

PLANE PARTITION DIAMONDS AND GENERALIZATIONS

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Abstract

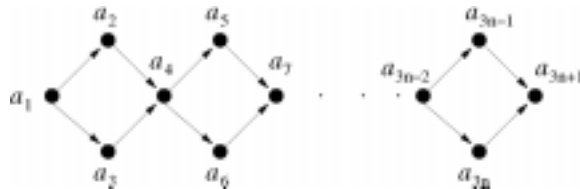
In this note we generalize the plane partition diamonds of Andrews, Paule, and Riese to plane partition polygons and plane tree diamonds and show how to compute their generating functions.

1. Introduction

In [1], Andrews, Paule, and Riese introduce the family of plane partition diamonds. A *plane partition diamond* of length n is a sequence of length $3n + 1$ of nonnegative integers $a = (a_1, \dots, a_{3n+1})$ satisfying, for $0 \leq i \leq n - 1$,

$$a_{3i+1} \geq a_{3i+2}, \quad a_{3i+1} \geq a_{3i+3}, \quad a_{3i+2} \geq a_{3i+4}, \quad a_{3i+3} \geq a_{3i+4}.$$

This is shown graphically below.



The configuration $(7, 5, 5, 5, 4, 5, 2, 1, 1, 0, 0, 0, 0)$ is a plane partition diamond of length 4 :

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Using partition analysis, the authors in [1] find the generating function of these configurations. Let \mathcal{D}_n be the set of all plane partition diamonds of length n . Their result is as follows :

Theorem 1 [1] For $n \geq 1$,

$$D_n(x_1, \dots, x_{3n+1}) := \sum_{a \in \mathcal{D}_n} \prod_{i=1}^{3n+1} x_i^{a_i} = \prod_{i=1}^{3n+1} \frac{1}{1 - X_i} \prod_{i=1}^n \frac{1 - X_{3i-2} X_{3i}}{1 - X_{3i}/x_{3i-1}},$$

where $X_k = x_1 \dots x_k$.

Note that when $x_i = q$, $1 \leq i \leq 3n + 1$, Theorem 1 gives

$$\sum_{a \in \mathcal{D}_n} q^{a_1 + \dots + a_{3n+1}} = \frac{\prod_{i=1}^n (1 + q^{3i-1})}{\prod_{i=1}^{3n+1} (1 - q^i)}.$$

In this note we give a combinatorial proof of this generating function in Section 2 and exhibit some natural generalizations in Sections 3 and 4.

2. Plane Partition Diamonds

We give a combinatorial proof of Theorem 1. Let $\tilde{\mathcal{D}}_n$ be the set of diamonds in \mathcal{D}_n such that $a_{3n+1} = 0$ and let $\tilde{D}_n(x_1, \dots, x_{3n})$ be the associated generating function. First we study $\tilde{\mathcal{D}}_1$.

Lemma 2

$$\tilde{D}_1(x_1, x_2, x_3) = \frac{1 - x_1^2 x_2 x_3}{(1 - x_1)(1 - x_1 x_2)(1 - x_1 x_3)(1 - x_1 x_2 x_3)}.$$

Proof. These configurations are such that $a_1 \geq a_2 \geq 0$, $a_1 \geq a_3 \geq 0$. We consider three cases :

- $a_1 \geq a_2 \geq a_3$. These configurations correspond to partitions into at most three parts which have generating function

$$S_1(x_1, x_2, x_3) = 1/((1 - x_1)(1 - x_1 x_2)(1 - x_1 x_2 x_3)).$$

- $a_1 \geq a_3 \geq a_2$. These configurations have generating function

$$S_2(x_1, x_2, x_3) = 1/((1 - x_1)(1 - x_1x_3)(1 - x_1x_2x_3)).$$

- $a_1 \geq a_3 = a_2$. These configurations have generating function

$$S_3(x_1, x_2, x_3) = 1/((1 - x_1)(1 - x_1x_2x_3)).$$

It easy to see that \tilde{D}_1 is equal to $S_1 + S_2 - S_3$. □

We next show how to get from \tilde{D}_n to D_n for $n \geq 1$. (This lemma is analogous to Corollary 2.2 in [1].)

