 1$, when $m_1 = 3, 4$. In the latter case, each term in any identity produced has a common $Q(m_1, 1)$ that is removed by Step 9.

Choosing L too small will not lose any true identities (because any true identity holds mod x^L) but may introduce some bogus identities.

Step 2. The intervals in this step imply the algorithm finds Q's in their reduced form. These intervals are also open because at the endpoints the value of a Q factor is zero (cf. [4, (15)]).

Also, each invariant I_0 falls in the interval $[\,0,\, 3m_2^{\,3}\,)$. We choose the value of α, however, so that I is in $[0, 9m_1m_2)$. There is no loss in doing this because an identity with invariant

$I_0 = I + 9m_1m_2\alpha$ will also appear as an identity with invariant I, with the exception that the identity with invariant I will have a factor of x^α that cancels out of each term.

Moreover note that the Invariant Conjecture makes the search finite. Without it, there is no upper bound on the value of α.

Finally, the values of I_0 (and hence I) need not be integers. They may be half-integers or quarter-integers.

Step 3. Each triple in a C_I is associated with its product (2.8), a possible term in a Q^2 identity balanced at (m_1, m_2); and since $(n_1, n_2) \in \left(0, \frac{m_1}{2}\right) \times \left(0, \frac{m_2}{2}\right)$, the value α is the minimum exponent occurring in the power series representation of $x^\alpha\, Q(m_1, n_1)\, Q(m_2, n_2)$. Consequently, any identity among these reduced triples in C_I must contain two terms with the same α.

Step 4. It is useful to note for a reduced $T(k, l)$ with $0 \le l \le k$, the quadratic exponent function $v(n) = kn^2 + ln$ is a monotone increasing sequence $v(0) \le v(-1) \le v(1) \le v(-2) \le v(2) \le \cdots$.

Step 6. The idea here is to look for dependencies among the Q^2 terms in the family by finding dependencies among the first L terms of their series. It is clear that the sum or difference of two Q^2 identities balanced at (m_1, m_2) will be another Q^2 identity of the same kind.

Step 7. The B_i's produced in this step are clearly all bases of S. The basis $\widehat{B}$ contains vectors with a reduced number of 1's in them, so the identities associated with them have fewer terms. The vectors in $\widehat{B}$ also have fewer 1's in common with each other, so their associated identities have fewer terms in common with each other. Further, the sequence of basis transformations as well as the basis $\widehat{B}$ are not necessarily unique. These depend on the sorting algorithm. This step can be skipped if all that is desired is a basis from which all mod 2 identities can be constructed.

Step 8. Each vector $\mathbf{v} \in \widehat{B}$ has at least two 1's in it.

Step 9. When this algorithm was first programmed and run, we had no idea how many tentative Q^2 identities might be produced. We were hardly prepared, however, for the deluge of identities that came pouring out. An inspection of the results showed that almost all of them were trinomials each of whose terms had a common, non-zero factor Q. When this factor was divided through the identity, the resulting Q identity was linear, and so could be ignored.

As it happens, each three-term linear identity that was found in this way was an instance of the following balanced two-parameter family (see [4, (35)]): For $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$, we have

$$Q(3m, n) = Q(3m, m - n) - x^n Q(3m, m + n). \tag{3.5}$$

This linear family provides us with an opportunity to test Conjecture I_T. If we multiply such an identity through by $Q(r, s)$ for any $r \in \mathbb{Z}^+$ and $s \in \mathbb{Z}$, then Conjecture I_T should hold for each of the balanced Q^2 identities. That it does is readily verified.

Step 10. An identity with smaller values exists by sending $x^d \to x$.

Step 11. This step changes the invariant value when the smallest value of α is positive.

Step 12. It should be pointed out in the final step of the algorithm that the procedure is designed only to produce identities with coefficients that are ± 1. As a result, many mod 2 series do not lift to an identity of this kind. (See Section 7c.)

At the end of this step, either a tentative identity is made or it is not, and either all triples in the mod 2 identity are used or they are not. There are, therefore, four outcomes of interest to us.

(1) If a tentative identity is made and all triples are used, we consider this successful and proceed to the Proof Algorithm.

(2) If no identity is made but all triples are used, the mod 2 identity cannot be lifted to a tentative integer identity using ± 1 coefficients.

(3) If a tentative identity is made but not all triples in the mod 2 identity are used, then the given mod 2 identity is reducible. In this case we use only the triples that make the identity and ignore the other triples. (These other triples form a mod 2 identity and could also be sent to Step 9 for processing, but we do not do this in our search.)

(4) If no identity is produced and not all triples are used, then there is no value k_{1c} that can be found in the current range of exponents. So we increase the number of exponents and try again. In practice, the value of N was initially chosen to be 500. When this proved insufficient, it was replaced by $L = 10000$.

Steps 4-8 and 12. These steps constitute an approach which may be used to search for possible dependencies in a given set of power series with integer coefficients regardless of their origin. For example, we may use these steps to search for tentative T^2 identities.

4. The Search Algorithm in Action ... an Example

To illustrate the Search Algorithm, we find all tentative Q^2 identities balanced at $(m_1, m_2) = (14, 70)$. These are listed in Table 1 using the abbreviation

$$(\alpha, n_1, n_2) \stackrel{def}{=} x^\alpha Q(m_1, n_1) Q(m_2, n_2). \tag{4.1}$$

The invariant value I in the first column indicates the family from which the identity came.

Table 1. Q^2 Identities Balanced at (14, 70)

#	I	Q^2 Identity Balanced at (14, 70)
1	0	(0,3,5)+(3,1,25) = (0,5,15)
2	441	(0,2,13)+(1,5,8)+(10,4,33) = (0,3,12)+(1,1,18)+(3,6,3)
3	441	(0,3,2)+(1,5,22)+(3,2,27) = (0,6,17)+(1,4,23)+(7,1,32)
4	1449	(0,2,9)+(1,2,19)+(2,4,21) = (0,3,14)+(1,5,16)+(5,5,26)
5	1764	(0,3,9)+(3,5,1)+(7,5,29)+(8,1,31) = (0,1,11)+(1,3,19)
6	2205	(0,1,10)+(1,4,5)+(5,6,25)+(7,3,30) = (0,2,15)+(2,5,20)
7	2709	(0,1,14)+(1,3,4)+(3,3,24) = (0,4,11)+(3,6,21)+(8,4,31)
8	3969	(0,1,4)+(0,5,6)+(1,2,1)+(3,3,26) = (0,4,19)+(0,6,11)
9	3969	(0,4,9)+(6,2,29)+(9,6,31)+(11,5,34) = (0,3,16)+(3,1,24)
10	4221	(0,1,8)+(1,6,13)+(6,5,28) = (0,2,7)+(2,1,22)+(6,6,27)
11	6741	(0,4,7)+(2,2,23)+(9,2,33) = (0,5,12)+(2,5,2)+(5,3,28)
12	7056	(0,1,17)+(1,1,3)+(2,3,23) = (0,5,13)+(5,5,27)+(9,3,33)
13	7749	(0,1,6)+(1,2,21)+(3,6,1) = (0,5,14)+(7,6,29)+(10,1,34)
14	8001	(0,3,18)+(1,4,3)+(5,1,28) = (0,4,17)+(1,6,7)+(8,3,32)

Step 1. Put $m_1 = 14$ and $m_2 = 70$.

Step 2. Compute the families of triples C_I. It turns out there are 63 of these here.

Step 3. In this example, we will work through Steps 4 – 12 for the family C_{441}, a family containing the 12 triples listed in Table 2. The second and third identities in Table 1 come from these.

Table 2. Q^2 Terms for Family $I = 441$.

$$R_i(x) = (\alpha_i, n_{1i}, n_{2i}) = x^{\alpha_i} Q(14, n_{1i}) Q(70, n_{2i})$$

i	α_i	n_{1i}	n_{2i}
1	0	2	13
2	0	3	12
3	1	1	18
4	1	5	8
5	2	3	2
6	2	6	17

i	α_i	n_{1i}	n_{2i}
7	3	4	23
8	3	5	22
9	3	6	3
10	5	2	27
11	9	1	32
12	10	4	33

Steps 4, 5, and 6. We need to compute the mod 2 coefficients of the series expansion of each term R_i, to be placed in the ith column of the series matrix A. For the first column of A, using (3.4) and (2.1) we find that

$$R_1(x) = Q(14,2)Q(70,13) \equiv 1 + x^2 + x^{10} + x^{13} + x^{15} + x^{20} + x^{22} + x^{23} + \cdots \pmod 2.$$

Thus, the first column of A begins $1,0,1,0,0,0,0,0,0,0,1,0,0$, while the next eleven columns respectively contain the coefficients of the series $R_2, \ldots, R_{12}$. The first 13 rows of A are displayed below, followed by its mod 2 row-reduced form in which all rows after the twelfth are zero.

$$\begin{bmatrix}
1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0\\
1&0&1&0&1&1&0&0&0&0&0&0\\
0&1&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&1&1&0&0&0&1&1&0&0\\
0&0&0&1&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&1&1&0&1&0&0\\
0&1&0&0&0&1&0&1&1&0&0&0\\
0&0&0&1&0&0&1&0&1&0&1&0\\
1&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&1&1&0&0&1&0&0&0
\end{bmatrix}
\longrightarrow
\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&1\\
0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&1&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

(Computing the series to degree L is usually overkill, as in this example where only the first thirteen rows are needed to row reduce it correctly.) By standard linear algebra techniques, we find that the independent columns are 12 and 11, which yield the two 12-component vector basis $B =$
$\{(1,1,1,1,0,0,0,0,1,0,0,1)^T,\ (0,0,0,0,1,1,1,1,0,1,1,0)^T\}$ for the null space of A.

