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A REFINEMENT OF A PARTITION THEOREM OF SELLERS

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Abstract

Abel's identity

$$a_1b_1+a_2b_2+\cdots+a_mb_m = (b_1+b_2+\cdots+b_m)a_m+(b_1+\cdots+b_{m-1})(a_{m-1}-a_m)+\cdots+b_1(a_1-a_2)$$

is used to give a refinement of a recent theorem of Sellers.

1. Introduction

Recently Sellers ([1], [2]) proved via partition analysis the following theorem.

Theorem 1. Let $K = (k_2, k_3, k_4, \cdots)$ be an infinite vector of nonnegative integers with $k_2 \geq 1$. Define $p(n; K)$ as the number of partitions of n of the form $p_1+p_2+p_3+p_4+\cdots$ with $p_1 \geq p_2 \geq p_3 \geq p_4 \cdots \geq 0$ and $p_1 \geq k_2p_2 + k_3p_3 + k_4p_4 + \cdots$. Then, for all $n \geq 0$, $p(n; K)$ equals the number of partitions of n whose parts must be $1's$ or of the form $(\sum_{i=2}^{m} k_i) + (m-1)$ for some integer $m \geq 2$.

The main result of this note is a refinement of Sellers's theorem. To state the theorem, we will first introduce some notation.

Definition 1. Let j be a nonnegative integer and let $K = (k_2, k_3, k_4, \cdots)$ be a sequence of nonnegative integers with $k_2 \geq 1$.

Define

(1) $K_1 := \emptyset$ and $K_m := (k_2, k_3, \cdots, k_m)$ for $m \geq 2$.

(2) $S_j(n; K_1) :=$ the set of all partitions of n with exactly the part j,

and for $m \geq 2$

$S_j(n;K_m) :=$ the set of all partitions of n with $n = p_1 + p_2 + \cdots + p_m$,
$p_1 \geq p_2 \geq \cdots \geq p_m \geq 1$, and $p_1 = j + k_2p_2 + k_3p_3 + \cdots + k_mp_m$.

(3) $T_j(n;K_1) :=$ the set of all partitions of n with 1 appearing exactly j times and with largest part ≤ 1,

and for $m \geq 2$

$T_j(n;K_m) :=$ the set of all partitions in which the only parts appeaing are $1, k_2+1, k_2+k_3+2, \cdots, k_2+k_3+\cdots+k_m+m-1$, the largest part equals $k_2+k_3+\cdots+k_m+m-1$, and 1 appears exactly j times.

(4) $S(n;K_m) := \bigcup_{j\geq 0} S_j(n;K_m)$,
$T(n;K_m) := \bigcup_{j\geq 0} T_j(n;K_m)$,
$S(n;K) := \bigcup_{m\geq 1} T(n;K_m)$,
$T(n;K) := \bigcup_{m\geq 1} T(n;K_m)$.

(5) We will use the following notation for a partition $p_1f_1 + p_2f_2 + \cdots + p_mf_m$ of n:

$$n = p_1 \cdot (f_1) + p_2 \cdot (f_2) + \cdots + p_m \cdot (f_m),$$

where f_i denotes the multiplicity of the part p_i and $p_1 \geq p_2 \geq \cdots \geq p_m$.

With the notation in Definition 1, we can now state our main result:

Theorem 2. We have

$$|S_j(n;K_m)| = |T_j(n;K_m)|,$$

for integers $j, n \geq 0$ and $m \geq 1$.

Remark. Sellers's Theorem 1 says that

$$\sum_{j\geq 0}\sum_{m\geq 1} |S_j(n;K_m)| = \sum_{j\geq 0}\sum_{m\geq 1} |T_j(n;K_m)|,$$

and this follows immediately from Theorem 2.

In Section 2, Abel's transform will be defined. In Section 3, Abel's transform will be used to give a short proof of the main result. In Section 4, two examples will be given.

