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ON COVERING NUMBERS

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Abstract

A positive integer n is called a covering number if there are some distinct divisors $n_1, \ldots, n_k$ of n greater than one and some integers $a_1, \ldots, a_k$ such that $\mathbb{Z}$ is the union of the residue classes $a_1(\text{mod } n_1), \ldots, a_k(\text{mod } n_k)$. A covering number is said to be primitive if none of its proper divisors is a covering number. In this paper we give some sufficient conditions for n to be a (primitive) covering number; in particular, we show that for any $r = 2, 3, \ldots$ there are infinitely many primitive covering numbers having exactly r distinct prime divisors. In 1980 P. Erdős asked whether there are infinitely many positive integers n such that among the subsets of $D_n = \{d \geqslant 2 : d \mid n\}$ only D_n can be the set of all the moduli in a cover of $\mathbb{Z}$ with distinct moduli; we answer this question affirmatively. We also conjecture that any primitive covering number must have a prime factorization $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ (with $p_1, \ldots, p_r$ in a suitable order) which satisfies $\prod_{0<t<s}(\alpha_t + 1) \geqslant p_s - 1$ for each $1 \leqslant s \leqslant r$, with strict inequality when $s = r$.

–Dedicated to Prof. R. L. Graham for his 70th birthday

1. Introduction

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, $a(\text{mod } n) = \{a + nx : x \in \mathbb{Z}\}$ is called a residue class with modulus n. If every integer lies in at least one of the residue classes

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$a_1(\text{mod } n_1), \ldots, a_k(\text{mod } n_k)$, then we call the finite system

$$A = \{a_i(\text{mod } n_i)\}_{i=1}^k \tag{1.0}$$

a *cover* of $\mathbb{Z}$ (or *covering system*), and $n_1, \cdots, n_k$ its *moduli*. If (1.0) forms a cover of $\mathbb{Z}$ but none of its proper subsystems does, then (1.0) is said to be a *minimal cover* of $\mathbb{Z}$.

In the 1930s P. Erdős (cf. [E50]) invented the concept of a cover of $\mathbb{Z}$ and gave the following example

$$\{0(\text{mod } 2),\ 0(\text{mod } 3),\ 1(\text{mod } 4),\ 5(\text{mod } 6),\ 7(\text{mod } 12)\}$$

whose moduli 2, 3, 4, 6, 12 are distinct. Covers of $\mathbb{Z}$ with distinct moduli are of particular interest and they have some surprising applications (see, e.g., [F] and [S00]). For problems and results concerning covers of $\mathbb{Z}$ and their generalizations the reader may consult [E97], [FFKPY], [Gu], [PS], [S03], [S04] and [S05].

Here is a famous open conjecture.

The Erdős–Selfridge Conjecture. *If* (1.0) *forms a cover of* $\mathbb{Z}$ *with the moduli* $n_1, \ldots, n_k$ *distinct and greater than one, then* $n_1, \ldots, n_k$ *are not all odd.*

Following J. A. Haight [H] we introduce the following concept.

Definition 1.1. A positive integer n is called a *covering number* if there is a cover of $\mathbb{Z}$ with all the moduli distinct, greater than one and dividing n.

Erdős' example shows that $2^2 \cdot 3 = 12$ is a covering number. By density considerations, if n is a covering number then $\sum_{1<d\,|\,n} 1/d \geqslant 1$; it follows that none of $2, 3, \ldots, 11$ is a covering number. Moreover, Example 3 of [S96] indicates that $2^{n-1}n$ is a covering number for every $n = 3, 5, 7, \ldots$.

In the direction of the Erdős–Selfridge conjecture, S. Guo and Z. W. Sun [GS] proved that any odd and squarefree covering number should have at least 22 distinct prime divisors.

