CENTRAL AND LOCAL LIMIT THEOREMS FOR EXCEDANCES BY CONJUGACY CLASS AND BY DERANGEMENT

Lane Clark
Department of Mathematics, Southern Illinois University Carbondale, Carbondale, IL 62901
lclark@math.siu.edu

Received: 11/15/00, Revised: 9/14/01 Accepted: 1/31/02, Published: 2/8/02

Abstract

We give central and local limit theorems for the number of excedances of a uniformly distributed random permutation belonging to certain sequences of conjugacy classes and belonging to the sequence of derangements.

1. Introduction

For $n \in \mathbb{P}$ and $0 \le k \le D_n \in \mathbb{N}$, let $b(n,k) \in [0,\infty)$ and $B_n := b(n,0)+\cdots+b(n,D_n) > 0$. We say the array $\{b(n,k) : n \ge 1,\ 0 \le k \le D_n\}$ satisfies a *central limit theorem with mean* μ_n *and variance* σ_n^2 provided

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sum_{k \le \lfloor (x)_n \rfloor} \frac{b(n,k)}{B_n} - \Phi(x) \right| = 0, \tag{1}$$

where $(x)_n := x\sigma_n + \mu_n$. Equivalently, we say $\{b(n,k)\}$ is *asymptotically normal.* Further, the array $\{b(n,k) : n \ge 1,\ 0 \le k \le D_n\}$ satisfies a *local limit theorem on* $S \subseteq \mathbb{R}$ if and only if

$$\lim_{n\to\infty} \sup_{x\in S} \left| \sigma_n \frac{b(n, \lfloor (x)_n \rfloor)}{B_n} - \phi(x) \right| = 0. \tag{2}$$

In general, a central limit theorem for a sequence of random variables gives (1) from which (2) follows under certain conditions (see Theorems 1, 3 and 4 and Lemma 2). We note that (1) is equivalent to pointwise convergence in view of the uniform continuity of $e^{-t^2/2}$ and [11; Theorem 1 of Section 9] (see Bender [1]).

A permutation of $[n]$ is a bijection $\sigma : [n] \to [n]$ which we write as $\sigma = \big(\sigma(1), \ldots, \sigma(n)\big)$. We denote the set of permutations of $[n]$ by $\mathfrak{S}[n]$. A permutation $\sigma \in \mathfrak{S}[n]$ is a *derangement* of $[n]$ provided $\sigma(i) \neq i$ for all $i \in [n]$. Let $\mathfrak{D}[n]$ denote the set of derangements of $[n]$.

Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_p \geq 1)$ of n $(= \lambda_1 + \cdots + \lambda_p) \in \mathbb{P}$, let $m_i = m_i(\lambda) := \big|\{j \in [p] : \lambda_j = i\}\big|$ for $i \in [n]$. Here we write $\lambda = 1^{m_1}2^{m_2}\cdots n^{m_n}$. We say $\sigma \in \mathfrak{S}[n]$ is of *cycle type* λ if and only if the decomposition of σ into a product of disjoint cycles has precisely $m_i(\lambda)$ cycles of length i for $i \in [n]$. We denote the set of all permutations of $[n]$ of cycle type λ by $\mathfrak{S}_\lambda[n]$. The $\mathfrak{S}_\lambda[n]$, where λ is a partition of n, are the conjugacy classes of $\mathfrak{S}[n]$. It is readily seen (see [7]; p. 233]) that $|\mathfrak{S}_\lambda[n]| = n!/(1^{m_1}2^{m_2}\cdots n^{m_n})(m_1!\, m_2! \cdots m_n!)$.

We say $i \in [n]$ is an *excedance* (respectively, a *descent*) of $\sigma \in \mathfrak{S}[n]$ provided $\sigma(i) > i$ (respectively, $i \neq n$ and $\sigma(i) > \sigma(i+1)$). Let $\mathrm{E}(\sigma)$ (respectively, $\mathrm{Des}(\sigma)$) denote the *set of excedances* (respectively, *descents*) of σ and $e(\sigma) := |\mathrm{E}(\sigma)|$ (respectively, $d(\sigma) := |\mathrm{Des}(\sigma)| + 1$ and $\mathrm{maj}(\sigma) := \sum\{i : i \in \mathrm{Des}(\sigma)\}$).

