

ON OPTIMAL PLAY IN THE GAME OF HEX

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Abstract

Hex is a two person game played on an $n \times n$ board in which the players take turns trying to construct paths from one side of the board to the other. It is known that there exists a winning strategy for the first player, but no one has yet been able to find such a strategy for any board larger than 9×9 . Despite this, we ask the following two questions: “*what is the shortest path with which player one can guarantee a win?*” and “*what is the minimal number of moves player one must make to guarantee a win?*” We give lower bounds on answers to these questions and conclude with a number of conjectures and “challenges.”

1. A Brief History of Hex

In 1942, Piet Hein invented a game he called Polygon while contemplating the (then unsolved) four-color theorem. The game was later independently re-discovered by John Nash while he was a graduate student at Princeton University and it quickly became popular there (and at other schools) under the names of John and Nash. A couple of years later, in 1950, Parker Brothers began producing the game as “Game of Hex.”



Figure 1. A picture of the original game marketed by Parker Brothers [1].
For copies of the original instructions see [7].

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With so many names given to the same game, one might wonder why “Hex” is the name that endured. The fact that Parker Brothers put its marketing muscle behind that name is clearly a large contributing factor, but Martin Gardner’s 1957 *Scientific American* article [5], *Concerning the game of Hex, which may be played on the tiles of the bathroom floor*, is undoubtedly another. This article and future writings by Gardner on the game were quite popular and brought the game to the attention of a vast number of mathematicians and computer scientists. In the article, Gardner states that

Hex may well become one of the most widely played and thoughtfully analyzed new mathematical games of the century.

So what exactly is this game of Hex, how does one play and what makes the game so special? First, Hex is a two person game played on an $n \times n$ board of hexagonal tiles arranged in a rhombus. We will denote the game of hex played on an $n \times n$ board as $Hex(n)$. The Parker Brothers board shown in Figure 1 is an 11×11 board and is considered the “standard” size board. Two smaller 4×4 boards are illustrated below.

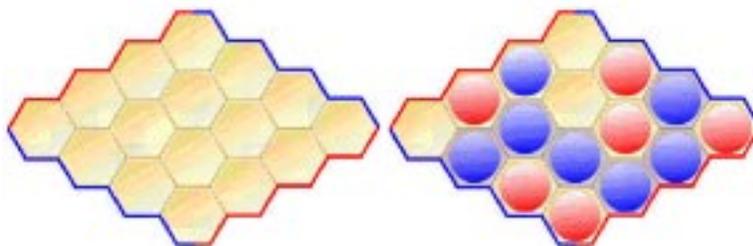


Figure 2. An empty 4×4 Hex board (left) and a 4×4 board containing a winning path for blue (right).

Each player is assigned a pair of opposite sides of the board and given a set of counters. The players take turns placing one of their counters on any unoccupied tile of the board until one player has a path consisting of their own counters connecting their two sides of the board. Throughout the discussion here, one player is designated blue, the other red and we color the sides of the board and the counters to indicate to whom they belong. We also make the arbitrary choice of always letting the blue player go first. Figure 2 above illustrates a winning game of $Hex(4)$ for blue.

Hein and Nash each proved independently that Hex cannot end in a tie and moreover, that there exists a first player winning strategy. Proofs of these facts can be found in [4], [6] and [10], among other places. (Most notably, in [4], David Gale proves that the inability of Hex to end in a tie is equivalent to the Brouwer fixed point theorem.) For small n , it is a simple matter to work out first player winning strategies for $Hex(n)$, but as n grows, this task quickly becomes quite difficult. Jing Yang was first to find a first

player winning strategy for $Hex(7)$ [11] and seems to have recently found such strategies for $Hex(8)$ and $Hex(9)$ [13]. The search is still on for a winning strategy for $Hex(n)$ when $n > 9$.

We note that while we will mostly be concerned with Hex played on $n \times n$ boards, the game can also be played on asymmetric, $m \times n, m \neq n$, boards as well. However, in this case, the player with the smaller side can always win and a winning strategy is quite easy to describe (see [6], for example).

2. Introduction to the problem

Despite the fact that no one is able to describe winning strategies for $Hex(n)$ when n is large, we examine two possible measures of *optimal* winning strategies. In particular, we ask the following two questions:

Question 1 *What is the shortest path with which player one can guarantee a win?*

Question 2 *What is the minimal number of moves player one must make to guarantee a win?*

To address the first question, we let $Hex(n, l)$ denote the game of $Hex(n)$ with the added constraint that player one wins only by constructing a path of length less than or equal to l . For example, the winning path in Figure 2 would constitute a win for blue in $Hex(4, 7)$ but not in $Hex(4, 5)$. Furthermore, if Figure 2 represents the state of the board during a $Hex(4, 5)$ game, play would continue since the $Hex(4)$ winning path is not with a path of length less than or equal to 5 and since blue still has a chance of winning with such a path. However, since it is red's move next, the game should conclude in a draw on the very next turn. This illustrates that $Hex(n, l)$ can end in a tie for some values of n and l , but note that because player one has a winning strategy for $Hex(n)$, the second player can never have a winning strategy for $Hex(n, l)$.

Now suppose we define $\lambda(n)$ to be

$$\lambda(n) = \min\{l \mid \text{player one has a winning strategy for } Hex(n, l)\}.$$

Question 1 is then equivalent to:

Question 3 *What is the value of $\lambda(n)$?*

Unfortunately, for n large, computing $\lambda(n)$ is almost certainly more difficult than finding a winning strategy. Therefore, instead of trying to calculate the value of $\lambda(n)$

precisely, we look to finding bounds for $\lambda(n)$. This task also proves to be quite challenging. In particular, it is trivial to see that $\lambda(n)$ satisfies $n \leq \lambda(n) \leq \lceil n^2/2 \rceil$ and $\lambda(n) = n$ for $n < 4$, but significantly improving these bounds appears to be far from trivial. In the following sections, we focus our attention on the lower bound and prove:

Theorem 4 $\lambda(n) > n$ for all $n \geq 4$.