If (1.0) is a cover of $\mathbb{Z}$ with $n_1 \leqslant \cdots \leqslant n_{k-1} < n_k$, then $\sum_{i=1}^{k-1} 1/n_i \geqslant 1$ by Theorem I (iv) of Sun [S96]. So, a necessary condition for $n \in \mathbb{Z}^+$ to be a covering number is that

$$\frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d} \geqslant 2 + \frac{1}{n}, \tag{1.1}$$

where $\sigma(n)$ is the sum of all positive divisors of n. However, as shown by Haight [H], there does not exist a constant $c > 0$ such that $n \in \mathbb{Z}^+$ is a covering number whenever $\sigma(n)/n > c$.

Let (1.0) be a cover of $\mathbb{Z}$, and set $w(r) = |\{1 \leqslant i \leqslant k : r \equiv a_i \pmod{n_i}\}|$ for $r = 0, \ldots, N-1$, where $N = [n_1, \ldots, n_k]$ is the least common multiple of $n_1, \ldots, n_k$. By Theorem 5(ii) and Example 6 of [S01],

$$\sum_{\substack{1\leqslant i\leqslant k\\ \gcd(x+a_i,n_i)=1}} \frac{1}{\varphi(n_i)} = \sum_{\substack{0\leqslant r<N\\ \gcd(x+r,N)=1}} \frac{w(r)}{\varphi(N)} \geqslant \sum_{\substack{0\leqslant r<N\\ \gcd(x+r,N)=1}} \frac{1}{\varphi(N)} = 1 \quad \text{for all } x \in \mathbb{Z},$$

where φ is Euler's totient function. If $1 < n_1 < \cdots < n_k$ and $x \equiv -a_i \pmod{n_i}$ for all those $i \in I = \{1 \leqslant j \leqslant k : n_j \text{ is a prime}\}$ (such an integer x exists by the Chinese Remainder Theorem), then

$$\sum_{\substack{i=1\\ i\notin I}}^{k} \frac{1}{\varphi(n_i)} \geqslant \sum_{\substack{1\leqslant i\leqslant k\\ \gcd(x+a_i,n_i)=1}} \frac{1}{\varphi(n_i)} \geqslant 1.$$

Thus, if $n \in \mathbb{Z}^+$ is a covering number then we have

$$\sum_{\substack{d\mid n\\ d \text{ is composite}}} \frac{1}{\varphi(d)} \geqslant 1. \tag{1.2}$$

Throughout this paper, for a predicate P we let $[\![P]\!]$ be 1 or 0 according as P holds or not. For a real number x, as usual we use $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote the greatest integer not exceeding x and the least integer greater than or equal to x, respectively.

Our first theorem in this paper gives a sufficient condition for covering numbers.

Theorem 1.1. *Let $p_1, \ldots, p_r$ be distinct primes, and let $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+$. Suppose that*

$$\prod_{0<t<s} (\alpha_t + 1) \geqslant p_s - [\![r \neq s]\!] \qquad \text{for all } s = 1, \ldots, r. \tag{1.3}$$

Then $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is a covering number.

Remark 1.1. As usual the empty product $\prod_{0<t<1} (\alpha_t + 1)$ is regarded as 1, thus (1.3) implies that $p_1 = 2 \leqslant r$.

The Erdős–Selfridge conjecture can be viewed as the converse of the following result.

Corollary 1.1. *Let $p_1 = 2 < p_2 < \cdots < p_r$ $(r > 1)$ be distinct primes. Then there are $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+$ such that $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is a covering number.*

Proof. For $t = 1, \ldots, r-1$ we set

$$\alpha_t = \left\lceil \frac{p_{t+1} - [\![t \neq r-1]\!]}{p_t - 1} \right\rceil - 1.$$

Then

$$\prod_{0<t<s}(\alpha_t+1) \geqslant \prod_{0<t<s}\frac{p_{t+1}-[\![t+1\neq r]\!]}{p_t-[\![t\neq r]\!]} = \frac{p_s-[\![s\neq r]\!]}{p_1-[\![1\neq r]\!]} = p_s-[\![r\neq s]\!]$$

for all $s=1,\ldots,r$. Thus, by Theorem 1.1, $p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ is a covering number. □

In contrast with Corollary 1.1, we have the following second theorem.

