GEOGRAPHY PLAYED ON AN N-CYCLE TIMES A 4-CYCLE

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Abstract

In the game of Geography, two players alternate moving a coin along the edges of a directed graph. The coin may not be moved to the same vertex more than once, and the last player who is able to move wins. We analyze this game played on the cartesian product of cycles $C_n \times C_4$.

1. Introduction

Geography, as described in [2] and [3], is a game for two players, played on a directed graph. A coin is initially placed on a vertex and, on his turn, a player moves the coin along a directed edge to an adjacent previously unoccupied vertex. The players move alternately and, under normal play, the last player who is able to move wins.

Some particular cases of Geography have been studied. For example, *Kotzig's (or Modular) Nim* is played on a directed graph with vertex set $\{0, 1, 2, \ldots, n-1\}$. There is a directed edge from x to y if and only if $y - x \pmod n$ is contained in a given set of integers, called the *move set*. Many instances of this game have been solved; we refer the reader to [1] and [4] for the results.

We will assume that the reader is familiar with standard terminology of combinatorial game theory, as described in [1]. In particular, for an impartial game G, the *outcome class* of G, denoted $\mathcal{G}(G)$, is $\mathcal{P}$ if the previous player has a winning strategy, and $\mathcal{N}$ if the next player wins. We will refer to our two players as Alice and Bob, with Alice playing first.

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We now define the graphs to be considered here. Let C_n denote the directed cycle with vertex set $V(C_n) = \{0, 1, \ldots, n-1\}$ and edge set $E(C_n) = \{(0,1), (1,2), \ldots, (n-2, n-1), (n-1, 0)\}$ where (x, y) denotes an edge directed from x to y. The *cartesian product* $C_m \times C_n$ has vertex set $V(C_m) \times V(C_n)$ and there is a directed edge from (a, b) to (c, d) if and only if $a = c$ and (b, d) is a directed edge in C_n, or $b = d$ and (a, c) is a directed edge in C_m.

Nowakowski and Poole [5] investigated the game of Geography played on a cartesian product of two directed cycles. In particular, the following theorem may be found in [5].

Theorem 1 *If $n = 2$, or if both n and m are even, then $\mathcal{G}(C_n \times C_m) = \mathcal{N}$.*

Furthermore, the main result of [5] is the determination of the outcome class of $C_n \times C_3$ for all positive integers n.

2. Geography played on $C_n \times C_4$

In this paper, we consider the game of Geography played on $C_n \times C_4$. The following is an immediate consequence of Theorem 1.

Theorem 2 *If n is even then $\mathcal{G}(C_n \times C_4) = \mathcal{N}$.*

Therefore, we need only consider the case in which n is odd and in the sequel we will assume that this is the case. The rest of this paper is devoted to proving the following result.

Theorem 3 *If n is an odd positive integer then*

$$\mathcal{G}(C_n \times C_4) = \begin{cases} \mathcal{P} & \text{if } n \equiv 11 \pmod{12}, \\ \mathcal{N} & \text{otherwise.} \end{cases}$$

We will visualize the graph $C_n \times C_4$ as a rectangular board having 4 rows and n columns. The rows are numbered $0, 1, 2, 3$ from top to bottom and the columns are numbered $0, 1, 2, \ldots, n-1$ from left to right: the ordered pair (c, r) refers to the cell at column c and row r. We will assume, without loss of generality, that the coin is initially placed at $(0, 0)$ and will denote this on the diagrams by X. A legal move is to move right (in the first coordinate) or down (in the second coordinate), including "wrap around" moves. Every time play reaches the last column, we will say that one *pass* has occurred. We will denote the players' moves by A and B for Alice and Bob, respectively. It will be convenient, on occasion, to refer to the *ith* move by player Y; this will be done using the notation Y_i. Further, a move that is forced upon player Y will be denoted $Y!$ and parentheses will be used when it does not matter which of the two available moves is played.