Remark 5 *We note that it was known [6] as early as 1959 that $\lambda(4) \leq 5$ and that $\lambda(5) \leq 7$. Together with the theorem, this implies that $\lambda(4) = 5$ and $6 \leq \lambda(5) \leq 7$. It has also been known since that time that all but four opening moves for the first player result in a loss for that player in a game of Hex(4). In Section 7, we give a Hex(4) second player winning strategy for 10 of the 12 possible opening moves which admit such a strategy. A second player winning strategy for the remaining 2 opening moves is outlined in the appendix.*

Turning our attention to Question 2, we define the game $Hex_d(n)$ to be the game of $Hex(n)$ with the added constraint that player one must win by playing at most d counters and define $\delta(n)$ to be

$$\delta(n) = \min\{d \mid \text{player one can guarantee a win at } Hex_d(n)\}.$$

Therefore, Question 2 is equivalent to:

Question 6 *What is the value of $\delta(n)$?*

This question is related to a more common measure called the *depth* of a game—the minimal number of total moves (made by both players) necessary to guarantee a win. The depth of $Hex(n)$ is then $2\delta(n) - 1$. As in the case for $\lambda(n)$, it appears that (in general) computing $\delta(n)$ precisely is intractable. Therefore, we again focus on establishing a nontrivial lower bound. We clearly have that $\delta(n) \geq \lambda(n)$, but we can in fact say a bit more. We prove:

Theorem 7 *For all n , $\delta(n) \geq n + \lfloor n/4 \rfloor$.*

3. Preliminaries.

We begin by establishing some conventions and proving a proposition central to Hex strategy.

We will reference a tile on the Hex board by an ordered pair indicating the tile's row number and column number as demonstrated in Figure 3 below.

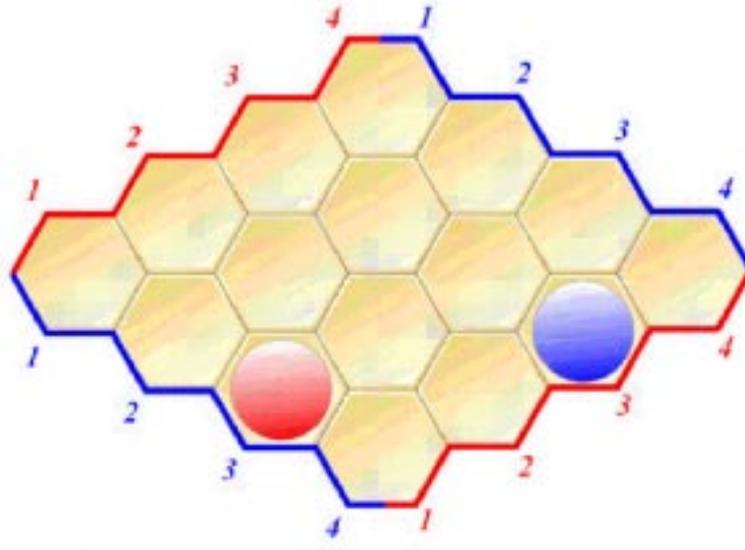


Figure 3. The red counter is in tile $[3, 1]$, while the blue counter is in tile $[4, 3]$.

We formally define a *path* to be an ordered set of adjacent tiles, denoted $P = \langle [b_1, r_1], [b_2, r_2], \dots, [b_m, r_m] \rangle$. (Note that the *length* of the path P is m .) A path will be called *complete* if all of the tiles in the path are occupied by one player's counters and *incomplete* otherwise. To be clear, an incomplete path may contain tiles that are occupied by both players (in which case, the path can never be completed) or contain only tiles that are either unoccupied or occupied by one player's counters.

Now, suppose that during a game of $Hex(n, l)$, it is blue's turn, but placement of a red counter in tile $[b, r]$ would guarantee a win for red through the path P_1 or P_2 and placement of a red counter in tile $[b', r']$ would guarantee a win for red through the path Q_1 or Q_2 . If R is the set of tiles

$$R = (P_1 \cup P_2) \cap (Q_1 \cup Q_2),$$

then the only way blue can retain a chance at winning is by placing a counter in one of the tiles in R . For example, suppose we are faced with the board in Figure 4 during a game of $Hex(4, 4)$. If we let $[b, r] = [3, 2]$, $[b', r'] = [3, 3]$, $P_1 = \langle [1, 4], [2, 3], [3, 2], [4, 1] \rangle$, $P_2 = \langle [1, 4], [2, 3], [3, 2], [4, 2] \rangle$, $Q_1 = \langle [1, 4], [2, 3], [3, 3], [4, 2] \rangle$ and $Q_2 = \langle [1, 4], [2, 3], [3, 3], [4, 3] \rangle$, then we have precisely the condition described above.

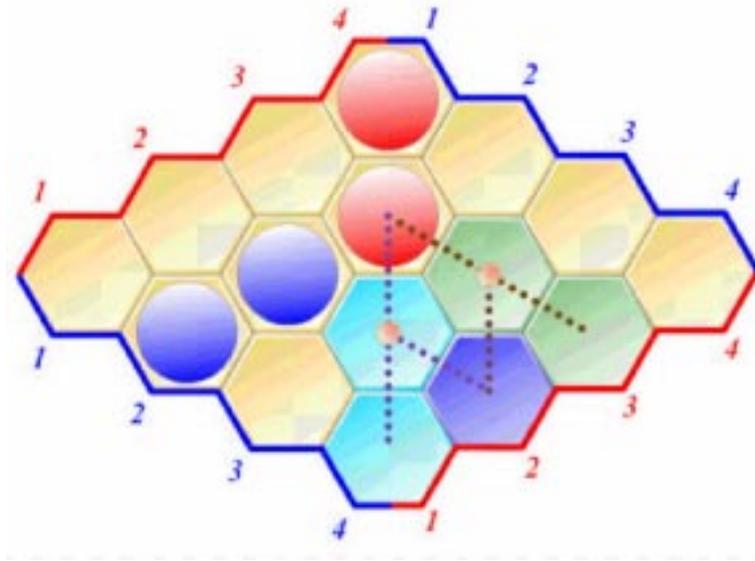


Figure 4. Red is guaranteed a win in this game of $Hex(4, 4)$ if a blue counter is not placed in tile $[4, 2]$ on the next turn.

The principle above generalizes to the following:

Proposition 8 *Let n and l be fixed positive integers. Let T_1 and T_2 denote distinct sets of tiles or red edges. Suppose that during a game of $Hex(n, l)$, $S = \{[b_1, r_1], [b_2, r_2], \dots, [b_k, r_k]\}$ is a set of unoccupied tiles such that, given the current state of the Hex board, we have for each i , $1 \leq i \leq k$:*

1. $\{P_{ij}\}_{j=1}^{a_i}$ is a set of incomplete paths from T_1 to T_2 ;
2. placement of a red counter in tile $[b_i, r_i] \in S$ guarantees that red can complete one of the paths in the set $\{P_{ij}\}_{j=1}^{a_i}$.