Theorem 1.2. *Let $\alpha_1,\ldots,\alpha_r\in\mathbb{Z}^+$. Then $p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ is a covering number for some distinct primes $p_1<\cdots<p_r$, if and only if one of the following* (i)–(iii) *holds.*

(i) $r=2\leqslant\alpha_1$; (ii) $r=3$ *and* $\max\{\alpha_1,\alpha_2\}\geqslant 2$; (iii) $r\geqslant 4$.

Definition 1.2. A covering number is called a *primitive covering number* if none of its proper divisors is a covering number.

Our third theorem provides a sufficient condition for primitive covering numbers.

Theorem 1.3. *Let $p_1=2<p_2<\cdots<p_r$ $(r>1)$ be distinct primes. Suppose further that $p_t-1\mid p_{t+1}-1$ for all $0<t<r-1$, and $p_r\geqslant(p_{r-1}-2)(p_{r-1}-3)$. Then*

$$p_1^{\frac{p_2-1}{p_1-1}-1}\cdots p_{r-2}^{\frac{p_{r-1}-1}{p_{r-2}-1}-1}p_{r-1}^{\lfloor\frac{p_r-1}{p_{r-1}-1}\rfloor}p_r$$

is a primitive covering number.

Remark 1.2. By Theorem 1.3, the number $2\cdot3\cdot5\cdot7=210$ is a primitive covering number; Erdős ever constructed a cover of $\mathbb{Z}$ whose moduli are all the 14 proper divisors of 210 (cf. [Gu] or [GS]).

Corollary 1.2. *For any $r=2,3,\ldots$ there are infinitely many primitive covering numbers having exactly r distinct prime divisors.*

Proof. By Dirichlet's theorem (cf. [R, pp. 237–244]), for any $m\in\mathbb{Z}^+$ there are infinitely many primes p such that $m\mid p-1$. So, the desired result follows from Theorem 1.3. □

As an application of Theorem 1.3 and its proof, here we give our last theorem.

Theorem 1.4. (i) *An integer $n>1$ with at most two distinct prime divisors is a primitive covering number if and only if $n=2^{p-1}p$ for some odd prime p.*

(ii) *A positive integer $n\equiv 0 \pmod 3$ with exactly three distinct prime divisors is a primitive covering number if and only if $n=2\cdot3^{(p-1)/2}p$ for some prime $p>3$.*

(iii) *If $p > 5$ is a prime, then both $2^3 5^{\lfloor (p-1)/4 \rfloor} p$ and $2 \cdot 3 \cdot 5^{\lfloor (p-1)/4 \rfloor} p$ are primitive covering numbers. If $p > 7$ is a prime, then $2 \cdot 3^2 7^{\lfloor (p-1)/6 \rfloor} p$ is a primitive covering number, and so is $2^5 7^{\lfloor (p-1)/6 \rfloor} p$ provided that $p \neq 13, 19$.*

Remark 1.3. Note that $2^5 7^2 \cdot 13$ and $2^5 7^3 \cdot 19$ are both covering numbers by Theorem 1.1. But we don't know whether they are primitive covering numbers.

The following corollary provides an affirmative answer to a question of Erdős [E80].

Corollary 1.3. *There are infinitely many positive integers n such that among the subsets of $D_n = \{d \geqslant 2 : d \mid n\}$ only D_n can be the set of all the moduli in a cover of $\mathbb{Z}$ with distinct moduli.*

Proof. Let p be one of the infinitely many odd primes. By Theorem 1.4(i), $2^{p-1}p$ is a primitive covering number.