The cell (c, r) is called a *sink* if it has not yet been occupied but its followers, $(c + 1, r)$ and $(c, r + 1)$, have both been played. Thus a player who is able to reach a sink wins the game. An important notion, introduced in [5], is that of a *closing-off sequence.* In the game $C_n \times C_4$, such a sequence of moves has one of the following forms.

$$\begin{array}{rclcccccccc} S_1 &=& (a,i) &\to& (a+1,i) &\to& (a+1,i+1) &\to& (a+1,i+2) &\to& (a+1,i+3) \\ S_2 &=& (a,i) &\to& (a,i+1) &\to& (a+1,i+1) &\to& (a+1,i+2) &\to& (a+1,i+3) \\ S_3 &=& (a,i) &\to& (a,i+1) &\to& (a,i+2) &\to& (a+1,i+2) &\to& (a+1,i+3) \end{array}$$

Notice that a closing-off sequence creates a sink at $(a, i+3)$ (assuming that it has not yet been occupied) and that the game will end no later than at this cell on the next pass. As we will see in the next section, a player who is the first to complete a closing-off sequence on the first pass wins the game.

3. Preliminary Lemmas

In this section, we establish two lemmas, namely Lemma 2 and Lemma 3, which present conditions under which one player may force a win. They will be used repeatedly in the proof of Theorem 3.

Lemma 1 *If the first two moves of the game are in the first column then Alice has a winning strategy.*

Proof. We have that Alice's first move is to $(0, 1)$ and Bob replies to $(0, 2)$. Suppose now that Alice moves to $(0, 3)$: Bob is then forced to respond with $(1, 3)$.

Alice's winning strategy now is to always play down. Note that this move is always available to her and that Bob is always the first player to reach a new column. Eventually then, Bob will be the first to play on the last column and, as the game cannot wrap around to the first column, Alice will win. □

Therefore, should Alice begin the game by moving down to $(0, 1)$, Bob is forced to respond by moving across to $(1, 1)$. In the sequel then, we will assume that, on the first pass, neither $(0, 2)$ nor $(0, 3)$ are occupied.

Lemma 2 *A player who completes a closing-off sequence on the first pass, and is the first to do so, has a winning strategy.*

Proof. Suppose that Alice is the first player to complete a closing-off sequence and does so on the first pass by moving to $(a + 1, i)$. This move creates a sink at (a, i) and we show that Alice wins the game by moving to this vertex on the second pass.

First, notice that since n is odd, colouring the vertices in a checkerboard fashion shows that Alice will be the player to reach (a, i) on the second pass.

Second, it must be shown that the game will not end before Alice reaches (a, i). We observe that, since the game cannot proceed beyond column a on the second pass, it will end earlier if and only if another sink is both created and occupied *on the same pass*. In order for this to happen, the players must make four consecutive moves (two moves each) in the same column. We claim that this cannot happen if Alice always moves across when it is possible to do so.

Clearly, up to the second last column of the first pass, Alice can always move across. At the last column, however, Alice may be forced to move down once (if Bob has played to $(n-1, 0)$, for example). However, play will not "go around" in the second coordinate in the last column because in order for this to happen, Bob must move to either $(n-1, 2)$ or $(n-1, 3)$. In either case, since we assume that neither $(0, 2)$ nor $(0, 3)$ are occupied on the first pass, the across move will be available to Alice. Finally, once the second pass has begun, it is impossible for play to go around in any one column since each column contains at least one occupied cell. Therefore, regardless of how the players move, play must continue until Alice reaches the sink at (a, i) and wins. □

Lemma 3 *If the last two moves on the first pass are to* $(n-2, 1)$ *and* $(n-1, 1)$ *then Alice wins.*

Proof. Colouring the cells in a checkerboard fashion, we see that, since n is odd, Bob must be the player to move to $(n-2, 1)$ and it is Alice's response to $(n-1, 1)$. Thus a sink is created at $(n-1, 0)$. Bob's response will be to either $(n-1, 2)$ or $(0, 1)$. In either event, Alice's winning strategy is to reply to $(0, 2)$ which, by the assumption following Lemma 1, will be unoccupied.