Lemma 3

$$D_n(x_1, \dots, x_{3n+1}) = \frac{\tilde{D}_n(x_1, \dots, x_{3n})}{(1 - X_{3n+1})}.$$

Proof. Starting with a diamond $a = (a_1, \dots, a_{3n+1})$ in \mathcal{D}_n , corresponding to the term $\prod_{i=1}^{3n+1} x_i^{a_i}$, we can associate to it in $\tilde{\mathcal{D}}_n$ the diamond $\tilde{a} = (a_1 - a_{3n+1}, a_2 - a_{3n+1}, \dots, a_{3n} - a_{3n+1}, 0)$, corresponding to the term $\prod_{i=1}^{3n+1} x_i^{a_i} / X_{3n+1}$. □

Finally we decompose a diamond in \mathcal{D}_n into a diamond in $\tilde{\mathcal{D}}_1$ and a diamond in \mathcal{D}_{n-1} .

Lemma 4 For $n > 1$,

$$D_n(x_1, \dots, x_{3n+1}) = \tilde{D}_1(x_1, x_2, x_3)D_{n-1}(x_1x_2x_3x_4, x_5, x_6, \dots, x_{3n+1}).$$

Proof. Given a diamond $a = (a_1, a_2, a_3, 0)$ in $\tilde{\mathcal{D}}_1$ and a diamond $b = (b_1, \dots, b_{3(n-1)+1})$ in \mathcal{D}_{n-1} , we map them to the diamond $c = (c_1, \dots, c_{3n+1})$ with $c_i = a_i + b_1$ for $1 \leq i \leq 3$ and $c_i = b_{i-3}$ for $4 \leq i \leq 3n + 1$. It is easy to check that $c \in \mathcal{D}_n$ and that this map is reversible. □

Combining the three lemmas gives the proof of Theorem 1.

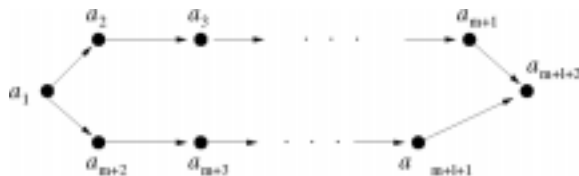
3. Generalization : The Plane Partition Polygons

Let us now generalize. Let m and l be integers. An (m, l) -gon is a sequence of length $m + l + 2$ of nonnegative integers $a = (a_1, \dots, a_{m+l+2})$ such that

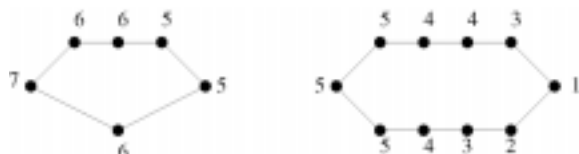
$$a_j \geq a_{j+1}, \quad 1 \leq j \leq m; \quad a_{m+1} \geq a_{m+l+2}$$

$$a_1 \geq a_{m+2}; a_{m+k} \geq a_{m+k+1}, 2 \leq k \leq m + l + 1,$$

as illustrated below.



The case $m = l = 1$ corresponds to the diamonds of length 1. Two examples are shown below, a $(3, 1)$ -gon $(7, 6, 6, 5, 6, 5)$ and a $(4, 4)$ -gon, $(5, 5, 4, 4, 3, 5, 4, 3, 2, 1)$.



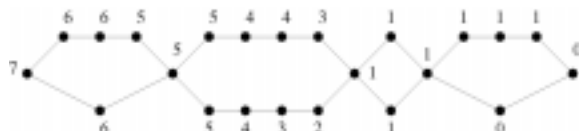
Given two lists of natural numbers of length $n > 0$, $s = (s_1, \dots, s_n)$ and $v = (v_1, \dots, v_n)$, below we will define a plane partition polygon as a sequence of integers satisfying constraints corresponding to a linear arrangement of (s_j, v_j) -gons, $1 \leq j \leq n$. In order to reference the starting index of each (s_j, v_j) -gon in the sequence of integers, define ℓ_j for $0 \leq j \leq n$ by $\ell_j = j + 1 + \sum_{i=1}^j (s_i + v_i)$.