Let (1.0) be any minimal cover of $\mathbb{Z}$ with $1 < n_1 < \cdots < n_k$ and $[n_1, \ldots, n_k] = 2^{p-1}p$. We want to show that $\{n_1, \ldots, n_k\} = \{d > 1 : d \mid 2^{p-1}p\}$. By a conjecture of Š. Znám proved by R. J. Simpson [Si], we have

$$k \geqslant 1 + f([n_1, \ldots, n_k]) = 1 + (p-1)(2-1) + (p-1) = 2p - 1,$$

where the Mycielski function $f : \mathbb{Z}^+ \to \mathbb{Z}$ is given by $f(\prod_{t=1}^r p_t^{\alpha_t}) = \sum_{t=1}^r \alpha_t (p_t - 1)$ with $p_1, \ldots, p_r$ distinct primes and $\alpha_1, \ldots, \alpha_r$ nonnegative integers (cf. [S90] and [Z]). On the other hand,

$$k \leqslant |\{d > 1 : d \mid 2^{p-1}p\}| = |\{2^\alpha p^\beta : \alpha = 0, \ldots, p-1;\ \beta = 0, 1\}| - 1 = 2p - 1.$$

So $k = 2p - 1 = |\{d > 1 : d \mid 2^{p-1}p\}|$ and we are done. □

In the next section we are going to prove Theorems 1.1 and 1.2. Section 3 is devoted to our proofs of Theorems 1.3 and 1.4. To conclude this section we propose the following conjecture concerning the converse of Theorem 1.1.

Conjecture 1.1. *Any primitive covering number can be written in the form $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_1, \ldots, p_r$ distinct primes and $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+$, such that (1.3) is satisfied.*

Remark 1.4. Actually the author made this conjecture on July 16, 1988. Since (1.3) implies $p_1 = 2$, Conjecture 1.1 is stronger than the Erdős–Selfridge conjecture.

2. Proofs of Theorems 1.1 and 1.2

For $n \in \mathbb{Z}^+$ let $d(n)$ denote the number of distinct positive divisors of n. If n has the factorization $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ where $p_1, \ldots, p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+$, then it is well known that $d(n) = \prod_{t=1}^{r}(\alpha_t + 1)$.

Proof of Theorem 1.1. For each $s = 1, \ldots, r$, since

$$d(p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}}) = \prod_{0<t<s} (\alpha_t + 1) \geqslant p_s - [\![r \neq s]\!]$$

there exist $p_s - [\![r \neq s]\!]$ distinct positive divisors $d_1^{(s)}, \ldots, d_{p_s - [\![r\neq s]\!]}^{(s)}$ of $\prod_{0<t<s} p_t^{\alpha_t}$. Let $\mathcal{A}$ be the system consisting of $0(\text{mod } d_{p_r}^{(r)} p_r^{\alpha_r})$ and the following $\sum_{s=1}^{r} \alpha_s(p_s - 1)$ residue classes:

$$jp_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha-1} (\text{mod } d_j^{(s)} p_s^{\alpha}) \quad (\alpha = 1, \ldots, \alpha_s;\ j = 1, \ldots, p_s - 1;\ s = 1, \ldots, r).$$

Then all the moduli of $\mathcal{A}$ are distinct. Observe that

$$\begin{aligned} &\bigcup_{j=1}^{p_s-1} jp_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha-1} (\text{mod } d_j^{(s)} p_s^{\alpha}) \\ \supseteq &\bigcup_{j=1}^{p_s-1} jp_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha-1} (\text{mod } p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha}) \\ = &0(\text{mod } p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha-1}) \setminus 0(\text{mod } p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha}) \end{aligned}$$

and

$$\begin{aligned} &\bigcup_{\alpha=1}^{\alpha_s} \left(0(\text{mod } p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha-1}) \setminus 0(\text{mod } p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha})\right) \\ &= 0(\text{mod } p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}}) \setminus 0(\text{mod } p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha_s}). \end{aligned}$$