2. Abel's Identity, Abel's Transform, and the Conjugation of Partitions

Abel's identity states that, for $a_i, b_i \in \mathbb{R}$, one has

$$a_1b_1 + a_2b_2 + \cdots + a_mb_m = (b_1 + b_2 + \cdots + b_m)a_m + (b_1 + b_2 + \cdots b_{m-1})(a_{m-1} - a_m) + \cdots + b_1(a_1 - a_2), \quad (1)$$

which can be readily verified by induction.
Let $p_1 \cdot (f_1) + p_2 \cdot (f_2) + \cdots + p_m \cdot (f_m)$ be a partition of n.
Define Abel's transform α by

$$\alpha : p_1 \cdot (f_1) + p_2 \cdot (f_2) + \cdots + p_m \cdot (f_m) \to \begin{array}{l} (f_1 + f_2 + \cdots + f_m) \cdot (p_m) \\ +(f_1 + f_2 + \cdots + f_{m-1}) \cdot (p_{m-1} - p_m) \\ + \cdots + f_1 \cdot (p_1 - p_2) \end{array} \quad (2)$$

It follows from (1) that α^2 is the identity map. Hence α is a bijection. Indeed, α is the conjugation of partitions. This bijection, together with other two, will provide a bijection between the two sets $S_j(n; k_m)$ and $T_j(n; k_m)$ in the next section.

3. Proof of Theorem 2

By use of Abel's transform α, we are in a position to prove Theorem 2.

Proof of Theorem 2. Let $n = p_1 + p_2 + \cdots + p_m$ be a partition in $S_j(n; k_m)$. Write $n = j + p_2 \cdot (k_2 + 1) + p_3 \cdot (k_3 + 1) + \cdots + p_m \cdot (k_m + 1)$. Combining the sequence of bijections

$$\begin{array}{ll} & n = j + p_2 \cdot (k_2 + 1) + p_3 \cdot (k_3 + 1) + \cdots + p_m \cdot (k_m + 1) \\ \to & n - j = p_2 \cdot (k_2 + 1) + p_3 \cdot (k_3 + 1) + \cdots + p_m \cdot (k_m + 1) \\ \stackrel{\alpha}{\to} & n - j = (k_2 + k_3 + \cdots + k_m + (m-1)) \cdot p_m + (k_2 + k_3 + \cdots + k_{m-1} + (m-2)) \cdot \\ & \quad (p_{m-1} - p_m) + \cdots + (k_2 + 1) \cdot (p_2 - p_3) \\ \to & n = (k_2 + k_3 + \cdots + k_m + (m-1)) \cdot p_m + (k_2 + k_3 + \cdots + k_{m-1} + (m-2)) \cdot \\ & \quad (p_{m-1} - p_m) + \cdots + (k_2 + 1) \cdot (p_2 - p_3) + 1 \cdot (j), \end{array}$$

we see that the last partition is in $T_j(n; k_m)$. Hence $|S_j(n; k_m)| = |T_j(n; k_m)|$. □

4. Examples

Example 1. The case $K = (2, 1, 1, 1, \cdots)$ and $n = 12$.
In this case, we have $(k_2 + 1, k_2 + k_3 + 2, k_2 + k_3 + k_4 + 3, \cdots) = (3, 5, 7, 9, 11, \cdots)$,

$$S(12; K) = \left\{ \begin{array}{l} 12 \\ 11+1, 10+2, 9+3, 8+4 \\ 10+1+1, 9+2+1, 8+3+1, 8+2+2 \\ 9+1\cdot(3), 8+2+1+1, 7+2+2+1 \\ 8+1\cdot(4), 7+2+1\cdot(3) \\ 7+1\cdot(5) \end{array} \right\},$$

and

$$T(12; K) = \left\{ \begin{array}{l} 1\cdot(12) \\ 3+1\cdot(9), 3+3+1\cdot(6), 3\cdot(3)+1\cdot(3), 3\cdot(4) \\ 5+1\cdot(7), 5+3+1\cdot(4), 5+3+3+1, 5+5+1+1 \\ 7+1\cdot(5), 7+5, 7+3+1+1 \\ 9+1\cdot(3), 9+3 \\ 11+1 \end{array} \right\}.$$