Step 7. Since the two vectors in B are already orthogonal, this step is unnecessary here. Hence $B = \widehat{B}$.

Step 8. These give the two tentative mod 2 identities:

$$R_1 + R_2 + R_3 + R_4 + R_9 + R_{12} \equiv 0 \pmod 2 \tag{4.2}$$

and

$$R_5 + R_6 + R_7 + R_8 + R_{10} + R_{11} \equiv 0 \pmod 2. \tag{4.3}$$

Step 9. Since neither of these mod 2 identities have the same value for all n_1 or n_2, neither will be discarded as a linear identity.

Step 10. For (4.2), $\gcd(m_1, n_{12}) = \gcd(14, 3) = 1$, so it is primitive. For (4.3), $\gcd(m_1, n_{15}) = 1$, so it is also primitive.

Step 11. This step applies only to (4.3). Since all terms have $\alpha_i \geq 2$, we divide all of its terms by x^2.

Step 12. For the purposes of this example, we consider only (4.2). This step to of the algorithm transforms the mod 2 congruence (4.2) into an actual equation over the integers. We first expand the six Q^2 terms in (4.2) into power series using (3.4). Here we use only the terms up to degree L. As it happens, the dependency between these series can be found using only a small number of low-degree terms. Here the six series are listed below up to degree 40. Our problem is to place these on one or the other side of an equals sign so the resulting series on the two sides agree up to degree L. (It is worth mentioning that the coefficients of the series are all very small due in part to their having been generated by a quadratic exponent.)

$$R_1 = Q(14,2)Q(70,13) = 1 - x^2 - x^{10} - x^{13} + x^{15} + x^{20} + x^{22} + x^{23} - x^{33} - x^{35} - x^{36} + \cdots$$
$$R_2 = Q(14,3)Q(70,12) = 1 - x^3 - x^8 - x^{12} + x^{15} + x^{19} + x^{20} + x^{23} - x^{31} - x^{35} - x^{40} \cdots$$
$$R_3 = xQ(14,1)Q(70,18) = x - x^2 - x^{13} + x^{18} - x^{19} + x^{20} + x^{26} + x^{31} - x^{33} - x^{35} + \cdots$$
$$R_4 = xQ(14,5)Q(70,8) = x - x^5 - x^6 - x^9 + x^{13} + 2x^{14} - x^{22} + x^{30} - x^{38} + \cdots$$
$$R_9 = x^3Q(14,6)Q(70,3) = x^3 - x^5 - x^6 + x^8 - x^9 + x^{12} + x^{13} - x^{16} + x^{35} - x^{38} + \cdots$$
$$R_{12} = x^{10}Q(14,4)Q(70,33) = x^{10} - 2x^{14} - x^{16} + x^{18} + x^{20} + x^{26} - x^{30} + x^{36} - x^{40} + \cdots$$

We begin our search for an equation whose mod 2 reduction is (4.2) by choosing $S = R_1$, because its first nonzero term has the minimum exponent 0. Since R_2 is the only other series that begins with 1, we are forced to subtract R_2 to "balance" the zero degree terms. So far

$$S = R_1 - R_2 = -x^2 + x^3 + x^8 - x^{10} + x^{12} - x^{13} - x^{19} + x^{22} + x^{31} - x^{33} - x^{36} + x^{40} + \cdots$$

Next, the series for R_3 is the only remaining series with a non-zero term $-x^2$, forcing us to subtract R_3 from S:

$$S = R_1 - R_2 - R_3 = -x + x^3 + x^8 - x^{10} + x^{12} - x^{18} - x^{20} + x^{22} - x^{26} + x^{35} - x^{36} + x^{40} + \cdots$$

The leading term $-x$ in S is now matched only in R_4, forcing us to add series R_4 to S:

$$S = R_1 - R_2 - R_3 + R_4 = x^3 - x^5 - x^6 + x^8 - x^9 - x^{10} + x^{12} + x^{13} + 2x^{14} - x^{18} - x^{20} - x^{26} + x^{30} + x^{35} - x^{36} - x^{38} + x^{40} + \cdots$$

It is now clear from the terms x^3 and $-x^{10}$ that we must subtract R_9 from S and then add series R_{12}, which ends up giving us a zero series out to degree L. (This can readily be checked here to degree 40.)

We now have our first tentative Q^2 identity:

$$R_1 + R_4 + R_{12} = R_2 + R_3 + R_9, \tag{4.4}$$

which is line 2 of Table 1. Similarly, if we use the lifting algorithm in Step 12 on the second tentative mod 2 identity, we get

$$R_5 + R_8 + R_{10} = R_6 + R_7 + R_{11}, \tag{4.5}$$

which is line 3 of Table 1.

5. Proving

The tentative Q^2 identities (over $\mathbb{Z}$) found by the search algorithm have been verified up to degree L by a power series verification. It is reasonable, therefore, to expect they will be found to be true by the method of this section.

The preliminary step in the proof method is to transform the tentative Q^2 identity into a tentative T^2 identity by replacing each Q^2 term by a sum of four T^2 terms. The proof algorithm then uses the following 6-parameter Fundamental T^2 Formula to produce a set of (true) T^2 identities that, when multiplied by appropriate powers of x and summed, produces the tentative T^2 identity. Put differently, identities generated from the Fundamental T^2 Formula "span" a space of identities, inside of which our tentative identity is shown to lie.

(a) The Fundamental T^2 Formula.

In a previous paper [11, Theorem 1] we introduced the following.

Theorem 5.1 (Fundamental T^2 Formula)**.** *Suppose that $m,\ u,\ v \in Z^+$ and that $e,\ f,\ k \in Q$, where $uv < 2m$. Then*

$$\sum_{n\in R_m} x^{\alpha_n} T(k_1, l_{1n})T(k_2, l_{2n}) = \sum_{n\in R'_m} x^{\alpha_n} T(k_1, l'_{1n})T(k_2, l'_{2n}), \tag{5.1}$$

where

$$\alpha_n = \frac{2vk}{m}n^2 + 2en, \qquad k_1 = uk, \qquad k_2 = (2m-uv)vk,$$

$$\begin{cases} l_{1n} = \dfrac{2uvk}{m}n + ue + f \\ l_{2n} = (2m-uv)\Big(\dfrac{2vk}{m}n + e\Big) - vf \end{cases} \qquad \begin{cases} l'_{1n} = \dfrac{2uvk}{m}n + ue - f \\ l'_{2n} = (2m-uv)\Big(\dfrac{2vk}{m}n + e\Big) + vf, \end{cases}$$

and R_m and R'_m are any complete residue systems (mod m).

At present no formula is known that produces balanced Q^2 identities in the way that Theorem 5.1 produces balanced T^2 identities. Consequently, the first step in the proof method is to use (3.4) to transform a tentative Q^2 equation balanced at (m_1, m_2) into a tentative T^2 equation balanced at $(k_1, k_2) = (\frac{3}{2}m_1, \frac{3}{2}m_2)$. The resulting T^2 equation is then proved using a combination of identities produced by Theorem 5.1.

Before we can describe how all possible choices of the parameters in Theorem 5.1 can efficiently be found, we give some technical lemmas about the parameters m, u, v, k, e, f.

Lemma 5.1. *In Theorem 5.1, we have (i) $um \mid 4vk_1$, (ii) $m \mid 2vk_1$, and (iii) $m \mid 2k_2$.*

Proof. Since $\alpha_n \in \mathbb{Z}$ for all n, we have (i) $\alpha_1 + \alpha_{-1} = \dfrac{4vk}{m} = \dfrac{4vk_1}{um} \in \mathbb{Z}$. Also, we know that l_{jn} must have the same half-parity as k_j for all n and $j = 1, 2$.. Hence, (ii) $l_{11} - l_{10} = \dfrac{2vk_1}{m} \in \mathbb{Z}$ and (iii) $l_{21} - l_{20} = \dfrac{2k_2}{m} \in \mathbb{Z}$. □

Lemma 5.2. *In Theorem 5.1, we have the five inclusions:* $e \in \frac{1}{4}\mathbb{Z}$, $\frac{2vk}{m} + 2e \in \mathbb{Z}$, $f \in \frac{1}{2}\mathbb{Z}$, $ue + f + k_1 \in \mathbb{Z}$, *and* $(2m - uv)e + vf + k_2 \in \mathbb{Z}$.