If an integer x is not in the residue class $0(\text{mod } d_{p_r}^{(r)} p_r^{\alpha_r})$, then $x \not\equiv 0 \pmod{p_1^{\alpha_1} \cdots p_r^{\alpha_r}}$ and hence

$$x \in 0(\text{mod } 1) \setminus 0(\text{mod } p_1^{\alpha_1} \cdots p_r^{\alpha_r}) = \bigcup_{s=1}^{r} \left(0\left(\text{mod } \prod_{0<t<s} p_t^{\alpha_t}\right) \setminus 0\left(\text{mod } \prod_{t=1}^{s} p_t^{\alpha_t}\right)\right).$$

Therefore $\mathcal{A}$ does form a cover of $\mathbb{Z}$. $\square$

Remark 2.1. In the proof of Theorem 1.1, we make use of some basic ideas in [Z] and [S90].

Lemma 2.1. *Let $p_1, \ldots, p_r$ be distinct primes and $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+$. Suppose that $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is a covering number but $\prod_{0<t<r} p_t^{\alpha_t}$ is not. Then we must have $\prod_{0<t<r}(\alpha_t + 1) \geqslant p_r$.*

Proof. Let (1.0) be a minimal cover of $\mathbb{Z}$ with $1 < n_1 < \cdots < n_k$ and $[n_1, \ldots, n_k] \mid p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Since $\prod_{0<t<r} p_t^{\alpha_t}$ is not a covering number, p_r divides $[n_1, \ldots, n_k]$. Let $\alpha \in \mathbb{Z}^+$ be the largest integer such that p_r^{α} divides at least one of the moduli $n_1, \ldots, n_k$. Then we have

$$|\{1 \leqslant i \leqslant k : p_r^{\alpha} \mid n_i\}| \geqslant p_r$$

by [SS, Theorem 1] or [S96, Corollary 3]. Note that

$$|\{1 \leqslant i \leqslant k : p_r^\alpha \mid n_i\}| \leqslant |\{dp_r^\alpha : d \mid p_1^{\alpha_1} \cdots p_{r-1}^{\alpha_{r-1}}\}| = d\Big(\prod_{0<t<r} p_t^{\alpha_t}\Big) = \prod_{0<t<r}(\alpha_t+1).$$

So the desired result follows. □

Proof of Theorem 1.2. If (i) holds, then $2^{\alpha_1}3^{\alpha_2}$ is a covering number by Theorem 1.1 since $1 \geqslant 2-1$ and $\alpha_1+1 \geqslant 3$. If (ii) is valid, then $2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}$ is a covering number by Theorem 1.1, since $\alpha_1+1 \geqslant 3-1$ and $(\alpha_1+1)(\alpha_2+1) \geqslant (1+1)(2+1) > 5$. When (iii) happens (i.e., $r \geqslant 4$), letting $p_1, \ldots, p_r$ be the first r primes in the ascending order, we then have $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, hence $\prod_{s=1}^r p_s^{\alpha_s}$ is a covering number by Theorem 1.1, because $\alpha_1+1 \geqslant 3-1$, $(\alpha_1+1)(\alpha_2+1) \geqslant 5-1$, and $p_s < 2^{s-1} \leqslant \prod_{0<t<s}(\alpha_t+1)$ for $s \geqslant 4$ (by mathematical induction and Bertrand's postulate (cf. [R, pp. 220–221]) proved by Chebyshev).

Now suppose that there are distinct primes $p_1 < \cdots < p_r$ such that $n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ is a covering number. Let $d > 1$ be the smallest covering number dividing n. Then d is a primitive covering number. By Lemma 2.1, d cannot be a prime power. So $r \geqslant 2$. If $r = 2$ and $\alpha_1 = 1$, then $d = p_1p_2^\beta$ for some $\beta = 1, \ldots, \alpha_2$, thus by Lemma 2.1 we get the contradiction $1+1 \geqslant p_2 > p_1 \geqslant 2$. If $r = 3$ and $\alpha_1 = \alpha_2 = 1$, then $d = p_1p_2p_3^\gamma$ for some $\gamma = 1, \ldots, \alpha_3$, hence by Lemma 2.1 we have $(1+1)(1+1) \geqslant p_3 \geqslant 5$ which is impossible. Therefore one of (i)–(iii) holds. □

3. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Set

$$\alpha_1 = \frac{p_2-1}{p_1-1}-1, \ \ldots, \ \alpha_{r-2} = \frac{p_{r-1}-1}{p_{r-2}-1}-1 \ \text{ and } \ \alpha_{r-1} = \left\lfloor \frac{p_r-1}{p_{r-1}-1}\right\rfloor.$$

Then

$$\prod_{0<t<s}(\alpha_t+1) = \prod_{0<t<s}\frac{p_{t+1}-1}{p_t-1} = \frac{p_s-1}{p_1-1} = p_s-1$$

for $s = 1, \ldots, r-1$, and

$$\prod_{0<t<r}(\alpha_t+1) = \prod_{0<t<r-1}(\alpha_t+1) \times \left(\left\lfloor\frac{p_r-1}{p_{r-1}-1}\right\rfloor+1\right) > (p_{r-1}-1)\frac{p_r-1}{p_{r-1}-1} = p_r-1.$$

Thus $n = p_1^{\alpha_1}\cdots p_{r-1}^{\alpha_{r-1}}p_r$ is a covering number in light of Theorem 1.1.

Let $d > 1$ be the smallest covering number dividing n. It remains to show that $d = n$.

Suppose that p_s is the maximal prime divisor of d. If $s \neq r$, then $\prod_{0<t<s}(\alpha_t+1) = p_s-1 < p_s$ which contradicts Lemma 2.1. Therefore, d has the form $p_1^{\beta_1}\cdots p_{r-1}^{\beta_{r-1}}p_r$ where $\beta_t \in \{0, \ldots, \alpha_t\}$

for $t = 1, \dots, r-1$. By Lemma 2.1,

$$\prod_{0<t<r} (\beta_t + 1) \geqslant p_r.$$

If $\beta_{r-1} < \alpha_{r-1}$, then

$$\prod_{0<t<r} (\beta_t + 1) \leqslant \prod_{0<t<r-1} (\alpha_t + 1) \times \alpha_{r-1} = (p_{r-1} - 1) \left\lfloor \frac{p_r - 1}{p_{r-1} - 1} \right\rfloor \leqslant p_r - 1 < p_r.$$

So we must have $\beta_{r-1} = \alpha_{r-1}$.

Assume that $\beta_j < \alpha_j$ for some $1 \leqslant j \leqslant r-2$. Then

$$\prod_{t=1}^{r-1} (\beta_t + 1) \leqslant \prod_{t=1}^{r-2} (\alpha_t + 1) \times \frac{\alpha_j}{\alpha_j + 1} (\alpha_{r-1} + 1) = m,$$

where

$$\begin{aligned} m =& (p_{r-1} - 1) \left(1 - \frac{p_j - 1}{p_{j+1} - 1}\right) \left(\left\lfloor \frac{p_r - 1}{p_{r-1} - 1} \right\rfloor + 1 \right) \\ \leqslant& (p_{r-1} - 1) \left(1 - \frac{p_j - 1}{p_{j+1} - 1}\right) \left(\frac{p_r - 1}{p_{r-1} - 1} + 1 \right) \\ =& (p_{r-1} - 2 + p_r) \left(1 - \frac{p_j - 1}{p_{j+1} - 1}\right). \end{aligned}$$

Since

$$p_r \geqslant (p_{r-1} - 3)(p_{r-1} - 1 - 1) \geqslant (p_{r-1} - 3)\left(\frac{p_{j+1} - 1}{p_j - 1} - 1\right),$$

we have

$$(p_{r-1} - 2)\left(\frac{p_{j+1} - 1}{p_j - 1} - 1\right) - p_r < \frac{p_{j+1} - 1}{p_j - 1}$$

and hence

$$m \leqslant (p_{r-1} - 2)\left(1 - \frac{p_j - 1}{p_{j+1} - 1}\right) + p_r - p_r \frac{p_j - 1}{p_{j+1} - 1} < p_r + 1.$$