Let R be the set of tiles defined by:

$$R = \bigcap_{i=1}^k \bigcup_{j=1}^{a_i} P_{ij}$$

If on the next turn, a blue counter is not placed in one of the tiles in R , then red can guarantee completing one of the paths P_{ij} connecting T_1 and T_2 .

4. A lower bound for $\lambda(n)$.

We prove that $\lambda(n) > n$ for all $n \geq 4$ by induction on n and our proof of the basis step involves several cases. So that we may get to some of the more important ideas more quickly, we defer the proof of this basis step until Section 7. For now, we simply state:

Proposition 9 $\lambda(4) > 4$.

We define the n^2 tiles, $[i, j]$, on the $(n + 1) \times (n + 1)$ Hex board with $i > 1$ and $j < n + 1$ to be the *standard $n \times n$ sub-board*. The standard 4×4 sub-board is shown in Figure 5 below.

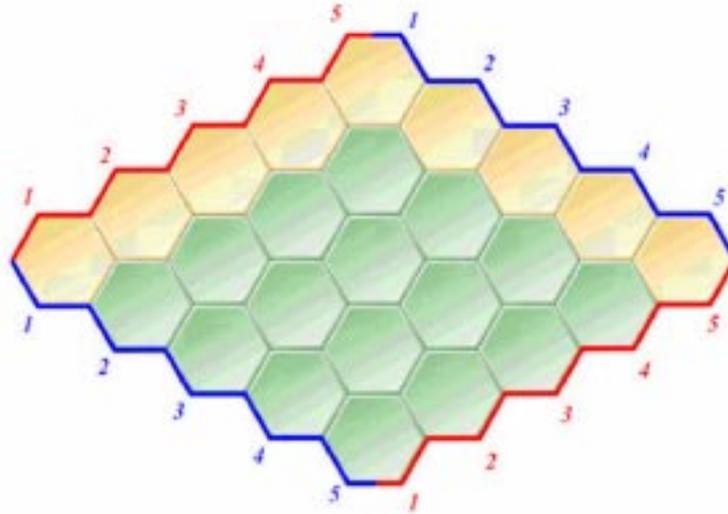


Figure 5. The 16 tiles of the standard 4×4 sub-board are marked green.

Theorem 10 *If $\lambda(n) > n$, then $\lambda(n + 1) > n + 1$.*

Proof: All $Hex(n + 1, n + 1)$ winning paths for blue must either

1. contain a $Hex(n, n)$ winning path for blue through the standard $n \times n$ sub-board;
- or
2. contain both $[1, n]$ and $[1, (n + 1)]$.

We assume $\lambda(n) > n$ and so red can prevent a winning $Hex(n, n)$ path for blue on the standard $n \times n$ sub-board. Whenever blue plays on the sub-board, red can play according

to the win-preventing strategy for $Hex(n, n)$. Whenever blue first plays outside of the sub-board, red can play in one of the unoccupied tiles $[1, n]$ or $[1, n + 1]$, thereby preventing any paths that must contain both. This then prevents a win for blue in $Hex(n + 1, n + 1)$.

□

Therefore, we now have:

Theorem 11 For all $n \geq 4$, $\lambda(n) > n$.

From this theorem, we get the following two corollaries:

Corollary 12 $\lambda(4) = 5$

Proof: Figure 6 below illustrates that blue can win $Hex(4, 5)$

□

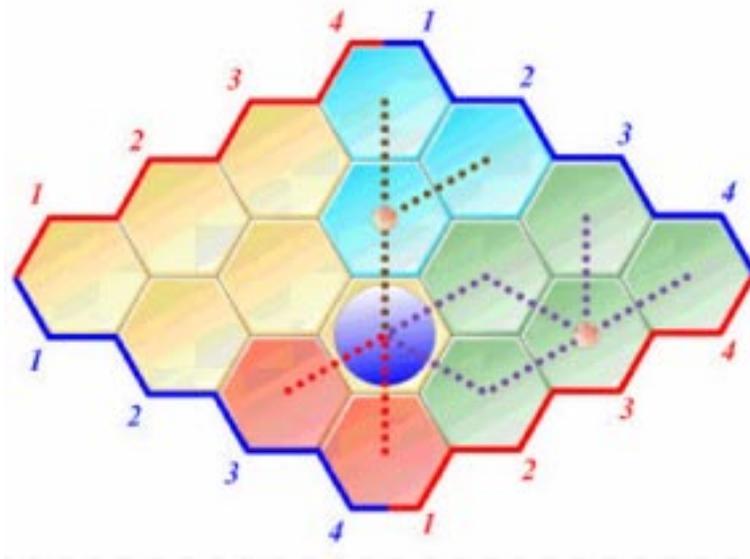


Figure 6. Blue can win $Hex(4)$ with a path of length 5 or less.

Corollary 13 $6 \leq \lambda(5) \leq 7$

Proof: If blue plays in tile $[3, 3]$, then blue can win with a path of length 7 or less.

□

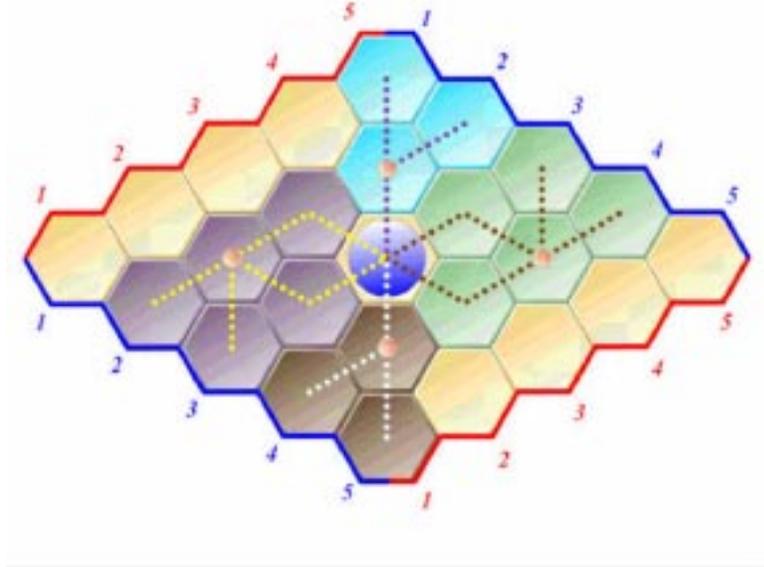


Figure 7. Blue can win $Hex(5)$ with a path of length 7 or less.