Proof. Since $\alpha_n \in \mathbb{Z}$ for all n, we have, $\alpha_1 - \alpha_{-1} = 4e \in \mathbb{Z}$ and $\alpha_1 = \frac{2vk}{m} + 2e \in \mathbb{Z}$. Furthermore, for all n and j =1,2, we know that l_{jn} and l'_{jn}, have the same half-parity as k_j, i. e., their sum and difference are integers. In particular, $l_{11} - l'_{11} = 2f \in \mathbb{Z}$, $l_{10} + k_1 = ue + f + k_1 \in \mathbb{Z}$, and $l'_{20} + k_2 = (2m - uv)e + vf + k_2 \in \mathbb{Z}$. □

Lemma 5.3. *The identities obtained by using* (e, f), $(-e, f)$, $(e, -f)$, $\left(\frac{2vk}{m} + e, f\right)$, *and* $\left(\frac{2vk}{m} - e, f\right)$ *in Theorem 5.1 are all the same except for a possible factor of a power of* x.

Proof. Replacing f by $-f$ switches (l_{1n}, l_{2n}) and (l'_{1n}, l'_{2n}) in Theorem 5.1. The proof for replacing e by $-e$ requires the following relationships: $l_{1n}(-e, f) = -l'_{1,-n}(e, f)$, $l_{2n}(-e, f) = -l'_{2,-n}(e, f)$, $l'_{1n}(-e, f) = -l_{1,-n}(e, f)$, $l'_{2n}(-e, f) = -l_{2,-n}(e, f)$, and $\alpha_n(-e, f) = \alpha_{-n}(e, f)$. The l values obtained for $(-e, f)$ can be reduced using the rule at (2.2) which does not affect the value of α_n. Finally, $-n$ can be replaced by n in summing over equivalent residue systems (mod m).

Next we consider the case of replacing e by $e + \frac{2vk}{m}$. Doing this we see that $\alpha_n\left(e + \frac{2vk}{m}, f\right) = \alpha_{n+1}(e, f) - \frac{2vk}{m}$, $l^{(\prime)}_{1n}\left(e + \frac{2vk}{m}, f\right) = l^{(\prime)}_{1,n+1}(e, f)$, and $l^{(\prime)}_{2n}\left(e + \frac{2vk}{m}, f\right) = l^{(\prime)}_{2,n+1}(e, f)$. Hence the identities produced from these two points are the same with the exception that the identity produced from $\left(e + \frac{2vk}{m}, f\right)$ must be multiplied by $x^{2vk/m}$. Replacing e with $-e$ in this last case completes the proof. □

Lemma 5.4. *All terms in* (5.1) *have an invariant value* I_T *which satisfies the equation*

$$k_2 u^2 e^2 + k_1 v^2 f^2 = \frac{uvI_T}{2m}. \tag{5.2}$$

Proof. We prove this for the $n = 0$ term on the left side of (5.1) by routine reduction of

$$\begin{aligned} I_T = k_1 l_{20}^2 + k_2 l_{10}^2 - 4k_1 k_2 \alpha_0 &= uk[(2m - uv)e - vf]^2 + (2m - uv)vk(ue + f)^2 \\ &= 2m[(2m - uv)kue^2 + vkf^2]. \end{aligned}$$

This implies

$$\frac{uvI_T}{2m} = (2m - uv)vku^2e^2 + ukv^2f^2 = k_2u^2e^2 + k_1v^2f^2.$$

The right-hand terms are produced by replacing f by $-f$ which leaves the desired relationship unchanged. The case for the nth left-hand term is left to the reader. □

(b) The Proof Algorithm

Steps 2 and 3 of the following algorithm are written in pseudo-code, where sub-levels are indicated by indentation.

Step 1. [Initialization] Input the tentative T^2 equation balanced at (k_1, k_2). Let $\widehat{I}$ be the I_T value of each term of this given equation, reduced modulo $4k_1k_2$. Let $\widehat{\alpha} = (I_T - \widehat{I})/4k_1k_2$. Multiply the input identity by $x^{\widehat{\alpha}}$ and reduce each term using (2.2).

Step 2. [Find all global parameters] Find all the possible values of the integer parameters m, u, v and rational k in Theorem 5.1 which give the values k_1, k_2 using the following algorithm written in pseudo-code.

for $m = 1$ to $2k_2$ such that $m \,|\, 2k_2$
  for $u = 1$ to $2m - 1$
    for $v = 1$ to $\left\lfloor \dfrac{2m-1}{u} \right\rfloor$
      if $mu \,|\, 4vk_1,\ m \,|\, 2vk_1,$ and $uk_2 = (2m - uv)vk_1$
      then append $\left(m, u, v, k = \dfrac{k_1}{u}\right)$ to the list of global parameters.

Step 3. [Find the local parameters e and f] For each m, u, v, k produced in the previous step, construct a list of triples (e, f, α) which produce an identity with the invariant I_T.

for $e = 0$ to $\dfrac{vk}{m}$, where $e \in \dfrac{1}{4}\mathbb{Z}$
  for $\alpha = 0$ to $\left\lfloor \dfrac{k_1 + k_2}{4} \right\rfloor - 1$
    $I_T = \widehat{I} + 4k_1k_2\alpha$
    $f = \sqrt{\dfrac{I_T}{2mvk} - \dfrac{(2m - uv)ue^2}{v}}$
    if $f \in \dfrac{1}{2}\mathbb{Z},\ \dfrac{2vk}{m} + 2e \in \mathbb{Z},\ ue + f + k_1 \in \mathbb{Z},$ and $(2m - uv)e - vf + k_2 \in \mathbb{Z}$
    then append the triple (e, f, α) to the list of triples for m, u, v, k

Step 4. [List all T^2 terms with invariant $\widehat{I}$] For each $(l_1, l_2) \in [0, k_1] \times [0, k_2]$ with l_i having the same half-parity as k_i, $i = 1, 2$, compute the invariant $I_0 = I_T(k_1, k_2, l_1, l_2, 0)$ using (3.2). If $I_0 \mod 4k_1k_2$ is the same as $\widehat{I}$, then append the triple $\left(\left\lfloor \dfrac{I_0}{4k_1k_2} \right\rfloor, l_1, l_2\right)$ to the list R.

Step 5. [Construct a matrix with all identities encoded] Let n be the number of triples found in R. We define M to be the $m \times n$ matrix whose rows have each different identity arising from Theorem 5.1 encoded. (At the end of this step, m will be the number of different identities coming from Theorem 5.1.) To begin, let M be empty. For each m, u, v, k from Step 2 and each (e, f, α) from Step 3 do the following.

> Let $\mathbf{v}$ be a length-n vector of integers and let $\mathcal{I}$ be the identity produced from m, u, v, k, e, f from Theorem 5.1. Multiply $\mathcal{I}$ by x^α and reduce each term using (2.2). (This ensures that each term in $\mathcal{I}$ is found in the list R.) Set the ith entry in $\mathbf{v}$ to be the number of times the ith triple in R is found on the left-hand side of $\mathcal{I}$ minus the number of times it is found on the right-hand side of $\mathcal{I}$. If either $\mathbf{v}$ or $-\mathbf{v}$ is a row in M then the identity produced from m, u, v, k, e, f, α has already been found. Otherwise, add $\mathbf{v}$ as the next row of M.

Step 6. [Put M into row-reduced, echelon form]

Step 7. [Prove the input identity] Let $\mathcal{I}_0$ be the T^2 identity exiting Step 1. As in Step 5, let $\mathbf{v}$ be a length-n integer vector whose ith entry is the number of times the ith triple in R is found on the left-hand side of $\mathcal{I}_0$ minus the number of times it is found on the right-hand side of $\mathcal{I}_0$. Append $\mathbf{v}$ to the bottom of M and row reduce again. If the row rank of M does not change, then $\mathcal{I}_0$ can be written as a linear combination of identities from the Fundamental T^2 Formula, thus proving the identity.

(c) Comments on the Proof Algorithm.

Step 1. The input to this algorithm is not restricted to balanced T^2 identities arising from the Q^2 Search Algorithm. The method can be used to prove *any* tentative T^2 equation, provided that (i) the equation is balanced and (ii) each term has the same invariant value I_T. In particular, this method was used to prove the familiar Ramanujan identity $H(x)G(x^{11}) - x^2G(x)H(x^{11}) = 1$,

$$\text{where} \quad G(x) = \prod_{n=0}^{\infty} \frac{1}{(1-x^{5n+1})(1-x^{5n+4})} \quad \text{and} \quad H(x) = \prod_{n=0}^{\infty} \frac{1}{(1-x^{5n+2})(1-x^{5n+3})},$$

once this equation had been rewritten as a balance T^2 identity. See [11, Section 3] for details. Also, it is possible to have empty input, which would happen when all T^2 terms obtained by the given Q^2 identity cancel each other.

Finally we should note that the value of $\widehat{\alpha}$ is zero when Step 11 of the Search Algorithm does not apply. Otherwise this step undoes that step. Multiplying the identity by $x^{\widehat{\alpha}}$ adjusts the invariant so that it lands in $[0, 4k_1k_2)$.

Step 2. To make our search finite we use Lemma 5.1 (a). To eliminate impossibilities, we use Lemma 5.1 (b) and (c) and the relation $\dfrac{k_2}{k_1} = \dfrac{(2m-uv)vk}{uk}$, or $uk_2 = (2m-uv)vk_1$.