We claim that

$$m = (p_j - 1) \left(\frac{p_{r-1} - 1}{p_j - 1} - \frac{p_{r-1} - 1}{p_{j+1} - 1} \right) \left(\left\lfloor \frac{p_r - 1}{p_{r-1} - 1} \right\rfloor + 1 \right) \neq p_r.$$

In fact, m is composite when $j > 1$; if $j = 1$ then

$$\frac{p_{r-1} - 1}{p_j - 1} - \frac{p_{r-1} - 1}{p_{j+1} - 1} = p_{r-1} - 1 - \frac{p_{r-1} - 1}{p_2 - 1} \geqslant \frac{p_{r-1} - 1}{2} > 1$$

unless $p_{r-1} = 3$ in which case

$$m = \left\lfloor \frac{p_r - 1}{3 - 1} \right\rfloor + 1 = \frac{p_r + 1}{2} < p_r.$$

In view of the above,

$$p_r \leqslant \prod_{t=1}^{r-1}(\beta_t+1) \leqslant m < p_r.$$

This leads to a contradiction.

By the above, $\beta_j = \alpha_j$ for all $j = 1, \ldots, r-1$, and thus $d = n$. We are done. □

Proof of Theorem 1.4. (i) If $p > 2$ is a prime, then $2^{p-1}p = 2^{\lfloor (p-1)/(2-1)\rfloor}p$ is a primitive covering number by Theorem 1.3 in the case $r = 2$.

By Lemma 2.1, any prime power cannot be a primitive covering number.

Now suppose that $n = p_1^{\alpha_1}p_2^{\alpha_2}$ is a primitive covering number, where $p_1 < p_2$ are two distinct primes, and $\alpha_1, \alpha_2 \in \mathbb{Z}^+$. Then

$$2 < \frac{\sigma(n)}{n} < \left(1 + \frac{1}{p_1} + \frac{1}{p_1^2} + \cdots\right)\left(1 + \frac{1}{p_2} + \frac{1}{p_2^2} + \cdots\right) = \frac{p_1}{p_1 - 1} \cdot \frac{p_2}{p_2 - 1}.$$

If $p_1 > 2$, then

$$2 < \frac{p_1}{p_1 - 1} \cdot \frac{p_2}{p_2 - 1} \leqslant \frac{3}{3-1} \cdot \frac{5}{5-1} = \frac{15}{8} < 2$$

which leads to a contradiction. So $p_1 = 2$. Observe that $\alpha_1 + 1 \geqslant p_2$ by Lemma 2.1. Therefore n is a multiple of $2^{p_2-1}p_2$. Since both $2^{p_2-1}p_2$ and n are primitive covering numbers, we must have $n = 2^{p_2-1}p_2$.

(ii) If $p > 3$ is a prime, then

$$2 \cdot 3^{\frac{p-1}{2}}p = 2^{\frac{3-1}{2-1}-1}3^{\frac{p-1}{3-1}}p$$

is a primitive covering number by Theorem 1.3 in the case $r = 3$.

Now assume that $n = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$ is a primitive covering number with $n \equiv 0 \pmod 3$, where $p_1 < p_2 < p_3$ are distinct primes, and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}^+$. If $p_1 \geqslant 3$, then