As in the proof of Lemma 2, once the second pass has begun, it is impossible for play to go around in any one column since each column contains at least one occupied cell. Therefore, play will eventually reach the second last column and the first player to do so will be Alice at $(n-2, 3)$, or Bob at $(n-2, 0)$ or $(n-2, 2)$. If Alice moves to $(n-2, 3)$ then Bob's next move is to either $(n-1, 3)$ or $(n-2, 0)$. In either event, Alice may reply to $(n-1, 0)$ and win. If Bob moves to $(n-2, 2)$ then Alice replies to $(n-2, 3)$ and has a winning strategy as described above. Finally, should Bob move to $(n-2, 0)$ then Alice replies to $(n-1, 0)$ and wins. □

4. Two Special Move Sequences

4.1 Playing the 12

Our analysis of the game $C_n \times C_4$ will depend upon the value of n modulo 12. The underlying reason for this is that, on the first pass, a player may force his opponent into repeating a block of moves comprising 12 columns. We call this *playing the* **12**.

Let us suppose that, on the first pass, Bob is the first player to reach column c and does so by playing to $(c, 0)$, denoted by B_0 on the diagram below. Alice may then dictate the play for the next 12 columns so that Bob will be forced to be the first player to reach the $(c + 12)th$ column by playing to $(c + 12, 0)$. The following diagram describes Alice's strategy. Recall that ! indicates a forced move and parentheses indicate that either move may be made.

B_0	A_1	(B_1)								(B_7)	A_8	$B_8!$
	(B_1)	A_2	$B_2!$	A_3	(B_3)							
				(B_3)	A_4	$B_4!$	A_5	(B_5)				
							(B_5)	A_6	$B_6!$	A_7	(B_7)	

Bob's first, third, fifth, and seventh moves may be either across or down. Notice, however, that Bob's other moves are forced. If, for example, B_2 is to $(2, 2)$ then, by Lemma 2, Alice wins by moving to $(2, 3)$ thereby completing a closing-off sequence.

4.2 Playing the 10

In a manner similar to playing the **12**, Alice may, at the beginning of the game, force a sequence of moves that comprises 10 columns. To do this, Alice plays down to $(0, 1)$ on her first move and, by the remark following Lemma 1, we may assume that Bob responds with $(1, 1)$.

The complete sequence of moves is called *playing the* **10** and is shown in the diagram below.

X								(B_6)	A_7	$B_7!$
A_1	$B_1!$	A_2	(B_2)							
		(B_2)	A_3	$B_3!$	A_4	(B_4)				
					(B_4)	A_5	$B_5!$	A_6	(B_6)	

Notice that, as in playing the **12**, Bob's third, fifth, and seventh moves are forced, in order to prevent a closing-off move from Alice.

5. Proof of Theorem 3

This section is devoted to the proof of our main result, Theorem 3. The argument is in six cases, depending upon the value of the odd integer n modulo 12.

5.1 The case $n \equiv 1 \pmod{12}$

5.11 $n = 1$

The game $C_1 \times C_4$ is trivially a first player win.

5.12 $n > 1$

In this case, $n = 13 + 12m$ for some $m \geq 0$. Alice's winning strategy is to begin by playing the **10**, followed by playing the **12** m times, and finally to finish with a block of 3 columns. The analysis is in two cases, according to Bob's second move in the final block of 3 columns.

CASE 1: Bob's second move in the final block of 3 is across

By Lemma 2, Alice wins by closing-off at $(n-1, 3)$.

X								(B)	A	$B!$	A	$\cdots$	A	$B!$	A	B
A	$B!$	A	(B)								(B)	$\cdots$				$A!$
		(B)	A	$B!$	A	(B)						$\cdots$				$B!$
					(B)	A	$B!$	A	(B)			$\cdots$	(B)			A

CASE 2: Bob's second move in the final block of 3 is down

After Bob moves to $(n-2, 1)$, Alice responds with $(n-1, 1)$ and wins by Lemma 3.

X								(B)	A	$B!$	A	$\cdots$	A	$B!$	A	
A	$B!$	A	(B)								(B)	$\cdots$			B	A
		(B)	A	$B!$	A	(B)						$\cdots$				
					(B)	A	$B!$	A	(B)			$\cdots$	(B)			

5.2 The case $n \equiv 3 \pmod{12}$

5.21 $n = 3$

Consider first the game $C_3 \times C_4$. Alice's winning strategy is to begin the game by playing across. If Bob's first move is down then Alice responds by moving across and wins by Lemma 3.