5. A lower bound for $\delta(n)$.

We now turn our attention back to our competing measure of optimal play, namely $\delta(n)$. As noted earlier, we clearly have $\delta(n) \geq \lambda(n)$, and in the case of $n = 4$, we have equality. Indeed, Figure 6 not only proves that $\lambda(4) = 5$, but also:

Theorem 14 $\delta(4) = 5$.

Similarly, Figure 7 illustrates that we have:

Theorem 15 $6 \leq \delta(5) \leq 7$.

Now consider the following lemma suggested to us by Carl Pomerance [8]:

Lemma 16 *Player one needs at least 5 counters to complete a winning path on an $n \times 4$ board.*

Proof: First observe that if player one is to complete a winning path on an $n \times 4$ board with 4 counters, then every counter placed must be part of the winning path. Now, suppose that player one's first move is in tile $[i, j]$. $[i, j]$ sits in precisely one 4×4 sub-board of the $n \times 4$ board which contains all possible winning paths of length 4 through

$[i, j]$. Let us call this sub-board H . Since $\delta(4) = 5$, player 2 can guarantee that it takes player one at least 5 counters to win $Hex(4)$ on H . But then, by definition of H , this means that player one cannot complete a winning path of length 4 that contains every counter placed. Therefore, player one needs at least 5 counters to complete a winning path on the $n \times 4$ board. \square

Therefore, we get:

Theorem 17 For all n , $\delta(n) \geq n + \lfloor n/4 \rfloor$.

Proof: Divide the $n \times n$ board into $n \times 4$ sub-boards, H_k , where H_k consists of the tiles $[i, j], 1 \leq i \leq n, 4(k-1) < j \leq 4k, 1 \leq k \leq \lfloor n/4 \rfloor$ and the $(n - 4\lfloor n/4 \rfloor) \times n$ sub-board H consisting of the tiles $[i, j], 1 \leq i \leq n, 4\lfloor n/4 \rfloor < j \leq n$. (See Figure 8 below, for example.)

Now, in order to win $Hex(n)$, player one must have a winning path across H and across each $H_k, 1 \leq k \leq \lfloor n/4 \rfloor$. Player one can complete a path across H with $(n - 4\lfloor n/4 \rfloor)$ counters, but for each k , player one needs at least 5 counters to complete a path across H_k by Lemma 16. Therefore, player one needs at least $5\lfloor n/4 \rfloor + (n - 4\lfloor n/4 \rfloor) = n + \lfloor n/4 \rfloor$ counters to complete a winning path in $Hex(n)$. \square

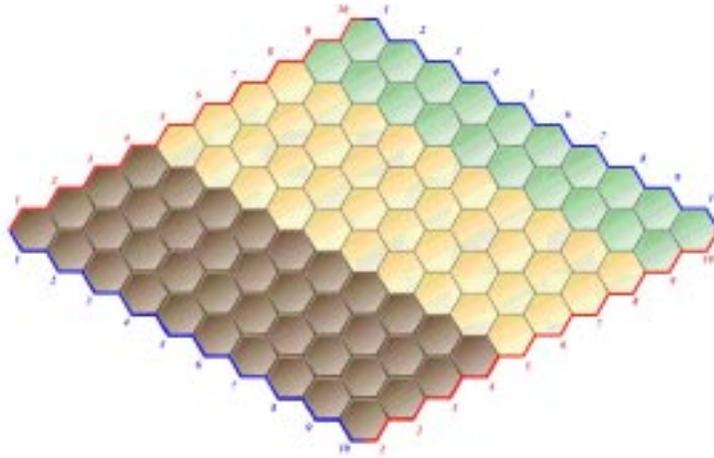


Figure 8. Subdivision of a 10×10 Hex board into two 10×4 sub-boards and one 10×2 sub-board as demanded in the proof of Theorem 17.

6. Conjectures and Questions

We now state a few conjectures and pose a few challenges to the interested reader. First, after playing $Hex(n)$ for various n , it seems to be the case that opening in tile $[\lceil n/2 \rceil, \lfloor n/2 \rfloor]$ (or equivalently $[\lfloor n/2 \rfloor, \lceil n/2 \rceil]$) is the strongest opening move. In particular, we conjecture:

Conjecture 18 *If $\lambda'(n)$ equals the minimum l such that player one can guarantee a win in $Hex(n, l)$ with an opening move in tile $[\lceil n/2 \rceil, \lfloor n/2 \rfloor]$, then $\lambda'(n) = \lambda(n)$.*

Playing $Hex(5)$ and $Hex(6)$ (especially with the conjecture above in mind) we believe:

Conjecture 19 $\lambda(5) = 7$ and $\lambda(6) = 9$.

It is within reason to expect to resolve this conjecture by an argument similar to the one demonstrating that $\lambda(4) > 4$ presented below. This is particularly true for showing $\lambda(5) > 6$. Consider for example that Figure 7 implies that 4 of the 13 possible opening moves (modulo rotational symmetry) in $Hex(5)$ admit a player two winning strategy. This leaves only 9 opening moves to consider, some of which are equally easy to dispose of. It would seem an ideal challenge for the committed student to try to dispose of these remaining opening moves. The case for $\lambda(6)$ is of course more complicated, but again, perhaps not impossible by the methods employed here. In this case, more so than in the $n = 5$ case, it might be more reasonable to write a computer program that helps in the endeavor. We similarly conjecture that:

Conjecture 20 $\delta(5) = 7$ and $\delta(6) = 9$.

As a result of Theorem 17, we have:

Theorem 21 $\lim_{n \rightarrow \infty} \delta(n) - n = \infty$.