Step 3. Lemma 5.3 shows that many points (e, f) produce the same identity. In fact, we may assume both e and f are positive and the points (e, f) are periodic with respect to e which have a period of $\dfrac{2vk}{m}$. (These points are also periodic with respect to f, but the proof is more difficult and unnecessary for our purposes.) This also shows that there are lines of symmetry at $e = \frac{vk}{m} + t\frac{2vk}{m}$ for all $t \in \mathbb{Z}$. Hence we may assume without loss of generality that $0 \leq f$ and $0 \leq e \leq \frac{vk}{m}$.

In the fourth line of the pseudocode we define f to be an ugly square root. This comes directly from (5.2). One could alternatively let f cycle through all half integers starting at zero and test if there is a value of $\alpha < \left\lfloor \frac{k_1+k_2}{4} \right\rfloor$ and hence an invariant $I_T = \widehat{I} + 4k_1k_2\alpha$ which satisfies (5.2). This upper bound on α, which comes from (3.2), makes this search finite.

Finally the conditions that $e \in \frac{1}{4}\mathbb{Z}$ and those found in the fifth line of the pseudocode come from Lemma 5.2.

Step 4. This is analogous to Step 2 of the Search Algorithm. We include the enpoints of the intervals of l_1 and l_2 because $T(k, 0)$ and $T(k, k)$ are not identically zero.

Step 6. We do not see any immediate reason why the row-reduced version of M must have all integer entries.

Step 7. If there are more than one tentative T^2 identity with the same (k_1, k_2), and invariant $(\widehat{I})$ that need to be proved, we suggest re-using matrix M after its initial row-reduction. In this case, only Steps 1 and 7 are needed.

6. The Proof Algorithm in Action ... an Example

We illustrate the Proof Algorithm with (4.4). Applying formula (3.4) with $m_1 = 14$ and $m_2 = 70$ to the six Q^2 terms in (4.4) and reducing the resulting T^2 terms, we get

$$\begin{aligned}
R_1 &= Q(14,2)Q(70,13) &&= S_1 - S_{41} - S_{11} + S_{48}\\
R_4 &= xQ(14,5)Q(70,8) &&= S_6 - S_{36} - S_{25} + S_{44}\\
R_{12} &= x^{10}Q(14,4)Q(70,33) &&= S_{37} - S_{45} - S_{46} + S_{52}\\
R_2 &= Q(14,3)Q(70,12) &&= S_2 - S_{40} - S_{15} + S_{49}\\
R_3 &= xQ(14,1)Q(70,18) &&= S_4 - S_{53} - S_9 + S_{55}\\
R_9 &= x^3Q(14,6)Q(70,3) &&= S_{14} - S_{26} - S_{24} + S_{34}
\end{aligned}$$

where the term S_i is the ith entry in Table 4.

Proving (4.4) is thus equivalent to proving the T^2 identity

$$\begin{aligned}
(6.1)\quad & S_1 + S_6 + S_9 + S_{15} + S_{24} + S_{26} + S_{37} + S_{40} + S_{44} + S_{48} + S_{52} + S_{53}\\
&= S_2 + S_4 + S_{11} + S_{14} + S_{25} + S_{34} + S_{36} + S_{41} + S_{45} + S_{46} + S_{49} + S_{55}.
\end{aligned}$$

Step 1. Input equation (6.1). Note that $I_T = \widehat{I} = 441$ and so $\widehat{\alpha} = 0$.

Step 2. The eight choices for the global parameters (m, u, v, k) which give $k_1 = 21$ and $k_2 = 105$ are (3, 1, 1, 21), (3, 1, 5, 21), (6, 2, 1, 10.5), (6, 2, 5, 10.5), (7, 3, 3, 7), (14, 6, 3, 3.5), (21, 4, 10, 5.25), and (21, 8, 4, 2.625).

Step 3. We illustrate this step with the global parameters $(m, u, v, k) = (3, 1, 1, 21)$. Let e range from 0 to $\frac{7}{2}$ inclusive, $\alpha = 0$ to 30, $I_T = 441 + 8820\alpha$, and $f = \sqrt{\frac{7}{2} + 70\alpha - 5e^2}$. The criteria in the fifth line of the pseudocode require not only that $e \in \frac{1}{2}\mathbb{Z}$, but also that e and f have the same half-parity. There are thirty-three triples (e, f, α) for this (m, u, v, k).

Step 4. It turns out that there are sixty-six T^2 terms in the family with invariant 441. These are listed in Table 3.

Table 3. Terms for family $I = 441$.

$$S_i(x) = [\alpha_i, l_{1i}, l_{2io}] = x^{\alpha_i} T(21, l_{1i}) T(105, l_{2i})$$

i	α_i	l_{1i}	l_{2i}	i	α_i	l_{1i}	l_{2i}	i	α_i	l_{1i}	l_{2i}
1	0	1	4	23	5	16	29	45	14	5	76
2	0	2	1	24	5	17	26	46	14	19	64
3	1	0	21	25	5	20	11	47	15	4	79
4	1	4	19	26	6	11	44	48	15	13	74
5	1	7	14	27	7	3	54	49	15	16	71
6	1	8	11	28	7	13	46	50	16	6	81
7	1	9	6	29	7	16	41	51	16	10	79
8	2	2	29	30	7	19	34	52	18	19	76
9	2	10	19	31	7	20	31	53	19	4	89
10	2	11	16	32	8	7	56	54	19	11	86
11	2	13	4	33	8	14	49	55	20	10	89
12	3	5	34	34	8	17	44	56	21	1	94
13	3	8	31	35	9	4	61	57	21	17	86
14	3	11	26	36	9	8	59	58	22	3	96
15	3	16	1	37	10	5	64	59	22	14	91
16	4	2	41	38	10	10	61	60	22	21	84
17	4	6	39	39	10	18	51	61	23	13	94
18	4	9	36	40	12	2	71	62	25	8	101
19	4	15	24	41	13	1	74	63	25	12	99
20	4	17	16	42	13	12	69	64	26	5	104
21	4	18	9	43	13	15	66	65	29	20	101
22	5	1	46	44	13	20	59	66	30	19	104

Table 4. Parameters of Identities.

#	(m,u,v,k)	e	f	α	#	(m,u,v,k)	e	f	α
1	(3, 1, 1, 21)	$\frac{1}{2}$	$\frac{3}{2}$	0	14	(6, 2, 1, 10.5)	$\frac{1}{4}$	$\frac{3}{2}$	0
2	(3, 1, 1, 21)	$\frac{1}{2}$	$\frac{17}{2}$	1	15	(6, 2, 1, 10.5)	$\frac{1}{4}$	$\frac{17}{2}$	1
3	(3, 1, 1, 21)	$\frac{1}{2}$	$\frac{53}{2}$	10	16	(6, 2, 1, 10.5)	$\frac{3}{4}$	$\frac{23}{2}$	2
4	(3, 1, 1, 21)	$\frac{1}{2}$	$\frac{67}{2}$	16	17	(6, 2, 1, 10.5)	$\frac{3}{4}$	$\frac{33}{2}$	4
5	(3, 1, 1, 21)	$\frac{3}{2}$	$\frac{23}{2}$	2	18	(6, 2, 1, 10.5)	$\frac{5}{4}$	$\frac{13}{2}$	1
6	(3, 1, 1, 21)	$\frac{3}{2}$	$\frac{33}{2}$	4	19	(6, 2, 1, 10.5)	$\frac{5}{4}$	$\frac{27}{2}$	3
7	(3, 1, 1, 21)	$\frac{3}{2}$	$\frac{37}{2}$	5	20	(7, 3, 3, 7)	$\frac{1}{2}$	$\frac{1}{2}$	0
8	(3, 1, 1, 21)	$\frac{3}{2}$	$\frac{47}{2}$	8	21	(7, 3, 3, 7)	$\frac{1}{2}$	$\frac{11}{2}$	1
9	(3, 1, 1, 21)	$\frac{5}{2}$	$\frac{13}{2}$	1	22	(7, 3, 3, 7)	$\frac{1}{2}$	$\frac{19}{2}$	3
10	(3, 1, 1, 21)	$\frac{5}{2}$	$\frac{27}{2}$	3	23	(7, 3, 3, 7)	$\frac{1}{2}$	$\frac{29}{2}$	7
11	(3, 1, 1, 21)	$\frac{5}{2}$	$\frac{43}{2}$	7	24	(7, 3, 3, 7)	$\frac{3}{2}$	$\frac{9}{2}$	1
12	(3, 1, 1, 21)	$\frac{5}{2}$	$\frac{57}{2}$	12	25	(14, 6, 3, 5.25)	$\frac{1}{4}$	$\frac{1}{2}$	0
13	(3, 1, 1, 21)	$\frac{7}{2}$	$\frac{7}{2}$	1	25	(14, 6, 3, 5.25)	$\frac{1}{4}$	$\frac{11}{2}$	1

Step 5. Altogether we find a total of 26 different identities this way. Table 4 lists the parameters (m, u, v, k) along with the values of e, f, and α for these identities. Our resulting M will thus be a 26×66 sparse matrix with small integer entries. The first identity is $S_2 + S_{49} + S_{42} = S_1 + S_{48} + S_{43}$. Hence, the first row of our matrix M contains all zeros except for the entries in columns $2, 49$, and 42 which contain 1 and the entries in columns $1, 48$, and 43 which contain -1.

Step 6. After putting the matrix into row-reduced, echelon form, it has 16 rows, so there are 16 identities in Table 4 which are linearly independent.