On the other hand, if Bob's first move is across then Alice's second move is forced to be down. If Bob's second move is down then Alice plays the closing-off move $(2, 3)$ and wins on her fourth move as shown below.

X	A_1	B_1
		$A_2!$
		B_2
$B_3!$	A_4	A_3

If Bob's second move is across then Alice wins on her fifth move as shown below.

X	A_1	B_1
B_2	A_3	$A_2!$
	$B_3!$	
A_5	A_4	$B_4!$

5.22 $n > 3$

We have $n = 15 + 12m$ for some $m \geq 0$. Alice's winning strategy is to open by playing the **10**, then to play the **12** m times, and finally to finish with a block of 5 columns. By moving to $(n-1, 1)$ Alice wins by Lemma 3.

X			$\cdots$		(B)	A	$B!$	A	$\cdots$	A	$B!$	A	(B)		
A	$B!$	A	$\cdots$					(B)	$\cdots$			(B)	A	$B!$	A
		(B)	$\cdots$						$\cdots$						
			$\cdots$	$B!$	A	(B)			$\cdots$	(B)					

5.3 The case $n \equiv 5 \pmod{12}$

Alice's winning strategy is to play the **12** as often as required and then to close with the block of 5 columns as shown below. By moving to $(n-1, 1)$, Alice wins by Lemma 3.

X	A	$\cdots$	(B)	A	$B!$	A	$\cdots$	(B)	A	$B!$	A	(B)		
	(B)	$\cdots$				(B)	$\cdots$				(B)	A	$B!$	A
		$\cdots$					$\cdots$							
		$\cdots$	A	(B)			$\cdots$	A	(B)					

5.4 The case $n \equiv 7 \pmod{12}$

5.41 $n = 7$

We consider first the game $C_7 \times C_4$. Observe that Bob's third move is forced to be down, otherwise Alice may move to $(6, 1)$ and win by Lemma 3. As shown below, Alice may create a sink by moving to $(3, 3)$ on the second pass.

X	A_1	(B_1)				
	(B_1)	A_2	$B_2!$	A_3		
				$B_3!$	A_4	$B_4!$
$B_5!$	$A_6!$	$B_6!$	A_7			A_5

After Bob responds to A_7 with either $(3, 0)$ or $(4, 3)$, Alice may force Bob's every move thereafter by moving to $(4, 0)$, $(6, 0)$, $(0, 1)$, $(1, 2)$ and finally to $(3, 2)$ at which point she wins.

5.42 $n > 7$

In this case, $n = 19 + 12m$ for some $m \geq 0$. The analysis is in two cases depending upon Bob's third move.

CASE 1: Bob's third move is down

Alice's winning strategy is to play the **12** $(m+1)$ times and then to finish with a block of 7 columns as shown below. Notice that Bob's penultimate move in the first pass must be down, otherwise Alice may move to $(n-1,1)$ and win by Lemma 3. Bob is forced to begin the second pass at $(0,3)$ and subsequently Alice creates a sink by moving to $(3,3)$.

X	A_1	(B_1)				$\cdots$	(B_7)	A_8	$B!$	A	(B)				
	(B_1)	A_2	$B_2!$	A_3		$\cdots$				(B)	A	$B!$	A		
				B_3	A_4	$\cdots$							$B!$	A	$B!$
$B!$	$A!$	$B!$	A			$\cdots$	A_7	(B_7)							A

On the second pass, the first player to reach the last column will be Alice at $(n-1,0)$ or Bob at $(n-1,1)$. In either case, Alice begins the third pass at $(0,1)$ and, two moves later, wins the game at $(3,2)$.

CASE 2: Bob's third move is across

Alice begins with a block of 14 columns. She then plays the **12** m times and concludes with a block of 5 columns as shown below. By Lemma 3, this is a winning strategy for Alice.