Definition 5 An (s, v) -plane partition polygon is a configuration $a = (a_1, a_2, \dots)$ of length ℓ_n such that

$$(a_{\ell_j}, a_{\ell_j+1}, \dots, a_{\ell_{j+1}})$$

is an (s_{j+1}, v_{j+1}) -gon for all $0 \leq j \leq n - 1$.

In the example pictured below, if $n = 4$ and $s = (3, 4, 1, 3)$ and $v = (1, 4, 1, 1)$ then $a = (7, 6, 6, 5, 6, 5, 5, 4, 4, 3, 5, 4, 3, 2, 1, 1, 1, 1, 1, 1, 1, 0, 0)$ is an (s, v) -plane partition polygon, as $(7, 6, 6, 5, 6, 5)$ is a $(3, 1)$ -gon $(5, 5, 4, 4, 3, 5, 4, 3, 2, 1)$ is a $(4, 4)$ -gon $(1, 1, 1, 1)$ is a $(1, 1)$ -gon and $(1, 1, 1, 1, 0, 0)$ is a $(3, 1)$ -gon.



Remark. The plane partition diamonds of length n are the (s, v) -plane partition polygons with $s_i = v_i = 1, 1 \leq i \leq n$.

Let $\mathcal{D}_{s,v}$ be the set of (s, v) -plane partition polygons and let $\tilde{\mathcal{D}}_{s,v}$ be the subset of $\mathcal{D}_{s,v}$ consisting of those configurations whose last entry is 0.

Remark. Here we do not exhibit the ℓ_n -variable generating function, although it should be possible to do so.

Let $D_{s,v}(z, q)$ and $\tilde{D}_{s,v}(z, q)$ be the generating functions

$$D_{s,v}(z, q) = \sum_{a \in \mathcal{D}_{s,v}} z^{a_1} q^{\sum_{i=1}^{\ell_n} a_i}, \quad \tilde{D}_{s,v}(z, q) = \sum_{a \in \tilde{\mathcal{D}}_{s,v}} z^{a_1} q^{\sum_{i=1}^{\ell_n} a_i}.$$

We show the following :

Theorem 6 *The generating function for $D_{s,v}(z, q)$ is :*

$$D_{s,v}(z, q) = \frac{\prod_{i=1}^n H_{s_i, v_i}(zq^{\ell_i-1}, q)}{(zq; q)_{\ell_n}},$$

with

$$H_{m,l}(z, q) = 1 + \sum_{k=1}^{\min(m,l)} z^k q^{k(k+1)} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} l \\ k \end{bmatrix}_q,$$

where $(z; q)_m = \prod_{i=0}^{m-1} (1 - zq^i)$ and $\begin{bmatrix} m \\ k \end{bmatrix}_q = (q^{m+1-k}; q)_k / (q; q)_k$.

To get a proof of Theorem 6, we use a generalization of the previous arguments for diamonds. First consider the case $n = 1$ and $s = (m)$ and $v = (l)$.

Lemma 7 *For $m, l \geq 0$*

$$\tilde{D}_{(m),(l)}(z, q) = \frac{H_{m,l}(z, q)}{(zq; q)_{m+l+1}}.$$

Proof. We know that if $a_1 = k$ then (a_2, \dots, a_{m+1}) is a partition into m nonnegative parts less than or equal to k , and $(a_{m+2}, \dots, a_{m+l+1})$ is a partition into l nonnegative parts less than or equal to k , and these sets have generating functions $\begin{bmatrix} m+k \\ k \end{bmatrix}_q$, and $\begin{bmatrix} l+k \\ k \end{bmatrix}_q$, respectively. So

$$\tilde{D}_{m,l}(z, q) = 1 + \sum_{k=1}^{\infty} z^k q^k \begin{bmatrix} m+k \\ k \end{bmatrix}_q \begin{bmatrix} l+k \\ k \end{bmatrix}_q.$$