Moreover, it seems reasonable to believe that both $\delta(n)$ and $\lambda(n)$ are strictly increasing. More directly, it seems highly unlikely that player one can guarantee a win in $Hex(n + 1)$ with a path of length equal to $\lambda(n)$ or by playing at most $\delta(n)$ counters. We have, however, been unsuccessful at proving:

Conjecture 22 *For all n , $\lambda(n + 1) > \lambda(n)$ and $\delta(n + 1) > \delta(n)$.*

One could imagine trying to prove this by induction. One approach would be to divide the board in the way described in the proof of Theorem 17. Unfortunately, this does not work as is evidenced by the fact that player one can guarantee a win with a path of length 4 on a 6×4 board (where player one has the shorter distance to span). In fact, we suspect that player one may be able to guarantee a win on an $n \times k$ board with a path of length k whenever n is sufficiently large. Similarly, we suspect that player one may be able to guarantee a win on an $n \times k$ board using at most $\Delta(k)$ counters, where $\Delta(k) < \delta(k)$, whenever n is sufficiently large. (We remark that we owe thanks to the referee for pointing out a flaw in an argument which professed to prove that player one required at least $\delta(k)$ counters to guarantee a win on an $n \times k$ board.)

Alternatively, one could try to prove Conjecture 22 by induction using the standard $n \times n$ sub-board in much the way that Theorem 10 is proved. In particular, in the case of $\lambda(n)$, observe that in a game of $Hex(n+1)$, we can prevent all winning paths on the standard $n \times n$ sub-board of length $\lambda(n) - 1$ (by definition of $\lambda(n)$). Therefore, a minimal winning path in $Hex(n+1)$ would necessarily have to contain tiles outside the standard $n \times n$ sub-board. But now, it seems plausible that player two could stop player one's paths that utilize these tiles that could be part of a minimal length winning path. However, all attempts to prove this have been unsuccessful.

Despite our inability to prove this conjecture and in light of Theorem 21, we suspect that perhaps $\lambda(n) - n$ also tends to infinity as n tends to infinity. Much less clear is whether or not $\lambda(n)/n$ or even $\lambda(n+1) - \lambda(n)$ tends to infinity as n tends to infinity. It is similarly unclear which of these, if any, hold for $\delta(n)$. Finally, we ask one last question for which we have no reasonable conjecture. Namely, what is the growth rate of $\lambda(n)$ relative to the growth rate of $\delta(n)$. These questions are summarized in the following (challenge) question:

Question 23 *What are the following limits:*

1. $\lim_{n \rightarrow \infty} \lambda(n) - n$, $\lim_{n \rightarrow \infty} \frac{\lambda(n)}{n}$ and $\lim_{n \rightarrow \infty} \lambda(n+1) - \lambda(n)$;
2. $\lim_{n \rightarrow \infty} \frac{\delta(n)}{n}$ and $\lim_{n \rightarrow \infty} \delta(n+1) - \delta(n)$;
3. $\lim_{n \rightarrow \infty} \delta(n) - \lambda(n)$ and $\lim_{n \rightarrow \infty} \frac{\delta(n)}{\lambda(n)}$?

7. A proof that $\lambda(4) > 4$.

First, observe that the Hex board has rotational symmetry and hence there are only $\lceil n^2/2 \rceil$ truly distinct first moves.

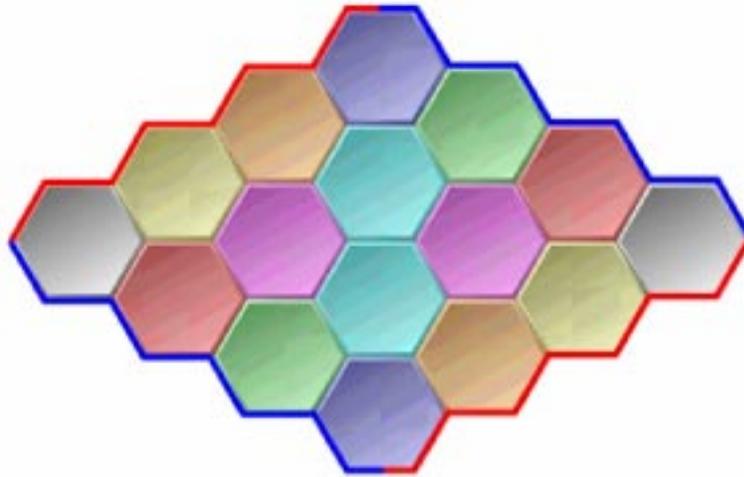


Figure 9. Tiles of the same color are equivalent first moves because of the rotational symmetry that exists in the board.

We now prove:

Proposition 24 $\lambda(4) > 4$.

Proof: We divide the proof into cases based on where blue's first counter is placed. The set of distinct first moves (modulo symmetry) can be organized into four categories. These categories are shown in Figure 10 below.

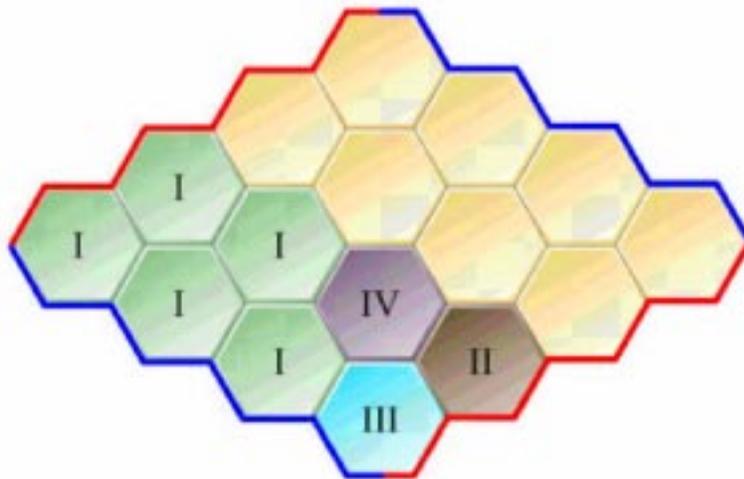


Figure 10. Four categories of distinct first moves (modulo the existing rotational symmetry).