Step 7. We append a row with 1 in columns $1, 6, 9, ...$ and -1 in columns $2, 4, 11, ...,$ the encoded equation (6.1). After performing another row-reduction, we see that the rank does not change. Therefore not only is (6.1) a true identity, but also we proved (4.4). The same happens with (4.5).

7. Search and Proof Results

We programmed the Search Algorithm of Section 3 and the Proof Algorithm of Section 5 as two C++ programs so that tentative integer identities found by the Search Program were fed into the Proof Program for verification. In this section we give some of the extensive data produced by the programs as well as some interesting statistics about certain classes of identities and proofs of identities. We consider briefly the problems of why some mod 2 identities do not lift to integer identities and why some the Proof Algorithm does not prove all tentative integer identities.

(a) Statistical Data.

The Search Algorithm was run for all pairs (m_1, m_2), where $5 \leq m_1 \leq 100$ and $m_1 \leq m_2 \leq 1000$. Of these 91056 pairs, we counted 208097572 families (C_I in Step 3 of the algorithm), of which 54729584 contained at least two triples with the same α. These latter could therefore be processed in the remaining steps of the algorithm.

For each of these families we constructed the mod 2 Maclaurin expansion associated with each triple, formed them into a matrix and performed the reduction as outlined in Steps 4 – 6. We used $L = 10000$, however, the largest value of L required was 418, which occurred at $(m_1, m_2) = (100, 900)$ and $I = 391500$. We then produced a basis of mod 2 identities, reduced as outlined in Step 7 of the Search Algorithm. We did not look at all linear combinations of these basis identities, which would have produced considerably more mod 2 identities. Rather, we considered the mod 2 identities that came from these basis elements. In spite of our efforts to reduce the amount of output, we were astonished to see that there were 123844352 mod 2 identities. On examination, 123709056 of these were discarded because they were really linear identities as described in Step 9 of the Search Algorithm. All linear identities had two or three terms.

Of the remaining (non-linear, mod 2) identities, 20172 were eliminated because they were not primitive (Step 10), while a further 17708 did not lift to an identity with ± 1 coefficients (Step 12). And for two other tentative identities (one with 18 terms and the other with 21 terms), using $N = 10000$ proved insufficient as in Step 12, comment. We were finally left with 97414 tentative identities that needed proofs.

When we transformed these Q^2 identities into T^2 identities, we found that in 108 cases all of the T^2-terms cancelled leaving a trivial identity. (All of these had four terms and are covered by Theorem 8.2.) Of the rest, we were able to prove 95889 identities using the Proof Algorithm, leaving a total of 1417 identities that could not be proved using our method. It should be noted that 28090 of these successful proofs involved a special infinite family of four-term identities identified in Theorem 8.1.

(b) Tables 5-9

We should point out that minor programming variations in implementing the steps of the algorithm may well change some of the counts given in the following five tables. In particular, adding a non-provable identity to one we can prove results in a non-provable identity so the number of provable identities depends on the choice of the basis found in Step 7 of the Search Algorithm.

In Table 5 we split all non-linear mod 2 identities (labelled Non-lin mod 2) by the number of terms in the identity, and we look at the number of n-term identities that are non-primitive (Not Prim), the number that do not lift to an integer identity (Not Lift), and the number that lift to an integer identity (Integer). Using the Proof Program, we further divide the

Table 5. Data on Identities classed by number of terms

# Terms	Non-lin mod 2	Not Prim	Not Lift	Integer	No Pf	Proof
2	125	113	12	0	0	0
3	1328	633	690	5	0	5
4	35235	5618	978	28639	0	28639
5	1345	597	610	138	0	138
6	11918	2446	359	9113	3	9110
7	4656	1282	953	2421	0	2421
8	20754	2642	573	17539	9	17530
9	4442	480	644	3318	4	3314
10	7459	1073	821	5565	14	5551
11	1370	371	710	289	4	285
12	9396	1733	922	6741	42	6699
13	1748	260	352	1136	0	1136
14	6124	635	1270	4219	6	4213
15	1235	91	1086	58	8	50
16	3540	403	793	2344	39	2305
17	400	24	263	113	0	113
18	3149	290	821	2037	26	2011
19	886	89	193	604	4	600
20	3185	361	649	2175	417	1758
21	1052	154	366	531	10	521
22	1663	142	570	951	42	909
23	322	56	73	193	5	188
24	3118	421	398	2299	105	2194
25	521	31	225	265	14	251
26	460	17	138	305	4	301
27	234	0	98	136	0	136
28	1070	89	380	601	216	385
29	154	0	44	110	2	108
30	925	75	233	617	29	588
31	138	0	62	76	0	76
32	604	4	158	442	53	389
33+	6740	42	2264	4434	361	4073
Total	135296	20172	17708	97414	1417	95997

integer identities into two classes: those that do not have proofs (No Pf) and those that have proofs (Proof), including the 108 trivial identities, using the Fundamental T^2 Formula (Theorem 5.1). Note that the values in the column labelled Non-lin mod 2 are the sum of the three columns Not Prim, Not Lift, and Integer, except for the two mod 2 identities we

Table 6. Data on Identities where m_1 divides m_2.

m_1	# Inv	Non-lin mod 2	Integer	Proof
5	4693	30	24	24
6	14693	7	0	0
7	4642	67	53	53
8	3957	79	75	75
9	7144	53	25	25
10	4489	158	111	111
11	4766	138	124	124
12	7385	71	36	36
13	5051	158	140	140
14	4793	247	178	178
15	4548	167	63	63
16	4377	339	263	263
17	5443	273	172	172
18	5543	268	174	174
19	5481	291	273	273
20	4432	514	405	405
21	3489	269	143	143
22	5335	506	390	390
23	5755	538	420	420
24	3709	292	158	158
25	5208	387	324	324
26	5485	571	449	449
27	2545	532	322	322
28	4716	991	773	773
29	5565	584	545	545
30	3652	554	319	319
31	5610	510	375	374
32	4846	893	616	616
33	2433	453	203	203
34	5689	714	484	484
35	4306	1053	865	865
36	2761	916	621	621
37	5184	510	507	503
38	5690	960	731	731
39	2060	500	273	273
40	4382	1620	1127	1127
41	4823	517	422	422
42	2754	701	379	379
43	4935	866	858	854
44	4664	1456	1143	1143
45	1820	1457	976	976
46	5156	1345	965	965
47	4418	931	814	811
48	2003	938	513	513
49	4133	1141	1002	1002
50	4909	1607	1195	1195
51	1680	564	296	296
52	4646	1631	1249	1249
53	3901	766	754	739
54	2046	1689	1052	1052
55	3662	1917	1541	1541
56	3896	3053	2206	2206
57	1548	736	383	383
58	5174	1530	1157	1157
59	3600	1069	1061	1059
60	1795	1933	1248	1248
61	3734	951	948	933
62	4879	1668	1321	1318
63	1293	1730	1268	1268
64	3667	1873	1215	1215
65	3212	1163	794	794
66	2058	1080	671	671
67	3446	915	639	639
68	4032	2227	1790	1790
69	1368	617	292	292
70	4151	2840	2135	2135
71	3646	1632	1485	1470
72	1418	2652	1615	1615
73	3181	1781	1673	1671
74	4345	1706	1469	1457
75	1164	936	598	598
76	3951	2522	1991	1991
77	2643	2383	2006	2006
78	1632	1312	852	852
79	2893	1413	1251	1248
80	2968	3332	2065	2065
81	1047	2381	1801	1801
82	4074	1769	1389	1389
83	3042	913	913	913
84	1399	2002	1154	1154
85	2665	2171	1317	1317
86	3976	2667	2028	2025
87	1124	974	452	452
88	3286	3839	2544	2544
89	3018	1800	1000	1000
90	1593	4074	2541	2541
91	2409	3167	2018	2018
92	3080	4153	3258	3258
93	1013	1250	522	520
94	3637	2540	1885	1885
95	2481	4194	3924	3924
96	1165	2016	1072	1072
97	2701	1821	1627	1622
98	3584	2738	2042	2042
99	1053	3543	2773	2773
100	3058	3524	2767	2767
Total	360506	128329	94080	93991

Table 7. Data on Identities where $d = \gcd(m_1, m_2) < m_1$

d	# Inv	Non-lin mod 2	Integer	Proof	d	# Inv	Non-lin mod 2	Integer	Proof
1	38949971	0	0	0	26	14366	140	83	83
2	7645585	0	0	0	27	5864	53	5	5
3	3391283	0	0	0	28	11991	186	123	123
4	1246995	16	0	0	29	10831	57	57	55
5	682948	127	24	0	30	8999	256	162	162
6	642455	6	0	0	31	10814	110	43	30
7	327633	196	40	17	32	11303	192	114	114
8	239985	180	123	83	33	5657	147	21	21
9	174777	30	4	0	34	5885	24	3	3
10	174880	138	52	0	35	3923	166	82	82
11	131285	250	95	20	36	2692	33	6	6
12	97402	56	2	0	37	5055	31	26	20
13	84475	263	61	18	38	5601	95	16	4
14	94272	296	151	98	39	2114	119	39	39
15	34273	87	10	4	40	4563	71	42	42
16	52811	489	366	200	41	4571	35	32	24
17	42274	293	148	76	42	2916	64	13	13
18	33248	202	63	44	43	4893	29	18	0
19	41090	407	246	92	44	4315	40	23	23
20	39825	319	221	86	45	1656	144	35	35
21	14048	154	31	18	46	5463	68	32	16
22	24346	309	199	63	47	4395	32	26	0
23	21779	263	156	47	48	1734	0	0	0
24	14291	229	88	66	49	3945	157	102	102
25	18753	262	105	38	50	4848	146	46	34
					Total	54369078	6967	3334	2006

were unable to classify (see lines 18 and 21), and the values in the column labelled Integer are the sum of the two columns labelled No Pf and Proof.