Given a partition $\lambda = 1^{m_1}2^{m_2}\cdots n^{m_n}$ of $n \geq 2$, let $a_\lambda(n,k) := |\{\sigma \in \mathfrak{S}_\lambda[n] : |\mathrm{Des}(\sigma)| = k\}|$ and $b_\lambda(n,k) := |\{\sigma \in \mathfrak{S}_\lambda[n] : e(\sigma) = k\}|$ for $k \in \mathbb{N}$. From [8; Theorem B], $a_\lambda(n,1) = \prod_{i=1}^{n} \binom{f_{i2}+m_i-1}{m_i}$ where $f_{i2} = \sum_{d|i} \mu(d)2^{i/d}/i$ for $i \in P$ when $m_1 \neq n$ and $n \geq 2$. Here, $a_\lambda(n,1) \in \mathbb{P}$ since $f_{i2} \in \mathbb{P}$ upon appealing to its definition, while, $a_\lambda(n,1) = (n-b+1)f_{b2}$ when $\lambda = 1^{n-b}b$ with $2 \leq b \leq n$. From [4; Theorem 3.1], $b_\lambda(n,1)$ is the coefficient of x in $Q_{\lambda,n}(x)$ (see Section 3). Properties of the Eulerian polynomials imply that $b_\lambda(n,1) = 0$ when $m_2 + \cdots + m_n \geq 2$ and $b_\lambda(n,1) = \binom{n}{b}$ when $\lambda = 1^{n-b}b$ with $2 \leq b \leq n$. Hence, $a_\lambda(n,1) \neq b_\lambda(n,1)$ when $m_2+\cdots+m_n \geq 2$ and $a_\lambda(n,1) \neq b_\lambda(n,1)$ for all but at most $b-1$ integers n when $\lambda = 1^{n-b}b$ with $2 \leq b \leq n$. Consequently, descents and excedances are **not** equidistributed over conjugacy classes in general. As is well known (see [14; p. 23]), they **are** equidistributed over $\mathfrak{S}[n]$.

In connection with the Betti numbers of certain varieties, Stanley [15] was interested in the symmetry and unimodality of the coefficients of some polynomials obtained by enumerating a set of permutations according to the number of excedances. Brenti [3], [4] showed these excedance polynomials are symmetric and unimodal when the set is a conjugacy class or is the set of derangements thus generalizing a conjecture and answering a question of [15]. Fulman [10] gave a central limit theorem for the coefficients of polynomials obtained by enumerating permutations belonging to certain sequences of conjugacy classes according to the number of descents. Our discussion in the previous paragraph shows that descents and excedances are not equidistributed over the conjugacy classes considered in [10] (see the remark after Theorem 3). We give both central and local limit theorems for the coefficients of excedance polynomials over certain sequences of conjugacy classes, including those of [10], and over the sequence of derangements (see Theorems 3 and 4). A more precise statement of the results and techniques is given in the next paragraph.

Using the method of moments, Fulman [10] recently gave a central limit theorem for $d(\sigma)$ with $\mu_n = (n-1)/2$, $\sigma_n^2 = (n-1)/12$ and for $\mathrm{maj}(\sigma)$ with $\mu_n = \binom{n}{2}/2$,

$\sigma_n^2 = n(n-1)(2n+5)/72$ when σ is a uniformly distributed random permutation in $\mathfrak{S}_{\lambda(n)}[n]$ for certain sequences $\lambda(n)$ of cycle types. In this paper, we give central and local limit theorems for $e(\sigma)$ with various μ_n, σ_n^2 when σ is a uniformly distributed random permutation in $\mathfrak{S}_{\lambda(n)}[n]$ for certain sequences $\lambda(n)$ of cycle types, including those of [10], and for $e(\sigma)$ with $\mu_n = (n-1)/2 + o(1)$, $\sigma_n^2 = 25n/12 + o(1)$ when σ is a uniformly distributed random permutation in $\mathfrak{D}[n]$ (see Theorems 3 and 4). Of course our results immediately give asymptotic formulas for the number of such permutations with a certain number of excedances. We use the method of Harper [12]. A slight extension of this method requiring only a nice factorization of the polynomials over $\mathbb{R}[x]$ is given in the next section (see Theorem 1). We refer the reader to the excellent survey of Pólya frequency sequences by Pitman [13].