X	A	(B)										(B)	A	$B!$	A	$\cdots$
	(B)	A	$B!$	A	B_3	A	(B)								(B)	$\cdots$
						(B)	A	$B!$	A	(B)						$\cdots$
									(B)	A	$B!$	A	(B)			$\cdots$

$\cdots$		(B)	A	$B!$	A	(B)		
$\cdots$					(B)	A	$B!$	A
$\cdots$								
$\cdots$	$B!$	A	(B)					

5.5 The case $n \equiv 9 \pmod{12}$

5.51 $n = 9$

We consider first $C_9 \times C_4$. The analysis is in two cases, depending upon Bob's first move.

CASE 1: Bob's first move is down

The analysis is in two subcases, depending upon Bob's third move.

CASE 1A: Bob's third move is across

Notice that Bob's fourth move must be down, otherwise Alice may play her fifth move at $(8,1)$ which wins by Lemma 3. Alice begins the second pass at $(0,2)$ which creates a sink.

X	A_1							
	B_1	A_2	$B_2!$	A_3	B_3	A_4		
A_6						$B_4!$	$A_5!$	$B_5!$

On the second pass, the first player to reach the last column is Alice at $(8, 0)$, or Bob at $(8, 1)$ or $(8, 3)$. If Bob plays to $(8, 1)$ then Alice replies to $(0, 1)$ and wins. If Alice plays to $(8, 0)$ then Bob is forced to reply to $(8, 1)$ after which Alice moves to $(0, 1)$ and wins. Finally, if Bob moves to $(8, 3)$ then Alice moves to $(8, 0)$ and wins on her next move as described above.

CASE 1B: Bob's third move is down

On the second pass, Alice may create a sink by moving to $(3, 3)$, as shown below.

X	A_1							
	B_1	A_2	$B_2!$	A_3				
				B_3	A_4	$B_4!$	A_5	(B_5)
$B_6!$	$A_7!$	$B_7!$	A_8				(B_5)	A_6

On the second pass, the first player to reach the last column is Alice at $(8, 0)$ or Bob at $(8, 1)$. In either case, Alice begins the third pass at $(0, 1)$ and is able to force a win at $(3, 2)$.

CASE 2: Bob's first move is across

Bob's fourth move is forced to be down lest Alice reply with $(8, 1)$ and win by Lemma 3. On the second pass, Alice may create a sink by moving to $(5, 3)$, as shown below.

X	A_1	B_1	A_2	(B_2)				
			(B_2)	A_3	$B_3!$	A_4		
						$B_4!$	A_5	$B_5!$
$B_6!$	$A_7!$	$B_7!$	$A_8!$	$B_8!$	A_9			A_6

On the second pass, the first player to reach the last column is Alice at $(8, 0)$ or Bob at $(8, 1)$. In either case, Alice begins the third pass at $(0, 1)$ and, three moves later, wins at $(5, 2)$.

5.52 $n > 9$

We have $n = 21 + 12m$ for some $m \geq 0$. We require two cases, according to Bob's first move.

CASE 1: Bob's first move is down

The analysis is in two subcases, depending upon Bob's third move.

CASE 1A: Bob's third move is across

Alice's winning strategy is to open with a block of 14 columns, followed by playing the **12** m times, and then to finish with a block of 7 columns, as shown below. Alice begins the second pass by moving to $(0, 2)$, thereby creating a sink at $(0, 1)$.

X	A											(B)	A	$B!$	A	$\cdots$
	B_1	A	$B!$	A	B_3	A	(B)								(B)	$\cdots$
A						(B)	A	$B!$	A	(B)						$\cdots$
									(B)	A	$B!$	A	(B)			$\cdots$

$\cdots$		(B)	A	$B!$	A	(B)				
$\cdots$					(B)	A	$B!$	A	(B)	
$\cdots$								(B)	A	$B!$
$\cdots$	$B!$	A	(B)							

The first player to reach the last column on the second pass is Alice at $(n-1, 0)$, or Bob at $(n-1, 1)$ or $(n-1, 3)$. If Bob has moved to $(n-1, 1)$ then Alice wins immediately by moving to $(0, 1)$. If Alice has moved to $(n-1, 0)$ then Bob is forced to $(n-1, 1)$ and Alice wins as above. Finally, if Bob has moved to $(n-1, 3)$ then Alice wins by moving to $(n-1, 0)$.