There appear to be more identities with an even number of terms than with an odd number of terms. Again we note that the number of terms in an identity depends on the choice of basis elements. The purpose of the joint reduction Step 7 of the Search Algorithm was to make the total number of terms in the basis identities as small as possible. A different implementation of this step could lead to different values in Table 5. The large number of four-term identities, 28639, includes 28198 cases of two infinite families of four-term identities, described in Section 8c. Similarly, we suspect that another infinite parametric family accounts for the large number of eight-term identities in Table 5.

Table 8. Data when m_1 divides m_2, $r = \dfrac{m_2}{m_1}$

r	Non-lin mod 2	Integer	Proof
1	631	252	252
2	6036	2999	2999
3	10019	7967	7957
4	36664	30203	30197
5	11448	9045	9045
6	600	340	336
7	14349	6383	6366
8	17322	13311	13311
9	528	410	404
10	3539	2425	2419
11	3259	2399	2393
12	3864	2919	2916
13	1359	1062	1048
14	695	154	146
15	1041	749	746
16	9842	8186	8180
17	355	223	223
18	26	0	0
19	278	209	209
20	1455	1044	1044
21	188	165	165
22	125	53	53
23	59	47	47
24	95	77	77
25	2050	1686	1686
26	28	28	28
28	561	228	228
29	15	15	15
30	20	12	12
31	32	32	32
32	642	498	498
33	21	21	21
34	18	0	0
35	231	215	215
37	19	19	19
39	25	25	25
40	157	131	131
44	41	35	35
45	18	18	18
48	39	22	22
49	214	177	177
50	55	18	18
52	28	20	20
55	68	48	48
56	27	21	21
60	11	9	9
64	106	79	79
68	9	9	9
75	6	2	2
80	17	15	15
85	16	16	16
91	18	18	18
98	8	0	0
100	21	12	12
112	4	4	4
121	8	7	7
125	9	9	9
128	5	5	5
169	1	1	1
196	2	1	1
200	2	2	2
Total	128329	94080	93991

Table 9. Frequency of identity space dimensions.

Dim	# Inv
0	54687283
1	21592
2	6731
3	3412
4	2559
5	1382
6	1398
7	884
8	671
9	623
10	763
11	377
12	475
13	152
14	139
15	166
16	89
17	79
18	125
19	69
20	108
21	50
22	51
23	42
24	59
25	25
26	24
27	46
28	16
29	8
30	44
31	11
32	15
33	22
34	5
35	13
36	9
37	1
38	2
39	12
40	6
41	4
42	3
43	2
44	4
45	3
46	3
47	6
49	2
53	1
57	1
64	1
65	1
66	2
67	3
69	2
73	1
76	3
77	1
91	1
92	1
94	1

It has been our experience over the years that Q^2 identities balanced at (m_1, m_2) are most likely to occur when m_1 divides m_2. Does the condition that m_1 divides m_2 increase the chance of finding a Q^2 identity? What happens if m_1 and m_2 overlap, but $\gcd(m_1, m_2)$ is less than m_1? What happens if m_1 and m_2 are relatively prime?

In order to answer these questions, we collected the data into two tables. In Table 6 we display some abridged data for the pairs (m_1, m_2) such that m_1 divides m_2. We provide the number of invariants (# Inv) that produced a family C_I which could be processed as in Step 3 of the Search Algorithm and the number of non-linear mod 2 identities produced. We further show the number of these that lifted to an integer equation, and in turn, the number of these we can prove using the Fundamental T^2 Formula. In Table 7 we deal with the pairs (m_1, m_2) where m_1 does not divide m_2, indexed by $d = \gcd(m_1, m_2)$. It seems for various reasons that the prime factors of m_1 and m_2 determine many features of the data we obtained.

Observe in Table 7 that no integer identity exists when d is 1, 2, 3, or 4. In particular, when $d = 1$, not one of the nearly 39 million invariant families leads to a single mod 2 identity, other than the linear identities. We raise the question: Do any non-linear Q^2 identities exist when m_1 and m_2 are relatively prime? When $\gcd(m_1, m_2) \geq 5$, a comparison of Tables 6 and 7 reveal some marked differences, confirming our opinion that the condition m_1 divides m_2 greatly increases the chances of finding and of proving a Q^2 identity. Table 6 shows that the 360506 possible invariant families produce 94080 possible integer identities, so that when m_1 divides m_2, the probability that an invariant family leads to an integer identity is approximately 26.1%, indicating that the Search Algorithm is successful in finding an actual identity over one fourth of the time. Furthermore, the Proof Algorithm is able to prove 93991 of these 94080 integer identities produced by the Search Algorithm, or roughly 99.9%.

Table 7 shows that genuine integer identities are much scarcer when the gcd of m_1 and m_2 lies strictly between 4 and m_1. Out of 3135244 invariant families, the Search routine succeeds in finding only 3334 integer identities, a success rate of one tenth of one percent. Moreover, the Proof Algorithm finds a proof for only 60.1% of the them (2004 out of 3334).

Another observation of Table 6 is that it is often the case that the Proof Algorithm proves every integer identity found by the Search Algorithm for a particular m_1. Indeed the Proof Algorithm is perfect up to $m_1 = 30$ and overall, for 81 of the 96 values of m_1 listed in Table 6. On the other hand, when $d = \gcd(m_1, m_2)$ is less than m_1, an examination of Table 7 shows that the results run the gamut between always finding a proof for a particular value of the gcd and striking out every time. The program proves all integer identities for 15 values of d and fails to find any proof at all for 6 other values of d. Naturally we exclude those values of $d > 4$ for which the Search Algorithm failed to find a single integer identity: $d = 6$ and 48.

Since the condition that m_1 divides m_2 seems so much of a criterion in finding and proving identities, we decided to investigate these cases more thoroughly, by considering the ratio $\frac{m_2}{m_1}$. In Table 8 we consider only pairs (m_1, m_2) where m_1 divides m_2. The index for the table is the quotient $r = \dfrac{m_2}{m_1}$. Rows where all entries are zero have been deleted from this table. Observe that we get a relatively larger number of identities, compared to the nearby rows, when this ratio is a square not divisible by 3, in particular, for $r = 4,\ 16,\ 25,\ 49,\ 64,\ 100,$ and 121. On the other hand, there are no non-linear identities of any kind for the square $r = 36$. Observe also that all potential identities have proofs when $r \geq 17$.

Finally, one can ask about the distribution of the dimension of the space of non-linear mod 2 identities found by the Search Algorithm for a given invariant. In table 9 we present the number of invariants that yield a given dimension. The majority of invariant families (over 54 million) do not produce any non-linear identities (dimension zero). As seen in Table 9, the majority of invariants which produce actual mod 2, non-linear identities result in a basis with a fairly small dimension. We were a bit surprised, however, to discover that one invariant led to a non-linear mod 2 identity space of dimension 94, therefore giving 2^{94} mod 2 identities. This occurs when $m_1 = 96$, $m_2 = 672$, and $I = 82944$.

(c) The Lifting Problem

There were 17708 primitive non-linear mod 2 identities that did not lift to the integers. However, since we use only ± 1 coefficients in our integer identities, it is possible for a mod 2 identity, which does not lift in Step 12 of the Search Algorithm, to lift to a true identity with coefficients other than ± 1, or one with Q_1, Q_2 or Q_3 in its terms [4, Definition 3], or even a non-balanced identity. Of course, it may be possible that it does not lift to any of these.

We were able, however, to find a proof for some of these recalcitrant tentative identities. Here is an example of a three-term mod 2 identity which does not lift to a Q^2 identity:

$$Q(6,1)Q(24,6) + Q(6,2)Q(24,4) + xQ(6,2)Q(24,8) \equiv 0 \pmod 2. \tag{7.1}$$

When we consider even multiples of other triples with the same invariant 324 as (7.1), we find the balanced identity

$$Q(6,1)Q(24,6) + xQ(6,2)Q(24,8) = Q(6,2)Q(24,4) + 2x^2Q(6,1)Q(24,10). \tag{7.2}$$

That (7.2) is true can be seen by first introducing $Q(24,2) = Q(24,6) - x^2Q(24,10)$, a case of (3.5), into (7.2) and then noting that the resulting four-term identity is an instance of (8.4) with $r = 6, s = 1$, and $t = 2$. Our algorithm, however, is not sophisticated enough to find this identity.