Let $\mathbb{N}$ denote the nonnegative integers; $\mathbb{P}$ the positive integers; $\mathbb{R}_0$ the nonnegative real numbers and $\mathbb{R}$ the real numbers. The collection of all polynomials in an indeterminate x whose coefficients are in the *set* A is denoted by $A[x]$. For $n \in \mathbb{P}$, $[n] := \{1, \ldots, n\}$. The cardinality of a set A is denoted by $|A|$. We denote the largest integer at most x by $\lfloor x \rfloor$.

The expectation of a random variable (r.v.) X is denoted by $\mathrm{E}(X)$ and its variance by $\mathrm{Var}(X)$. We write $X_n \xrightarrow{d} X$ when the sequence X_n of r.v.s converges in distribution to the r.v. X. For $x \in \mathbb{R}$, let

$$\phi(x) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \quad \text{and} \quad \Phi(x) := \int_{-\infty}^{x} \phi(t)\, dt\,.$$

We write $N(0,1)$ for a normally distributed r.v. with mean 0 and variance 1. We refer the reader to Comtet [7] for combinatorics and Durrett [9] for probability.

2. General Results

For completeness, we first formalize a slight extension of the method of Harper [12] requiring only a nice factorization of the polynomials over $\mathbb{R}[x]$, which we will use in the next section.

For $n \in \mathbb{P}$, let $Q_n(x) := P_{n,1}(x) \cdots P_{n,N(n)}(x)$ where $P_{n,j}(x) := \sum_{k=0}^{d_{n,j}} a(n,j,k)x^k \in \mathbb{R}_0[x] - \{0\}$ for $1 \le j \le N(n) \in \mathbb{P}$. Let $X_{n,1}, \ldots, X_{n,N(n)}$ be row-independent r.v.s with

$$\Pr(X_{n,j} = k) = \frac{a(n,j,k)}{P_{n,j}(1)} \quad (1 \le j \le N(n),\ 0 \le k \le d_{n,j})$$

(which, of course, exist), hence,

$$\mathrm{E}(X_{n,j}) = \frac{P_{n,j}'(1)}{P_{n,j}(1)} \quad \text{and} \quad \mathrm{E}(X_{n,j}^2) = \frac{P_{n,j}'(1) + P_{n,j}''(1)}{P_{n,j}(1)} \quad (1 \le j \le N(n)).$$

Let

$$X_n := X_{n,1} + \cdots + X_{n,N(n)}$$

hence,

$$\mu_n := \mathrm{E}(X_n) = \frac{Q_n'(1)}{Q_n(1)} \quad \text{and} \quad \sigma_n^2 := \mathrm{Var}(X_n) = \frac{Q_n'(1)}{Q_n(1)} + \frac{Q_n''(1)}{Q_n(1)} - \left(\frac{Q_n'(1)}{Q_n(1)}\right)^2 . \tag{3}$$

If $Q_n(x) := \sum_{k=0}^{D_n} b(n,k)x^k$ where $D_n := d_{n,1} + \cdots + d_{n,N(n)}$, then by row-independence

$$\Pr(X_n = k) = \frac{b(n,k)}{Q_n(1)} \quad (0 \le k \le D_n) .$$

Let

$$Y_{n,j} := \frac{X_{n,j} - \mathrm{E}(X_{n,j})}{\sigma_n} \quad \text{and} \quad Y_n := Y_{n,1} + \cdots + Y_{n,N(m)} = \frac{X_n - \mu_n}{\sigma_n} .$$