CASE 1B: Bob's third move is down

Alice's winning strategy is to play the **12** $(m+1)$ times, followed by a block of 9 columns, as shown below. On the second pass, Alice moves to $(3, 3)$ which creates a sink at $(3, 2)$.

X	A									(B)	A	$B!$	A	$\cdots$
	B_1	A	$B!$	A									(B)	$\cdots$
				B_3	A	$B!$	A	(B)						$\cdots$
$B!$	$A!$	$B!$	A				(B)	A	$B!$	A	(B)			$\cdots$

$\cdots$		(B)	A	$B!$	A	(B)						
$\cdots$					(B)	A	$B!$	A	(B)			
$\cdots$								(B)	A	$B!$	A	(B)
$\cdots$	$B!$	A	(B)								(B)	A

The first player to reach the last column on the second pass is Alice at $(n-1, 0)$ or Bob at $(n-1, 1)$. In either case, Alice may begin the third pass at $(0, 1)$ and is able to force a win at $(3, 2)$.

CASE 2: Bob's first move is across

The analysis is in two subcases, depending upon Bob's fourth move.

CASE 2A: Bob's fourth move is across

Alice's winning strategy is to open with a block of 16 columns, followed by playing the **12** m times, and then to close with a block of 5 columns, as shown below. By Lemma 3, Alice will win the game.

X	A	B_1	A	(B)										(B)	A	$B!$	$\cdots$
			(B)	A	$B!$	A	B_4	A	(B)								$\cdots$
								(B)	A	$B!$	A	(B)					$\cdots$
											(B)	A	$B!$	A	(B)		$\cdots$

$\cdots$		(B)	A	$B!$	A	(B)		
$\cdots$					(B)	A	$B!$	A
$\cdots$								
$\cdots$	$B!$	A	(B)					

CASE 2B: Bob's fourth move is down

Alice's winning strategy is to open with a block of 14 columns, followed by playing the **12** m times, and then to close with a block of 7 columns, as shown below. On the second pass, Alice moves to $(5, 3)$ which creates a sink at $(5, 2)$.

X	A	B_1	A	(B)								(B)	A	$B!$	A	$\cdots$
			(B)	A	$B!$	A									(B)	$\cdots$
A	(B)					B_4	A	$B!$	A	(B)						$\cdots$
(B)	A	$B!$	$A!$	$B!$	A				(B)	A	$B!$	A	(B)			$\cdots$

$\cdots$		(B)	A	$B!$	A	(B)				
$\cdots$					(B)	A	$B!$	A	(B)	
$\cdots$								(B)	A	$B!$
$\cdots$	$B!$	A	(B)							

On the second pass, the first player to reach the last column is Alice at $(n-1, 0)$ or Bob at $(n-1, 1)$ or $(n-1, 3)$. If Bob has moved to $(n-1, 3)$ then Alice will respond with $(n-1, 0)$ so that, in any case, Alice will begin the third pass at $(0, 1)$ and will win at $(5, 2)$.

5.6 The case $n \equiv 11 \pmod{12}$

This is the only case in which Bob wins. His winning strategy is to open with the block of 11 columns shown below, and then to play the **12** as often as required. On the first pass, after Bob plays to $(n-1, 0)$, Alice is forced to respond with $(n-1, 1)$. Bob may then play the closing-off move $(n-1, 2)$.

X	(A_1)								(A_7)	B_7	$A!$	B	$\cdots$
(A_1)	B_1	$A_2!$	B_2	(A_3)								(A)	$\cdots$
			(A_3)	B_3	$A_4!$	B_4	(A_5)						$\cdots$
						(A_5)	B_5	$A_6!$	B_6	(A_7)			$\cdots$

$\cdots$		(A)	B	$A!$	B	$\cdots$		(A)	B
$\cdots$					(A)	$\cdots$			$A!$
$\cdots$						$\cdots$			B
$\cdots$	$A!$	B	(A)			$\cdots$	$A!$	B	(A)

The proof of Theorem 3 is now complete.

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