Category I: Suppose blue is allowed to place counters in all the tiles of category I. Red can then move in position $[2, 3]$ as shown below:

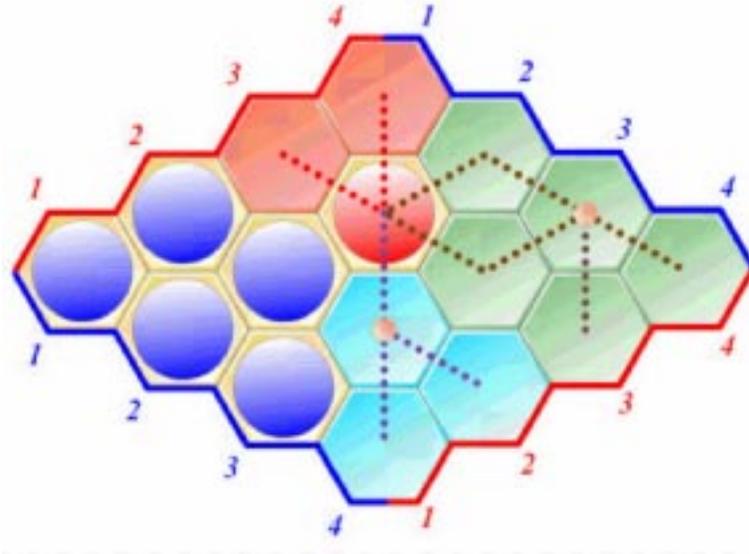


Figure 11. If blue places counters in all of the tiles in category I, red can respond by placing a counter in $[2, 3]$ and win at $Hex(4)$.

Figure 11 also illustrates that once red makes this move:

1. blue cannot then prevent a connection between this counter and the top red edge and
2. if a red counter is placed in $[3, 2]$: blue cannot prevent a connection between $[2, 3]$ and the bottom red edge through one of the paths:

$$P_1 = \langle [2, 3], [3, 2], [4, 1] \rangle \text{ or } P_2 = \langle [2, 3], [3, 2], [4, 2] \rangle.$$

- $[3, 4]$: blue cannot prevent a connection between $[2, 3]$ and the bottom red edge through one of the paths:

$$Q_1 = \langle [2, 3], [2, 4], [3, 4], [4, 4] \rangle, \quad Q_2 = \langle [2, 3], [2, 4], [3, 4], [4, 3] \rangle, \\ Q_3 = \langle [2, 3], [3, 3], [3, 4], [4, 4] \rangle \quad \text{or} \quad Q_4 = \langle [2, 3], [3, 3], [3, 4], [4, 3] \rangle.$$

Therefore, by proposition 8, blue cannot prevent a connection between the top red edge and the bottom red edge through tile $[2, 3]$. This proves that red can win at $Hex(4)$ with blue opening in category I and hence, blue cannot win at $Hex(4, 4)$ with such a first move.

Remark 25 Note that since $[3, 3]$ is adjacent to $[4, 3]$, the path Q_4 above could have been replaced by $Q'_4 = \langle [2, 3], [3, 3], [4, 3] \rangle$. This situation occurs often in subsequent arguments, but we make the choice of writing the longer path to make it easier to produce figures which agree with the written paths.

Category II: If blue first places a counter in $[4, 2]$, then red can place a counter in $[4, 1]$. This immediately eliminates the possibility of any winning path of length 4 for blue containing a tile in positions $[4, j]$, $1 \leq j \leq 4$.

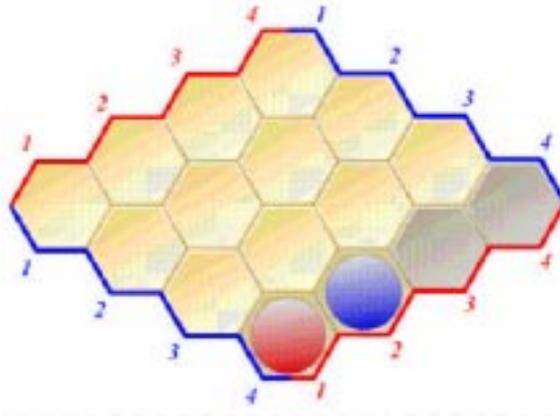


Figure 12. Blue’s move, $[4, 2]$, red’s response in tile, $[4, 1]$, and the resulting constriction of the board.

Therefore, if red can create a path from the top red edge to any of the tiles in position $[4, j]$, $1 \leq j \leq 4$, then blue cannot win with this opening move. This is equivalent to playing Hex on an $(n - 1) \times n$ board (in this case with $n = 4$) with red, the second player, having the smaller edge. As mentioned earlier, a second player winning strategy is known [6] for such asymmetric boards. Figure 13 below describes this strategy for the 3×4 board. In the figure, red need only play in the same color tile as blue.

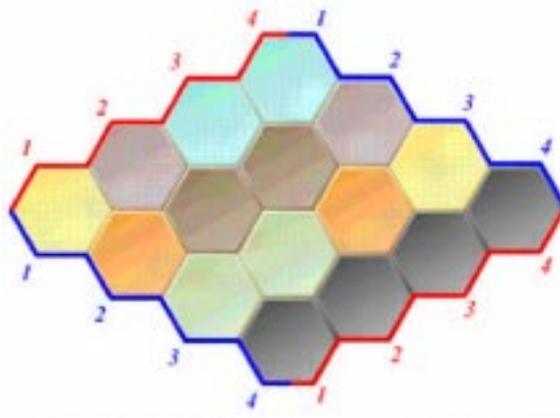


Figure 13. Assuming play made as shown in Figure 12, no tile in the bottom row can be used by blue in a winning path for $Hex(4, 4)$. But then, if red plays in the same color tile as blue in the resulting 3×4 sub-board, red can prevent a win for blue. (Tiles $[i, j]$ and $[i', j']$ in the 3×4 sub-board are colored the same if and only if $i + j + i' + j' = 9$ and $i - j - i' + j' = 1$.)

This rules out blue being able to win by first moving in category II.