Then $Y_{n,j}$ assumes the values $\big(k - \mathrm{E}(X_{n,j})\big)/\sigma_n$ with probabilities $a(n,j,k)/P_{n,j}(1)$ for $0 \le k \le d_{n,j}$. Let $M_n := \max_{1\le j\le N(n)} \big\{\mathrm{E}(X_{n,j}),\ |d_{n,j} - \mathrm{E}(X_{n,j})|\big\}$ and $G_{n,j}(x) := \Pr(Y_{n,j} \le x)$ be the distribution function of $Y_{n,j}$ for $1 \le j \le N(n)$. For all $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{j=1}^{N(n)} \int_{|y|>\epsilon} y^2 dG_{n,j}(y) = 0 ,$$

provided $\lim_{n\to\infty} M_n/\sigma_n = 0$. By the Lindeberg-Feller Theorem (see [9; pp. 98–101]), $Y_n \xrightarrow{d} N(0,1)$. We have proved the following central limit theorem which can be expressed in terms of the $b(n,k)$, X_n or Y_n since

$$\Pr(Y_n \le x) = \Pr(X_n \le \lfloor (x)_n \rfloor) = \sum_{k \le \lfloor (x)_n \rfloor} \frac{b(n,k)}{Q_n(1)}$$

for all $x \in \mathbb{R}$ where $(x)_n := x\sigma_n + \mu_n$.

Theorem 1. Suppose $\lim_{n\to\infty} M_n/\sigma_n = 0$. For each $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \sum_{k \le \lfloor (x)_n \rfloor} \frac{b(n,k)}{Q_n(1)} = \Phi(x)$$

where $(x)_n = x\sigma_n + \mu_n$. (As mentioned in the introduction, this is equivalent to (1).)

Remark. If $d_{n,j} \le 1$ for all $1 \le j \le N(n)$, then $M_n \le 1$ and $\sigma_n \to \infty$ as $n \to \infty$ suffices (Harper's method).

A sequence $a(0), \ldots, a(D)$ of real numbers is *log-concave* provided $a^2(j) \ge a(j-1)a(j+1)$ for all $1 \le j \le D-1$. It has *no internal zeros* if and only if there exist no indices

$0 \le i < j < k \le D$ with $a_i, a_k \neq 0$ but $a_j = 0$. The array $\{b(n,k) : n \ge 1,\ \ 0 \le k \le D_n\}$ of real numbers *has property* P provided the sequence $b(n,0), \ldots, b(n, D_n)$ has property P for all $n \in \mathbb{P}$. We require the following result of Canfield [5; Theorem II].

Lemma 2 (Canfield [5]). Suppose the array $\{b(n,k) : n \ge 1,\ \ 0 \le k \le D_n\}$ satisfies (1) and is log-concave with no internal zeros where $\sigma_n \to \infty$ as $n \to \infty$. Then the array $\{b(n,k) : n \ge 1,\ \ 0 \le k \le D_n\}$ satisfies (2) on $S = \mathbb{R}$.

3. Applications to Excedances

For $n \in \mathbb{P}$, the polynomial

$$A_n(x) := \sum_{\sigma \in \mathfrak{S}[n]} x^{d(\sigma)} := \sum_{k=1}^{n} A(n,k) x^k \in \mathbb{N}[x]$$

is called the *nth Eulerian polynomial* where $A_0(x) := 1$. The *Eulerian numbers* $A(n,k)$ satisfy the recurrence relation

$$A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k) \qquad (n,k \ge 2) \tag{4}$$

with the boundary conditions $A(n,1) = 1$ $(n \ge 1)$ and $A(n,k) = 0$ $(1 \le n < k)$ (see [7; pp. 240–246]). Then (4) together with $A_0(x) = 1$ gives

$$A_n(x) = nxA_{n-1}(x) + (x - x^2)A'_{n-1}(x) \qquad (n \ge 1).$$

Hence,

$$A'_n(x) = nA_{n-1}(x) + (nx - 2x + 1)A'_{n-1}(x) + (x - x^2)A''_{n-1}(x) \qquad (n \ge 1)$$

and, upon iteration together with $A_n(1) = n!$ $(n \ge 0)$, we have

$$A'_n(1) = \frac{(n+1)!}{2} \qquad (n \ge 1) \tag{5}$$

while,

$$A''_n(x) = (2n-2)A'_{n-1}(x) + (nx - 4x + 2)A''_{n-1}(x) + (x - x^2)A'''_{n-1}(x) \qquad (n \ge 1)$$

and, upon iteration together with (5), we have

$$A''_n(1) = (n+1)!\ \frac{3n-2}{12} \qquad (n \ge 2). \tag{6}$$

We first consider excedances by conjugacy class.