It is possible to give a pure combinatorial argument, but we will make use of the (third version of) Heine's transformation [2] :

$$\sum_{k \geq 0} \frac{(a, b; q)_k z^k}{(c, q; q)_k} = \frac{(abz/c; q)_{\infty}}{(z; q)_{\infty}} \times \sum_{k \geq 0} \frac{(c/a, c/b; q)_k (abz/c)^k}{(c, q; q)_k},$$

where $(a, b; q)_k = (a; q)_k(b; q)_k$. Setting $a = q^{m+1}$, $b = q^{l+1}$, $z = zq$, $c = q$ gives

$$\sum_{k=0}^{\infty} z^k q^k \begin{bmatrix} m+k \\ k \end{bmatrix}_q \begin{bmatrix} l+k \\ k \end{bmatrix}_q = \frac{1}{(zq; q)_{m+l+1}} \times \sum_{k=0}^{\infty} \frac{(q^{-m}, q^{-l}; q)_k (zq^{m+l+2})^k}{(q, q; q)_k} \dots$$

Finally, to get the result, we use that

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{(q^{-m}; q)_k}{(q; q)_k} \times (-1)^k q^{mk - k(k-1)/2}.$$

□

Now we need to go from $\tilde{\mathcal{D}}$ to \mathcal{D} .

Lemma 8 For any sequences s and v of length n

$$D_{s,v}(z, q) = \tilde{D}_{s,v}(z, q)/(1 - zq^{\ell_n}).$$

Proof. Starting with $a = (a_1, \dots, a_{\ell_n})$ in $\mathcal{D}_{s,v}$, corresponding to $z^{a_1}q^{a_1+\dots+a_{\ell_n}}$, associate $\tilde{a} = (a_1 - a_{\ell_n}, \dots, a_{\ell_n-1} - a_{\ell_n}, 0)$ in $\tilde{\mathcal{D}}_{s,v}$. corresponding to $z^{a_1}q^{a_1+\dots+a_{\ell_n}}/(zq^{\ell_n})^{a_{\ell_n}}$, and conversely. □

Finally we decompose the sequences.

Lemma 9 For any $n > 1$, $s = (s_1, \dots, s_n)$ and $v = (v_1, \dots, v_n)$,

$$D_{s,v}(z, q) = \tilde{D}_{s_1, v_1}(z, q)D_{s', v'}(zq^{s_1+v_1+1}, q),$$

with $s' = (s_2, \dots, s_n)$ and $v' = (v_2, \dots, v_n)$.

Proof. Starting with $a = (a_1, a_2, \dots)$ in $\tilde{\mathcal{D}}_{(s_1), (v_1)}$, and $b = (b_1, b_2, \dots)$ in $\mathcal{D}_{s', v'}$, we map them to the configuration $c = (c_1, \dots, c_{\ell_n})$ with $c_i = a_i + b_1$ for $1 \leq i \leq s_1 + v_1 + 1$ and $c_i = b_{i-s_1-v_1-1}$ for $s_1 + v_1 + 2 \leq i \leq \ell_n$. It is easy to check that $c \in \mathcal{D}_{s,v}$, and the mapping is reversible. □

Once again, combining the lemmas gives a proof of Theorem 6.

Example 1. The generating function of the plane partition diamonds of length n , is

$$\frac{\prod_{i=1}^n (1 + zq^{3i-1})}{(zq; q)_{3n+1}}$$

which is Theorem 1 with $x_1 = zq$ and $x_i = q$ for $i > 1$.