$$\begin{aligned}
\sum_{\substack{d \mid n \\ d \text{ is composite}}} \frac{1}{\varphi(d)} <& \sum_{s=1}^{3} \frac{1}{p_s - 1}\left(\frac{1}{p_s} + \frac{1}{p_s^2} + \cdots\right) \\
&+ \sum_{1 \leqslant s < t \leqslant 3} \frac{1}{(p_s-1)(p_t-1)}\left(1 + \frac{1}{p_s} + \frac{1}{p_s^2} + \cdots\right)\left(1 + \frac{1}{p_t} + \frac{1}{p_t^2} + \cdots\right) \\
&+ \frac{1}{(p_1-1)(p_2-1)(p_3-1)}\prod_{s=1}^{3}\left(1 + \frac{1}{p_s} + \frac{1}{p_s^2} + \cdots\right) \\
=& \sum_{s=1}^{3} \frac{1}{(p_s-1)^2} + \sum_{1 \leqslant s < t \leqslant 3} \frac{p_s p_t}{(p_s-1)^2(p_t-1)^2} + \frac{p_1 p_2 p_3}{(p_1-1)^2(p_2-1)^2(p_3-1)^2} \\
\leqslant& \frac{1}{(3-1)^2} + \frac{1}{(5-1)^2} + \frac{1}{(7-1)^2} + \frac{3 \cdot 5}{2^2 4^2} + \frac{3 \cdot 7}{2^2 6^2} + \frac{5 \cdot 7}{4^2 6^2} + \frac{3 \cdot 5 \cdot 7}{2^2 4^2 6^2} = \frac{1905}{2304}
\end{aligned}$$

and this contradicts (1.2). So $p_1 = 2$. Since $3 \mid n$, we have $p_2 = 3$. By part (i), $2^2 \cdot 3$ is a primitive covering number and hence it does not divide n. Therefore n has the form $2 \cdot 3^{\alpha} p$, where $p > 3$ is a prime and $\alpha \in \mathbb{Z}^+$.

By Lemma 2.1, $(1+1)(\alpha+1) \geqslant p$. Thus $\alpha \geqslant (p-1)/2$ and hence n is a multiple of $2 \cdot 3^{(p-1)/2} p$. As both $2 \cdot 3^{(p-1)/2} p$ and n are primitive covering numbers, we must have $n = 2 \cdot 3^{(p-1)/2} p$.

(iii) If $p > 5$ is a prime, then by Theorem 1.3 both

$$2^3 5^{\lfloor \frac{p-1}{4} \rfloor} p = 2^{\frac{5-1}{2-1}-1} 5^{\lfloor \frac{p-1}{5-1} \rfloor} p \quad \text{and} \quad 2 \cdot 3 \cdot 5^{\lfloor \frac{p-1}{4} \rfloor} p = 2^{\frac{3-1}{2-1}-1} 3^{\frac{5-1}{3-1}-1} 5^{\lfloor \frac{p-1}{5-1} \rfloor} p$$

are primitive covering numbers.

If p is a prime greater than 19, then $p > (7-2)(7-3)$, hence both

$$2 \cdot 3^2 \cdot 7^{\lfloor \frac{p-1}{6} \rfloor} p = 2^{\frac{3-1}{2-1}-1} 3^{\frac{7-1}{3-1}-1} 7^{\lfloor \frac{p-1}{7-1} \rfloor} p \quad \text{and} \quad 2^5 7^{\lfloor \frac{p-1}{6} \rfloor} p = 2^{\frac{7-1}{2-1}-1} 7^{\lfloor \frac{p-1}{7-1} \rfloor} p$$

are primitive covering numbers by Theorem 1.3. When $p \in \{11, 13, 17, 19\}$, we have

$$p > (7-3)\left(\max\left\{\frac{7-1}{3-1}, \frac{3-1}{2-1}\right\} - 1\right) = 8$$

and hence $2 \cdot 3^2 \cdot 7^{\lfloor \frac{p-1}{6} \rfloor} p$ is still a primitive covering number by the proof of Theorem 1.3. If p is 11 or 17, then

$$m = (7-1)\left(1 - \frac{2-1}{7-1}\right)\left(\left\lfloor \frac{p-1}{7-1} \right\rfloor + 1\right) < p + 1$$

and hence $2^5 7^{\lfloor \frac{p-1}{6} \rfloor} p$ is a primitive covering number by the proof of Theorem 1.3.

Combining the above we have shown Theorem 1.4. $\square$

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