Remark 26 *This is the only other first move (modulo symmetry) for blue which admits a second player winning strategy in Hex(4). The strategy outlined above is not sufficient to guarantee a win for red in Hex(4), but such a strategy is given in the appendix below.*

Category III: If blue first places a counter in tile $[4, 1]$, then red can respond by placing a counter in tile $[3, 2]$. Now, if a red counter is placed in tile $[2, 1]$, then blue cannot prevent a connection between $[3, 2]$ and the top red edge through one of the paths:

$$P_1 = \langle [3, 2], [3, 1], [2, 1], [1, 1] \rangle, \quad P_2 = \langle [3, 2], [2, 2], [2, 1], [1, 1] \rangle,$$

$$P_3 = \langle [3, 2], [3, 1], [2, 1], [1, 2] \rangle \quad \text{or} \quad P_4 = \langle [3, 2], [2, 2], [2, 1], [1, 2] \rangle.$$

Similarly, if a red counter is placed in tile $[2, 3]$, then blue cannot prevent a connection between $[3, 2]$ and the top red edge through one of the paths:

$$Q_1 = \langle [3, 2], [2, 3], [1, 3] \rangle \quad \text{or} \quad Q_2 = \langle [3, 2], [2, 3], [1, 4] \rangle.$$

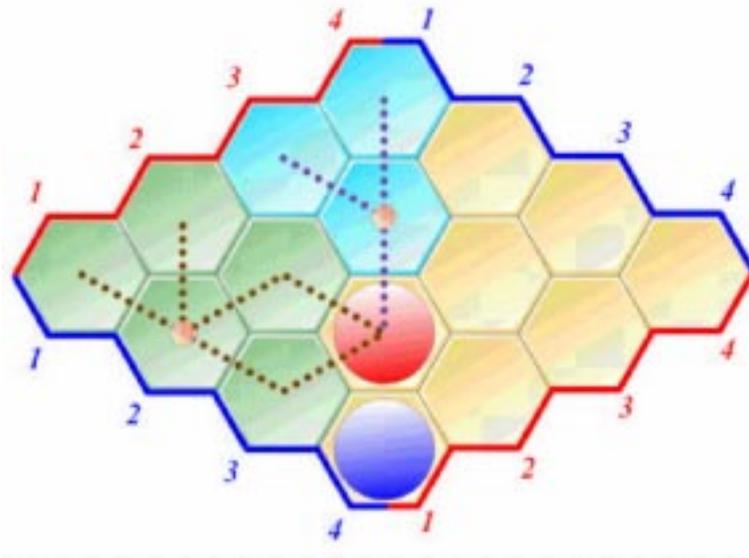


Figure 14. Blue's first move, $[4, 1]$, and red's response, $[3, 2]$, illustrating that red can guarantee a connection between $[3, 2]$ and the top red edge.

Therefore, by proposition 8, blue cannot prevent a connection between $[3, 2]$ and the top red edge. This implies that in order for blue to create a winning path of length 4, it must contain the tile $[4, 1]$ and tiles from the set $\{[4, 2], [4, 3], [4, 4], [3, 3], [3, 4], [2, 4]\}$. Hence, if blue's second move is not $[4, 2]$, then red can move there and prevent a win for blue. Consequently, blue's second move must be to place a counter in $[4, 2]$.

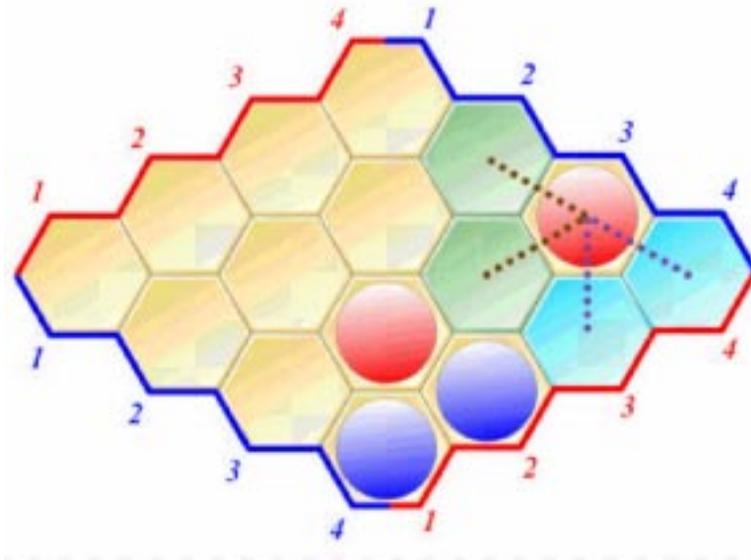


Figure 15. Blue's second move, $[4, 2]$, and red's response, $[3, 4]$, illustrating that red can prevent a win for blue.

But then, red can prevent the creation of a blue path of length 4 containing $[4, 1]$ by placing a counter in $[3, 4]$ as shown above.

Category IV: If blue first plays in position $[3, 2]$, red must respond by playing in tile $[2, 4]$. Now, the board divides into three parts as shown in Figure 16.

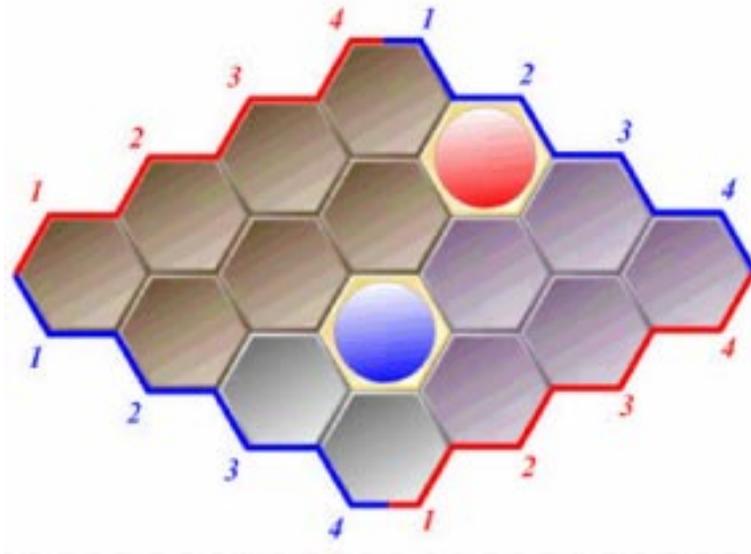


Figure 16. Blue's first move, $[3, 2]$, red's response, $[2, 4]$, and the resulting division of the board.

The only way blue can win with a path of length 4 that ends in tile $[3, 4]$ or tile $[4, 4]$ is through tiles $[i, j]$, $i > 2$ (tiles marked purple or gray). The only way blue can win with a path of length 4 that ends in tile $[1, 4]$ is through tiles $[i, j]$, $i < 3$

or $j = 1$ (tiles marked brown or gray). We now group tile $[3, 1]$ with the tiles $[i, j]$, $i < 3$ and call this group of tiles, group I. We exclude the tile $[3, 1]$ from the group of tiles $[i, j]$, $i > 2$ and call the resulting group of tiles, group II. This allows us to divide red's strategy for preventing blue from winning at $Hex(4, 4)$ into two distinct parts. Each time blue plays in one group of the board, red responds by playing in that same group as described below. Since a play by blue in $[4, 1]$ potentially effects the construction of a path through group I, we assume that blue has played in $[4, 1]$ when we outline red's groups I strategy. Likewise, since a play by blue in $[3, 1]$ potentially effects the construction of a path through group II, we assume that blue has played in $[3, 1]$ when we outline red's group II strategy.