Example 2.. The generating function of the plane partition hexagons of length n , that is $s_i = v_i = 2, 1 \leq i \leq n$ is

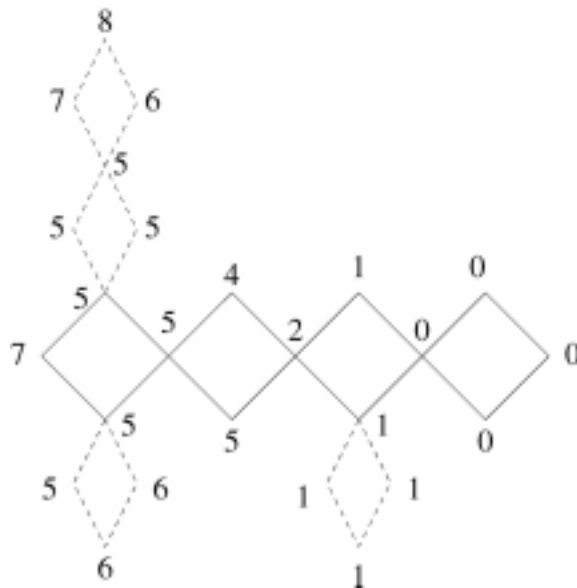
$$\frac{\prod_{i=1}^n (1 + zq^{5i-3}(1+q)^2 + z^2q^{10i-4})}{(zq; q)_{5n+1}}.$$

Example 3. The generating function of the plane partition octagons of length n , that is $s_i = v_i = 3, 1 \leq i \leq n$ is

$$\frac{\prod_{i=1}^n (1 + zq^{7i-5}(1+q+q^2)^2 + z^2q^{14i-8}(1+q+q^2)^2 + z^3q^{21i-9})}{(zq; q)_{7n+1}}.$$

4. Another Generalization : Plane Tree Diamonds

For a plane partition diamond $a = (a_1, \dots, a_{3n+1})$, say that a_{3n+1} is the *least* of the diamond because $a_{3n+1} \leq a_i$ for all i . Suppose we are given n and $t = (t_1, \dots, t_{3n+1})$, a sequence of non negative integers. Let a be any plane partition diamond of length n and let $d = (d_1, \dots, d_{3n+1})$ be any sequence of plane partition diamonds such that for $1 \leq i \leq n$, d_i has length t_i and least element a_i . Then we construct a *plane tree diamond* by attaching d_i to a_i for $1 \leq i \leq n$. For example, let $n = 4$ and $t = (0, 2, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0)$. Let $a = (7, 5, 5, 5, 4, 5, 2, 1, 1, 0, 0, 0, 0)$. Let d be defined by $d_1 = (7)$, $d_2 = (8, 7, 6, 5, 5, 5, 5)$, $d_3 = (6, 6, 5, 5)$, $d_4 = (5)$, $d_5 = (4)$, $d_6 = (5)$, $d_7 = (2)$, $d_8 = (1)$, $d_9 = (1, 1, 1, 1)$, $d_{10} = d_{11} = d_{12} = d_{13} = (0)$. The figure below shows the corresponding plane tree diamond, with a shown by solid lines and the d_i shown by dotted lines.



Let $T_{n,t}(q)$ be the generating function of these trees. Using the same kind of combinatorial arguments as in the previous sections, we get the following.

Theorem 10 *The generating function for plane tree diamonds is :*

$$T_{n,t}(q) = D_n(q^{3t_1+1}, q^{3t_2+1}, \dots, q^{3t_{3n+1}+1}) \prod_{i=1}^{3n+1} \tilde{D}_{t_i}(q).$$

The example given above has the generating function :

$$T_{4,(0,2,1,0,0,0,0,0,1,0,0,0,0)}(q) = \frac{1}{(1-q)(1-q^8)(q^{12}; q)_6(q^{21}; q)_5} \cdot \frac{(1-q^{14})(1-q^{28})(1-q^{37})(1-q^{46})}{(1-q^5)(1-q^{14})(1-q^{20})(1-q^{23})} \cdot \frac{(1+q^2)^3(1+q^5)}{(q; q)_3^3(q^3; q)_3}.$$

Note that this could be carried through with plane partition polygons also.

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- [1] G. Andrews, P. Paule and A. Riese, *Mac Mahon's Partition Analysis, VIII. Plane Partition Diamonds*, Adv. in Applied Maths **27**, 231-242 (2001).
- [2] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and its Applications, 35, Cambridge University Press, (1990).