We have the following partition of $S(12; K)$ and $T(12; K)$ into corresponding subsets.

$$\left\{ \begin{array}{l} S(12; K_1) = \{12\}, \\ T(12; K_1) = \{1\cdot(12)\}, \end{array} \right.$$
$$\left\{ \begin{array}{l} S(12; K_2) = \{11+1, 10+2, 9+3, 8+4\}, \\ T(12; K_2) = \{3+1\cdot(9), 3+3+1\cdot(6), 3\cdot(3)+1\cdot(3), 3\cdot(4)\}, \end{array} \right.$$
$$\left\{ \begin{array}{l} S(12; K_3) = \{10+1+1, 9+2+1, 8+3+1, 8+2+2\}, \\ T(12; K_3) = \{5+1\cdot(7), 5+3+1\cdot(4), 5+3+3+1, 5+5+1+1\}, \end{array} \right.$$
$$\left\{ \begin{array}{l} S(12; K_4) = \{9+1\cdot(3), 8+2+1+1, 7+2+2+1\}, \\ T(12; K_4) = \{7+1\cdot(5), 7+5, 7+3+1+1\}, \end{array} \right.$$
$$\left\{ \begin{array}{l} S(12; K_5) = \{8+1\cdot(4), 7+2+1\cdot(3)\}, \\ T(12; K_5) = \{9+1\cdot(3), 9+3\}, \end{array} \right.$$
$$\left\{ \begin{array}{l} S(12; K_6) = \{7+1\cdot(5)\}, \\ T(12; K_6) = \{11+1\}. \end{array} \right.$$

We further partition these subsets. For example, in $S(12; K_3)$ and $T(12; K_3)$ we have

$$\begin{array}{ll} S_7(12; K_3) = \{10+1+1\}, & T_7(12; K_3) = \{5+1\cdot(7)\}, \\ S_4(12; K_3) = \{9+2+1\}, & T_4(12; K_3) = \{5+3+1\cdot(4)\}, \\ S_1(12; K_3) = \{8+3+1\}, & T_1(12; K_3) = \{5+3+3+1\}, \\ S_2(12; K_3) = \{8+2+2\}, & T_2(12; K_3) = \{5+5+1+1\}. \end{array}$$

and all other $S_j(12; K_3)$ and $T_j(12; K_3)$ are empty.
We demonstrate, for example, $|S_1(12; K_3)| = |T_1(12; K_3)|$. Take $12 = 8+3+1$, a partition

in $S_1(12; K_3)$. Since $8 = 1 + 3 \cdot (2) + 1 \cdot (1)$, we have

$$\begin{aligned} 12 = 8 + 3 + 1 = 8 + 3 \cdot (1) + 1 \cdot (1) = 1 + 3 \cdot (3) + 1 \cdot (2) \quad &\rightarrow \quad 11 = 3 \cdot (3) + 1 \cdot (2) \\ &\overset{\alpha}{\rightarrow} \quad 11 = (3+2) \cdot 1 + 3 \cdot (3-1) \\ &\qquad\quad = 5 \cdot (1) + 3 \cdot (2) \\ &\rightarrow \quad 12 = 5 \cdot (1) + 3 \cdot (2) + 1 \cdot (1) \\ &\qquad\quad = 5 + 3 + 3 + 1 \end{aligned}$$

The corresponding Ferrers graph is

• • • • • • •
• • • $\qquad 12 = 8 + 3 + 1 = 8 + 3 \cdot (1) + 1 \cdot (1) = 1 + 3 \cdot (3) + 1 \cdot (2)$
• $\qquad$ since $8 = 1 + 3 \cdot (2) + 1 \cdot (1)$