Excedances by Conjugacy Class. For a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of $n \in \mathbb{P}$, let

$$Q_{\lambda,n}(x) := \sum_{\sigma \in \mathfrak{S}_\lambda[n]} x^{e(\sigma)} \in \mathbb{N}[x] \,.$$

Brenti [4; Theorem 3.1] showed that all

$$Q_{\lambda,n}(x) = |\mathfrak{S}_\lambda[n]| \, \prod_{i=1}^{p} \frac{A_{\lambda_i - 1}(x)}{(\lambda_i - 1)!} \,.$$

Of course, $Q_{\lambda,n}(1) = |\mathfrak{S}_\lambda[n]|$. Then

$$Q'_{\lambda,n}(x) = Q_{\lambda,n}(x) \sum_{i=1}^{p} \frac{A'_{\lambda_i - 1}(x)}{A_{\lambda_i - 1}(x)}$$

so that (5) gives

$$Q'_{\lambda,n}(1) = Q_{\lambda,n}(1) \, \frac{n - m_1(\lambda)}{2}$$

and

$$Q''_{\lambda,n}(x) = Q_{\lambda,n}(x) \left\{ \left(\sum_{i=1}^{p} \frac{A'_{\lambda_i - 1}(x)}{A_{\lambda_i - 1}(x)} \right)^2 + \sum_{i=1}^{p} \frac{A''_{\lambda_i - 1}(x)}{A_{\lambda_i - 1}(x)} - \sum_{i=1}^{p} \left(\frac{A'_{\lambda_i - 1}(x)}{A_{\lambda_i - 1}(x)} \right)^2 \right\}$$

so that (5) and (6) give

$$Q''_{\lambda,n}(1) = Q_{\lambda,n}(1) \left\{ \frac{n^2}{4} - \frac{m_1(\lambda)}{12}(6n - 5) + \frac{m_1^2(\lambda)}{4} - \frac{5n}{12} - \frac{m_2(\lambda)}{6} \right\} .$$

Hence, (3) gives

$$\mu_{\lambda,n} = \frac{n - m_1(\lambda)}{2} \quad \text{and} \quad \sigma^2_{\lambda,n} = \frac{n - m_1(\lambda) - 2m_2(\lambda)}{12} \,.$$

Since $A_n(x)$ has degree of $A_n(x)$ nonpositive real zeros for $n \in \mathbb{N}$ (see [7; p. 292]), each $Q_{\lambda,n}(x)$ has degree of $Q_{\lambda,n}(x)$ nonpositive real zeros. Hence, $M_{\lambda,n} \le 1$ and the coefficients of all $Q_{\lambda,n}(x)$ are log-concave with no internal zeros (see [2; Theorem 1.2.1]).

Now $Q_{\lambda,n}(x) = \sum_{k=0}^{n-1} b_\lambda(n,k) x^k$ where $b_\lambda(n,k)$ is the number of $\sigma \in \mathfrak{S}_\lambda[n]$ with $e(\sigma) = k$. Let $Z_{\lambda,n}(\sigma) = e(\sigma)$ where σ is chosen randomly from $\mathfrak{S}_\lambda[n]$ according to a uniform distribution. Then $\Pr(Z_{\lambda,n} = k) = b_\lambda(n,k)/|\mathfrak{S}_\lambda[n]|$ $(0 \le k \le n-1)$ so that $X_{\lambda,n} \overset{d}{=} Z_{\lambda,n}$. Hence, $\mathrm{E}(Z_{\lambda,n}) = \mu_{\lambda,n}$ and $\mathrm{Var}(Z_{\lambda,n}) = \sigma^2_{\lambda,n}$.