It is interesting that (7.1) can be lifted to the balanced Q_1Q_0 identity

$$Q_1(6,1)\,Q_0(24,6) = Q_1(6,2)\,Q_0(24,4) + xQ_1(6,2)Q_0(24,8). \tag{7.3}$$

To prove (7.3), convert it to a T_1T_0 identity using [4, Theorem 4] and then apply [4, Theorem 4] to the T_1 terms. The resulting T_0^2 identity, balanced at $(36,36)$, can then be proved using the Proof Algorithm as usual.

(d) Tentative Identities Without Proofs

The Proof Program found there were 1417 out of 97414 tentative identities (approximately 1.5%) which could not be proved using the Fundamental T^2 Formula. A simple example is the following six-term identity:

$$(7.4)\quad Q(28,1)\,Q(35,5) + x\,Q(28,9)\,Q(35,10) + x^3Q(28,3)\,Q(35,15) = Q(28,5)\,Q(35,10) + x^3Q(28,13)\,Q(35,5) + x^5Q(28,11)\,Q(35,15).$$

To prove this identity, we recall from [4, Theorem 9] the linear Q identity,

$$(7.5)\qquad Q(m,n) = Q(4m, m-2n) - x^nQ(4m, m+2n),$$

which is true for all $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}$. Now group the pairs of terms of (7.4) with the common factor $Q(35, n_2)$, $n_2 = 5, 10, 15$, and factor out the respective $Q(35, n_2)$'s. Then apply (7.5) to each of the three $Q(28, *)$ binomials. The resulting equation is (1.8). This same technique proves the other two six-term (tentative) identities without proofs (see Table 5, line 6) using (1.6) and (1.7).

Out of the nine eight-term (tentative) identities without proofs (see Table 5, line 8), eight collapse using (7.5) to the eight identities (8.16)–(8.21), (8.24), and (8.25). The remaining tentative identity, balanced at $(72, 90)$, may be transformed using (3.5) and (7.5) into a true identity balanced at $(18, 90)$.

Although this "contraction" technique proves many of the tentative identities which could not be established using the Proof Algorithm, it does not prove them all. The simplest example is the following tentative nine-term identity balanced at $(m_1, m_2) = (38, 95)$ using the notation (4.1):

$$(7.6)\quad (0,7,20) + (2,13,10) + (4,1,30) + (10,17,35) + (14,11,45) = (0,9,15) + (2,3,5) + (4,15,25) + (9,5,40).$$

Interestingly, the Proof Algorithm proves none of the six tentative identities balanced at $(38, 95)$, and obviously the "contraction" method does not work if the number of terms is odd.

8. New Q^2 Identities

(a) Two-term mod 2 identities

There are 39325 two-term mod 2 identities; however all but 125 are linear and occur only when $m_1 = m_2$. Not one of these 125 mod 2 identities lifts to an integer identity. Such is the case when $(m_1, m_2) = (5, 20)$. The triples $(0, 1, 6)$ and $(0, 2, 2)$ produce power series which are not equal but are congruent mod 2. All 125 two-term mod 2 identities occur when $\dfrac{m_2}{m_1} = 1, 2$, or 4. This raises the question: Are there any genuine non-linear two-term Q^2

identities? The answer is no. We can show that if $0 < a, c < \frac{m_1}{2}$ and $0 < b, d < \frac{m_2}{2}$, then

$$Q(m_1, a)Q(m_2, b) = Q(m_1, c)Q(m_2, d) \tag{8.1}$$

if and only if $a = c$ and $b = d$, or else $a = d$, $b = d$, and $m_1 = m_2$. The proof uses the product form of each Q [12, p. 1284] and examines the several thousand ways the factors on the two sides of (8.1) can be merged.

(b) Two More Three-term Identities

We found two new three-term identities:

$$Q(5, 1)Q(40, 2) + x^4Q(5, 1)Q(40, 18) = Q(5, 2)Q(40, 10). \tag{8.2}$$

$$Q(5, 2)Q(40, 6) + x^2Q(5, 2)Q(40, 14) = Q(5, 1)Q(40, 10) \tag{8.3}$$

The Proof Algorithm is able to verify both of these equations, making a total of five non-linear primitive three-term Q^2 identities in our search ranges.

Note that if we solve (8.2) and (8.3) for the ratio $\frac{Q(5,1)}{Q(5,2)}$, we get the equation

$$\frac{Q(40, 6) + x^2Q(40, 14)}{Q(40, 10)} = \frac{Q(40, 10)}{Q(40, 2) + x^4Q(40, 18)}.$$

Cross-multiplying and sending $x^2 \to x$ yields

$$(Q(20, 5))^2 = Q(20, 3)Q(20, 1)+xQ(20, 7)Q(20, 1)+x^2Q(20, 3)Q(20, 9)+x^3Q(20, 7)Q(20, 9),$$

an identity balanced at $(20, 20)$. (See Table 8, $r = 1$.) We find this identity interesting because it is one (of a few) that contains a square Q.

(c) Four-term Identities

Among the identities we found in our search are several types of four-term identities which we discuss next. The first is a two-parameter family of identities balanced at $(m_1, 4m_1)$ which was discovered from the search output because of its simple form. The two proofs of this identity we give illustrate two methods of proof we can use, the second being much more complicated than the first.

Theorem 8.1. *If $r \in \mathbb{Z}^+$ and $s, t \in \mathbb{Z}$, then*

$$\begin{aligned} Q(r, s)\, Q(4r, r - 2t) + x^sQ(r, t)\, Q(4r, r + 2s) = \\ Q(r, t)\, Q(4r, r - 2s) + x^tQ(r, s)\, Q(4r, r + 2t). \end{aligned} \tag{8.4}$$

The first proof rests on the commutativity of a product.

Proof 1. Using (7.5) twice, we find that

$$\begin{aligned} Q(m, s) \left[Q(4m, m - 2t) - x^tQ(4m, m + 2t)\right] = Q(m, s)Q(m, t) = \\ Q(m, t) \left[Q(4m, m - 2s) - x^sQ(4m, m + 2s)\right]. \end{aligned}$$

Equation (8.4) follows immediately. □

The second proof uses the method described in Section 5 which is based on the Fundamental T^2 Formula.

Proof 2. Define the following:

$$k_1 = \frac{3}{2}r, \qquad k_2 = 4k_1 = 6r, \qquad \sigma^\pm = \frac{r}{2} \pm 3s, \qquad \text{and} \qquad \tau^\pm = \frac{r}{2} \pm 3t.$$

Using our concise notation $[\alpha, l_1, l_2] = x^\alpha T(k_1, l_1)T(k_2, l_2)$ and $(\alpha, n_1, n_2) = x^\alpha Q(r, n_1)Q(4r, n_2)$, we apply (3.4) to transform each of the four Q^2 terms in (8.4) into four T^2 terms:

$$\begin{aligned}(0, s, r-2t) = {} & \left[0, \sigma^-, -2\tau^-\right] \\ & -x^s\left[0, \sigma^+, -2\tau^-\right] \\ & -x^t\left[k_1-\tau^+, \sigma^-, k_2-2\tau^+\right] \\ & +x^{s+t}\left[k_1-\tau^+, \sigma^+, k_2-2\tau^+\right]\end{aligned} \qquad \begin{aligned}(0, t, r-2s) = {} & \left[0, \tau^-, -2\sigma^-\right] \\ & -x^t\left[0, \tau^+, -2\sigma^-\right] \\ & -x^s\left[k_1-\sigma^+, \tau^-, k_2-2\sigma^+\right] \\ & +x^{s+t}\left[k_1-\sigma^+, \tau^+, k_2-2\sigma^+\right]\end{aligned}$$

$$\begin{aligned}(s, t, r+2s) = {} & x^s\left[0, \tau^-, -2\sigma^+\right] \\ & -x^{s+t}\left[0, \tau^+, -2\sigma^+\right] \\ & -\left[k_1-\sigma^-, \tau^-, k_2-2\sigma^-\right] \\ & +x^t\left[k_1-\sigma^-, \tau^+, k_2-2\sigma^-\right]\end{aligned} \qquad \begin{aligned}(t, s, r+2t) = {} & x^t\left[0, \sigma^-, -2\tau^+\right] \\ & -x^{s+t}\left[0, \sigma^+, -2\tau^+\right] \\ & -\left[k_1-\tau^-, \sigma^-, k_2-2\tau^-\right] \\ & +x^s\left[k_1-\tau^-, \sigma^+, k_2-2\tau^-\right]\end{aligned}$$

We now substitute these four expressions into equation (8.4) to get an identity containing all 16 of the above T^2 terms. This new identity may be broken into four groups, whose terms are identified by the exponent on the x in front of the term. (Four of the above terms have x^0 before them, four have x^s, four have x^t, and the remaining four have x^{s+t}.) Amazingly, all four of these groups have the structure

$$[0, \sigma, -2\tau] + [k_1-\tau, \sigma, k_2-2\tau] = [0, \tau, -2\sigma] + [k_1-\sigma, \tau, k_2-2\sigma] \tag{8.5}$$

where σ is either σ^+ or σ^- and τ is either τ^+ or τ^-.