$\downarrow$

• • •
• • •
• • • $\qquad 11 = 3 \cdot (3) + 1 \cdot (2)$
•
•

$\downarrow \quad \alpha$

• • • • •
• • • $\qquad 11 = 5 + 3 + 3$
• • •

$\downarrow$

• • • • •
• • • $\qquad 12 = 5 + 3 + 3 + 1$
• • •
•

Example 2. The case $K = (1, 1, 1, \cdots)$ and $n = 11$.
In this case, we have

$$\begin{aligned} &(k_2 + 1, k_2 + k_3 + 2, k_2 + k_3 + k_4 + 3, \cdots) = (2, 4, 6, 8, \cdots), \\ &S_3(11, K_3) = \{7 + 3 + 1, 7 + 2 + 2\}, \end{aligned}$$

and

$$T_3(11, K_3) = \{4 + 4 + 1 \cdot (3), 4 + 2 + 2 + 1 \cdot (3)\}.$$

We demonstrate $|S_3(11, K_3)| = |T_3(11, K_3)|$.

Since $7 = 3 + 3 \cdot (1) + 1 \cdot (1)$, we have

$$\begin{aligned} 11 = 7+3+1 = 7 + 3\cdot(1) + 1\cdot(1) &= 3 + 3\cdot(2) + 1\cdot(2) \\ &\rightarrow 8 = 3\cdot(2) + 1\cdot(2) \\ &\xrightarrow{\alpha} 8 = (2+2)\cdot 1 + 2\cdot(3-1) \\ &\quad = 4\cdot(1) + 2\cdot(2) \\ &\rightarrow 11 = 4\cdot(1) + 2\cdot(2) + 1\cdot(3). \end{aligned}$$

The corresponding Ferrers graph is

• • • • • • •
• • • $\qquad 11 = 7+3+1 = 7+3\cdot(1)+1\cdot(1) = 3+3\cdot(2)+1\cdot(2)$
• $\qquad$ since $7 = 3+3\cdot(1)+1\cdot(1)$

↓

• • •
• • • $\qquad 8 = 3\cdot(2)+1\cdot(2)$
•
•

↓ α

• • • •
• • $\qquad 8 = 4\cdot(1)+2\cdot(2)$
• •

↓

• • • •
• •
• • $\qquad 11 = 4\cdot(1)+2\cdot(2)+1\cdot(3)$
•
•
•

Next, since $7 = 3 + 2 \cdot (1) + 2 \cdot (1)$, we have

$$\begin{aligned} 11 = 7+2+2 = 7 + 2\cdot(1) + 2\cdot(1) &= 3 + 2\cdot(2) + 2\cdot(2) \\ &\rightarrow 8 = 2\cdot(2) + 2\cdot(2) \\ &\xrightarrow{\alpha} 8 = (2+2)\cdot(2) + 2\cdot(2-2) \\ &\quad = 4\cdot(2) \\ &\rightarrow 11 = 4\cdot(2) + 1\cdot(3). \end{aligned}$$

The corresponding Ferrers graph is

• • • • • • •
• • $\qquad 11 = 7 + 2 + 2 = 7 + 2 \cdot (1) + 2 \cdot (1) = 3 + 2 \cdot (2) + 2 \cdot (2)$
• • $\qquad$ since $7 = 3 + 2 \cdot (1) + 2 \cdot (1)$

$\downarrow$

• •
• • $\qquad 8 = 2 \cdot (2) + 2 \cdot (2)$
• •
• •

$\downarrow \quad \alpha$

• • • •
• • • • $\qquad 8 = 4 \cdot (2)$

$\downarrow$

• • • •
• • • •
• $\qquad 11 = 4 \cdot (2) + 1 \cdot (3)$
•
•

Therefore, $|S_3(11, K_3)| = |T_3(11, K_3)|$.

References

[1] J. A. Sellers, *Extending a recent result of Santos on partitions into odd parts*, INTEGERS: Electronic Journal of Combinatorial Number Theory 3 (2003), #A04.

[2] J. A. Sellers, *Corrigendum* to Article A4, Volume 3 (2003), INTEGERS 4 (2004), #A08.