For any sequence $\lambda(n)$ of partitions of n, now let

$$Q_n(x) := Q_{\lambda(n),n}(x) := \sum_{k=0}^{n-1} b(n,k) x^k$$

with $\mu_n := \mu_{\lambda(n),n}$ and $\sigma_n^2 := \sigma_{\lambda(n),n}^2$. As a consequence of Theorem 1 and Lemma 2, we have the following results for the number of excedances of $\sigma \in \mathfrak{S}_{\lambda(n)}[n]$ which can be expressed in terms of the $b(n,k)$, $X_{\lambda(n),n}$, $Y_{\lambda(n),n}$ or $Z_{\lambda(n),n}$.

Theorem 3. For any sequence $\lambda(n)$ of partitions of n with $\sigma_n \to \infty$ as $n \to \infty$, the array $\{b(n,k) : n \geq 1,\ 0 \leq k \leq n-1\}$ satisfies both a central limit theorem and a local limit theorem on $\mathbb{R}$ with the above μ_n and σ_n^2.

Remark. We do not require, as in [10], all $m_i\big(\lambda(n)\big) \to 0$ as $n \to \infty$ for our results to hold. If, however, all $m_i\big(\lambda(n)\big) \to 0$ as $n \to \infty$, then $\sigma_n \to \infty$ as $n \to \infty$, and we have both a central limit theorem and a local limit theorem on $\mathbb{R}$ with $\mu_n = n/2$ and $\sigma_n^2 = n/12$ (so not quite the same asymptotic distribution as $d(\sigma)$ in [10]). Here, descents and excedances are not equidistributed over $\mathfrak{S}_{\lambda(n)}[n]$ for **all** sufficiently large n.

We next consider excedances by derangement (which is not a conjugacy class). For $n \in \mathbb{N}$, let $s_n := \sum_{k=0}^{n} (-1)^k/k!$.

Excedances by Derangement. For $n \in \mathbb{P}$, let

$$Q_n(x) := \sum_{\sigma \in \mathfrak{D}[n]} x^{e(\sigma)} \in \mathbb{N}[x]$$

and $Q_0(x) := 1$. Brenti [3; proof of Proposition 5] showed that

$$\frac{A_n(x)}{x} = \sum_{m=0}^{n} \binom{n}{m} Q_m(x) \qquad (n \geq 1)\,.$$

By inversion (see [7; pp. 143–144])

$$Q_n(x) = (-1)^n + \sum_{m=1}^{n} (-1)^{n-m} \binom{n}{m} \frac{A_m(x)}{x} \qquad (n \geq 1)\,. \tag{7}$$

For $n \geq 3$, (5), (6) and (7) give

$$\begin{aligned}
Q_n(1) &= n!\, s_n\,,\\
Q_n'(1) &= \frac{n!}{2}\big\{(n-1)s_{n-1} + s_{n-2}\big\}\,,\\
Q_n''(1) &= \frac{n!}{12}\left\{(3n^2+13n+10)s_{n-2} + (6n+13)s_{n-3} + \frac{3(-1)^{n-3}}{(n-3)!}\right\}.
\end{aligned}$$

Hence, (3) and $s_{n-r}/s_n = 1 + o(n^{-1})$ for $r = 1, 2, 3$ give

$$\mu_n = \frac{n-1}{2} + o(1) \quad \text{and} \quad \sigma_n^2 = \frac{25n}{12} + o(1) \quad \text{as } n \to \infty\,.$$

Zhang [16] proved a conjecture of Brenti [3; p. 1140] by showing that $Q_n(x)$ has degree of $Q_n(x)$ distinct nonpositive real zeros for $n \in \mathbb{P}$. Hence, $M_n \leq 1$ and the coefficients of $Q_n(x)$ are log-concave with no internal zeros for $n \in \mathbb{P}$.

Now $Q_n(x) = \sum_{k=0}^{n-1} b(n,k)x^k$ where $b(n,k)$ is the number of $\sigma \in \mathfrak{D}[n]$ with $e(\sigma) = k$. Let $Z_n(\sigma) = e(\sigma)$ where σ is chosen randomly from $\mathfrak{D}[n]$ according to a uniform distribution. Then $\Pr(Z_n = k) = b(n,k)/|\mathfrak{D}[n]|$ $(0 \leq k \leq n-1)$ so that $X_n \stackrel{d}{=} Z_n$. Hence, $\mathrm{E}(Z_n) = \mu_n$ and $\mathrm{Var}(Z_n) = \sigma_n^2$ where $\sigma_n \to \infty$ as $n \to \infty$. As a consequence of Theorem 1 and Lemma 2, we have the following results for the number of excedances of $\sigma \in \mathfrak{D}[n]$ which can be expressed in terms of the $b(n,k)$, X_n, Y_n or Z_n.