Next we use the first part of (2.2) to negate the l_1 parts of the first and third terms of (8.5) and then we use the second part of (2.2) to reduce the l_1 part of the second and fourth terms. These four reductions result in the identity

$$[0, -\sigma, -2\tau] + [2k_1-\sigma-\tau, 2k_1-\sigma, k_2-2\tau] = [0, -\tau, -2\sigma] + [2k_1-\sigma-\tau, 2k_1-\tau, k_2-2\sigma]\,. \tag{8.6}$$

We derive (8.6) from Theorem 5.1 using the global parameters $m = 2$, $u = 1$, $v = 2$, $k = k_1$ and the residue systems $R_m = R'_m = \{0, 1\}$. From Theorem 5.1 we have

$$\alpha_n(e, f) = 2k_1n^2 + 2en \qquad \begin{aligned} l_{1,n}(e, f) &= 2k_1n + e + f \\ l_{2,n}(e, f) &= k_2n + 2e - 2f \end{aligned} \qquad \begin{aligned} l'_{1,n}(e, f) &= 2k_1n + e - f \\ l'_{2,n}(e, f) &= k_2n + 2e + 2f. \end{aligned}$$

With these values, observe that (8.6) is precisely equation (5.1) with $e + f = -\sigma$ and $e - f = -\tau$. Setting $(e, f) = \left(-\frac{1}{2}(\sigma+\tau), -\frac{1}{2}(\sigma-\tau)\right)$ completes the proof. □

Since the first proof shows that an infinite family of Q^2 identities arises from a linear Q identity, it is not surprising that another infinite family of four-term identities can be gotten from the other known linear Q identity (3.5). In this case there are nine identities where both m_1 and m_2 are multiples of 3. The following nine four-parameter identities cover all trivial (four-term) identities discovered during the search.

Theorem 8.2. *Let $m_1 = 3a$ and $m_2 = 3b$ for $a, b \in \mathbb{Z}^+$, and let $c, d \in \mathbb{Z}$. If we denote $(\alpha, n_1, n_2) = x^\alpha Q(m_1, n_1)Q(m_2, n_2)$, then we have the following four-term Q^2 identities balanced at (m_1, m_2):*

$$(0, a-c, d) + (d, c, b+d) = (0, c, b-d) + (c, a+c, d) \tag{8.7}$$

$$(0, c, d) + (c, a+c, d) + (d, a-c, b+d) = (0, a-c, b-d) \tag{8.8}$$

$$(0, a-c, d) + (c+d, a+c, b+d) = (0, c, d) + (c, a+c, b-d) \tag{8.9}$$

$$(0, a-c, b-d) = (0, c, d) + (c, a+c, b-d) + (d, c, b+d) \tag{8.10}$$

$$(0, c, b-d) + (c, a+c, b-d) = (0, a-c, d) + (d, a-c, b+d) \tag{8.11}$$

$$(0, a-c, b-d) = (0, c, b-d) + (c, a+c, d) + (c+d, a+c, b+d) \tag{8.12}$$

$$(0, c, d) + (d, a-c, b+d) = (0, c, b-d) + (c+d, a+c, b+d) \tag{8.13}$$

$$(0, a-c, d) + (d, c, b+d) + (c+d, a+c, b+d) = (0, a-c, b-d) \tag{8.14}$$

$$(c, a+c, d) + (d, a-c, b+d) = (c, a+c, b-d) + (d, c, b+d). \tag{8.15}$$

Proof. Consider the following three rational functions for $s \in \mathbb{Z}^+$ and $t \in \mathbb{Z}$:

$$F_1(s,t) = \frac{Q(3s, s-t) - x^t Q(3s, s+t)}{Q(3s,t)}$$

$$F_2(s,t) = \frac{Q(3s,t) + x^t Q(3s, s+t)}{Q(3s, s-t)}$$

$$F_3(s,t) = \frac{Q(3s, s-t) - Q(3s,t)}{x^t Q(3s, s+t)}.$$

Here we assume t is chosen so that the denominator of all F_i is non-zero. Using the linear identity (3.5) we see that each of the three functions F_1, F_2, and F_3 is equal to 1. For $i, j = 1, 2, 3$, we obtain identity number $i + 3(j-1)$ in the list of identities in the theorem by setting $F_i(a, c) = F_j(b, d)$. (The cases where any denominator is zero are easily verified.) □

Expanding any identity (8.7) – (8.15) into a T^2 identity using (3.4) results in all T^2 terms cancelling to leave the trivial identity.

There are many four-term identities which do not come from the previous two theorems. For example, consider $(m_1, m_2) = (8, 40)$. Since $\frac{m_2}{m_1} = 5$, not 4, Theorem 8.1 does not apply. Furthermore, 3 divides neither m_1 nor m_2 so Theorem 8.2 does not apply here either. The Q^2 search vprogram produced the six four term identities below (with invariants 252, 828, 1512, 1692, 2088, and 2268, respectively). A proof of each of these equations can be constructed using the Proof Algorithm in Section 5.

$$Q(8,1)Q(40,8) + xQ(8,1)Q(40,12) = Q(8,2)Q(40,7) + x^4Q(8,2)Q(40,17) \tag{8.16}$$

$$Q(8,1)Q(40,4) + xQ(8,2)Q(40,1) + x^3Q(8,1)Q(40,16) = Q(8,2)Q(40,9) \tag{8.17}$$

$$Q(8,1)Q(40,3) + xQ(8,1)Q(40,13) = Q(8,3)Q(40,7) + x^4Q(8,3)Q(40,17) \tag{8.18}$$

$$Q(8,2)Q(40,3) + xQ(8,2)Q(40,13) = Q(8,3)Q(40,8) + xQ(8,3)Q(40,12) \tag{8.19}$$

$$Q(8,1)Q(40,11) + xQ(8,3)Q(40,1) = Q(8,3)Q(40,9) + x^5Q(8,1)Q(40,19) \tag{8.20}$$

$$Q(8,2)Q(40,11) = Q(8,3)Q(40,4) + x^3Q(8,3)Q(40,16) + x^5Q(8,2)Q(40,19) \tag{8.21}$$

These identities seem to be related to rational functions in a way similar to the functions F_1–F_3 used in the proof of Theorem 8.2. If we group together terms with common factors, we see that the six equations neatly split into two groups. Equations (8.16), (8.18), and (8.19) give

$$\frac{Q(40,7) + x^4Q(40,17)}{Q(8,1)} = \frac{Q(40,8) + xQ(40,12)}{Q(8,2)} = \frac{Q(40,3) + xQ(40,13)}{Q(8,3)}, \tag{8.22}$$

while, equations (8.17),(8.20), and (8.21) produce

$$\frac{Q(40,9) - xQ(40,1)}{Q(8,1)} = \frac{Q(40,4) + x^3Q(40,16)}{Q(8,2)} = \frac{Q(40,11) - x^5Q(40,19)}{Q(8,3)}. \tag{8.23}$$

Not all four-term identities can be expressed in terms of fractions as in the the preceding examples. In that regard, the following two are “isolated curiosities” ([14, p. 335]):
(8.24)

$$Q(10,3)Q(110,11) + x^8Q(10,2)Q(110,44) = Q(10,4)Q(110,22) + x^2Q(10,1)Q(110,33)$$

and
(8.25)

$$Q(16,3)Q(112,7) + x^{11}Q(16,5)Q(112,49) = Q(16,7)Q(112,21) + x^2Q(16,1)Q(112,35).$$

References

[1] G. Andrews. An incredible formula of Ramanujan. *Aust. Math. Soc. Gazette*, 6:80–89, 1979.

[2] B. Berndt. *Ramanujan's Notebooks, Part III.* Springer-Verlag, New York, 1991.

[3] B. Berndt and H. Yesilyurt. Ramanujan's forty identities for the rogers-ramanujan functions and linear relations between them. *preprint.*

[4] R. Blecksmith and J. Brillhart. Linear quintuple-product identities. *Math. Comp.*, 72:1019–1033, 2003.

[5] R. Blecksmith, J. Brillhart, and I. Gerst. A computer–assisted investigation of Ramanujan pairs. *Math. Comp.*, 46:731–749, 1986.

[6] R. Blecksmith, J. Brillhart, and I. Gerst. Parity results for certain partition functions and identities similar to theta function identities. *Math. Comp.*, 48:29–38, 1987.

[7] R. Blecksmith, J. Brillhart, and I. Gerst. Some infinite product identities. *Math. Comp.*, 51:301–314, 1988.

[8] R. Blecksmith, J. Brillhart, and I. Gerst. On the mod 2 reciprocation of infinite modular–part products and the parity of certain partition functions. *Math. Comp.*, 54:345–376, 1990.

[9] R. Blecksmith, J. Brillhart, and I. Gerst. On a certain (mod 2) identity and a method of proof by expansion. *Math. Comp.*, 56:775–794, 1991.

[10] R. Blecksmith, J. Brillhart, and I. Gerst. New proofs for two infinite product identities. *Rocky Mountain J. Math.*, 22:819–823, 1992.

[11] R. Blecksmith, J. Brillhart, and I. Gerst. A fundamental modular identity and some applications. *Math. Comp.*, 61:83–95, 1993.

[12] R. Blecksmith, J. Brillhart, and I. Gerst. A lattice proof of a modular identity. *Rocky Mountain J. Math.*, 26:1275–1287, 1996.

[13] R. Blecksmith, J. Brillhart, and I. Gerst. A constructive theory of triple and quintuple product identities of the second degree. *Math. Comp.*, 67:707–814, 1998.

[14] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Oxford, 3rd edition, 1954.