Theorem 4. The array $\{b(n,k) : n \geq 1,\ 0 \leq k \leq n-1\}$ satisfies both a central limit theorem and a local limit theorem on $\mathbb{R}$ with the above μ_n and σ_n^2.

For $n \in \mathbb{P}$, we note for completeness that

$$\frac{A_n(x)}{x} = \sum_{\sigma \in \mathfrak{S}[n]} x^{e(\sigma)} := \sum_{k=0}^{n-1} b(n,k)x^k \in \mathbb{N}[x]\,.$$

Hence, we have a central limit theorem for the array $\{b(n,k) : n \geq 1,\ 0 \leq k \leq n-1\}$ with $\mu_n = (n-1)/2$ and $\sigma_n^2 = (n+1)/12$. Since the coefficients of $A_n(x)/x$ are log-concave with no internal zeros, we also have a local limit theorem on $\mathbb{R}$. (Compare with the analogous results of [6] for $A_n(x)$ with $\mu_n = (n+1)/2$ and $\sigma_n^2 = (n+1)/12$.)

Acknowledgment. I wish to thank the referee for comments and suggestions which have led to an improved version of this paper.

References

1. E.A. Bender, Central and local limit theorems applied to asymptotic enumeration, *J. Combin. Theory Ser. A* **15** (1973), 91–111.

2. F. Brenti, Unimodal, log-concave and Pólya frequency sequences in combinatorics, *Mem. Amer. Math. Soc.* **81** (1989), no. 413.

3. F. Brenti, Unimodal polynomials arising from symmetric functions, *Proc. Amer. Math. Soc.* **108** (1990), no. 4, 1133–1141.

4. F. Brenti, Permutation enumeration symmetric functions, and unimodality, *Pacific J. Math.* **157** (1993), no. 1, 1–28.

5. E.R. Canfield, Central and local limit theorems for the coefficients of polynomials of binomial type, *J. Combin. Theory Ser. A* **23** (1977), no. 3, 275–290.

6. L. Carlitz, D.C. Kurtz, R. Scoville and O.P. Stackelberg, Asymptotic properties of Eulerian numbers, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **23** (1972), 47–54.

7. L. Comtet, "Advanced Combinatorics," D. Reidel, Boston, MA, 1974.

8. P. Diaconis, M. McGrath and J. Pitman, Riffle shuffles, cycles, and descents, *Combinatorica* **15** (1995), no. 1, 11–29.

9. R. Durrett, "Probability: Theory and Examples," Wadsworth and Brooks/Cole, Pacific Grove, CA, 1991.

10. J. Fulman, The distribution of descents in fixed conjugacy classes of the symmetric groups, *J. Combin. Theory Ser. A* **84** (1998), no. 2, 171–180.

11. B.V. Gnedenko and A.N. Kolmogorov, "Limit Distributions for Sums of Independent Random Variables," Addison-Wesley, Cambridge, MA, 1954.

12. L.H. Harper, Stirling behavior is asymptotically normal, *Ann. Math. Statist.* **38** (1967), 410–414.

13. J. Pitman, Probabilistic bounds on the coefficients of polynomials with only real zeros, *J. Combin. Theory Ser. A* **77** (1997), no. 2, 279–303.

14. R. Stanley, "Enumerative Combinatorics," Vol. 1, Wadsworth & Brooks/Cole, Monterey, CA, 1986.

15. R. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, *Ann. New York Acad. Sci.* **576** (1989), 500–535.

16. X. Zhang, On q-derangement polynomials, in "Combinatorics and Graph Theory '95," Vol. 1 (Hefei), World Scientific Publishing, River Edge, NJ, 1995.