Group I strategy: If red places a counter in tile $[1, 4]$, then clearly blue could not complete a path using tiles only from group I. Therefore, blue must place a counter in that tile. Red must then respond by placing a counter in tile $[2, 3]$ as shown below.

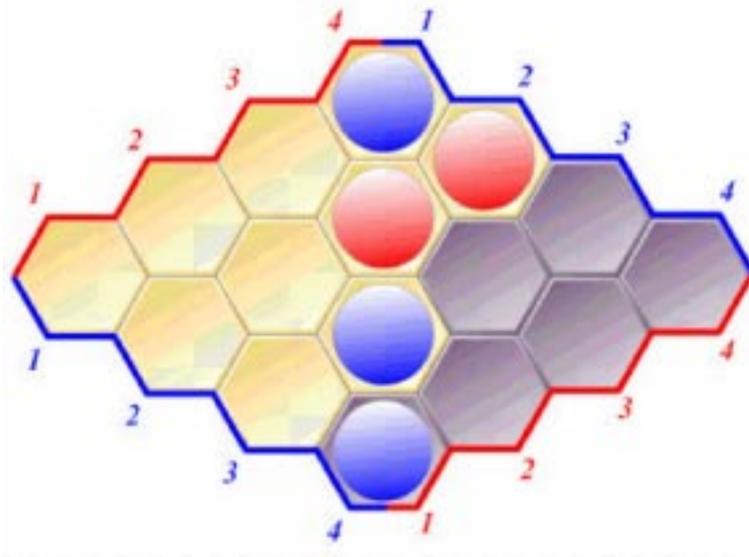


Figure 17. Blue's first move, $[1, 4]$, in the set of group I tiles and red's response, $[2, 3]$.

It is clear that blue must place a counter in tile $[1, 3]$ and that red must then place a counter in tile $[2, 1]$ as shown in Figure 18.

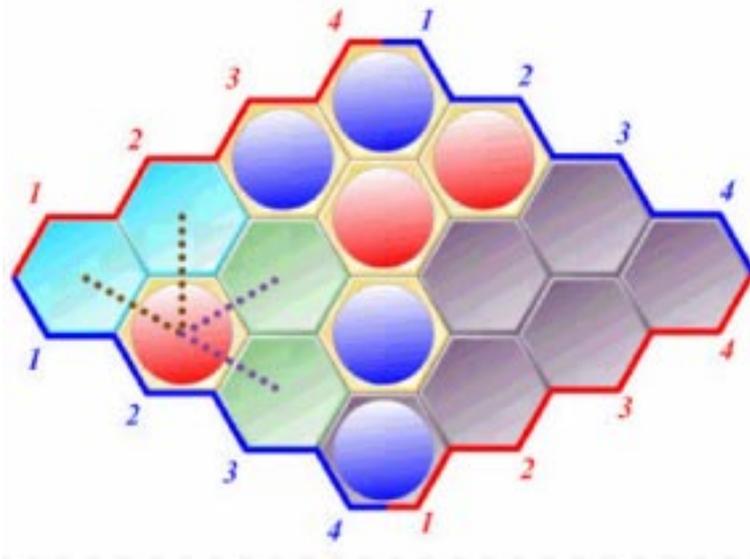


Figure 18. Blue's second move, $[1, 3]$ in the set of group I tiles and red's response, $[2, 1]$.

Now, blue cannot prevent red from creating a path from tile $[2, 1]$ to the top red edge, nor from tile $[2, 1]$ to either tile $[2, 2]$ or tile $[3, 1]$. But then blue cannot create a path of length 4 through the tiles in group I. Therefore, blue cannot create a path of length 4 containing tile $[1, 4]$.

Group II strategy: If red places a counter in tile $[3, 3]$, then red can guarantee a path from $[2, 4]$ to the bottom red edge through the path $P_1 = \langle [2, 4], [3, 3], [4, 2] \rangle$ or $P_2 = \langle [2, 4], [3, 3], [4, 3] \rangle$. Similarly, if red places a counter in tile $[3, 4]$, then red can guarantee a path from $[2, 4]$ to the bottom red edge through the path $Q_1 = \langle [2, 4], [3, 4], [4, 3] \rangle$ or $Q_2 = \langle [2, 4], [3, 4], [4, 4] \rangle$. These paths are illustrated in Figure 19 below.

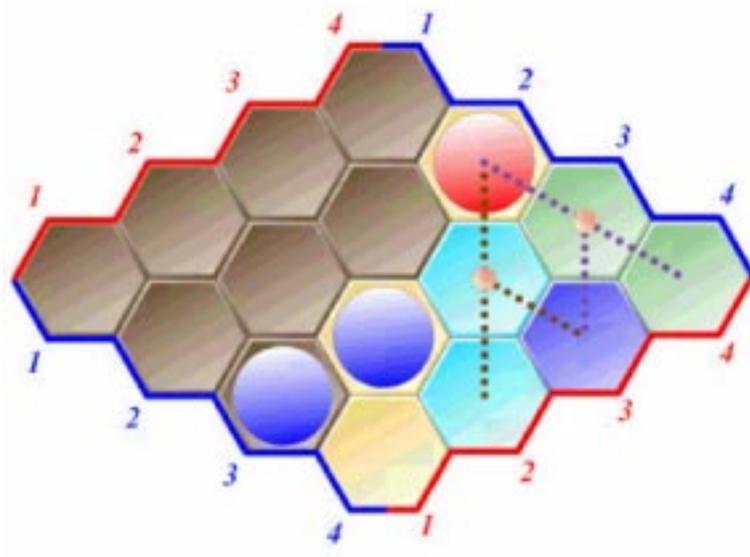


Figure 19. The paths which force the play of blue in tile $[4, 3]$.

If red creates such a path, then blue cannot create a winning path through the group II tiles. Therefore, by proposition 8, blue must place a counter in tile $[4, 3]$. Then red can respond by placing a counter in tile $[3, 3]$ leaving the board in the state below.

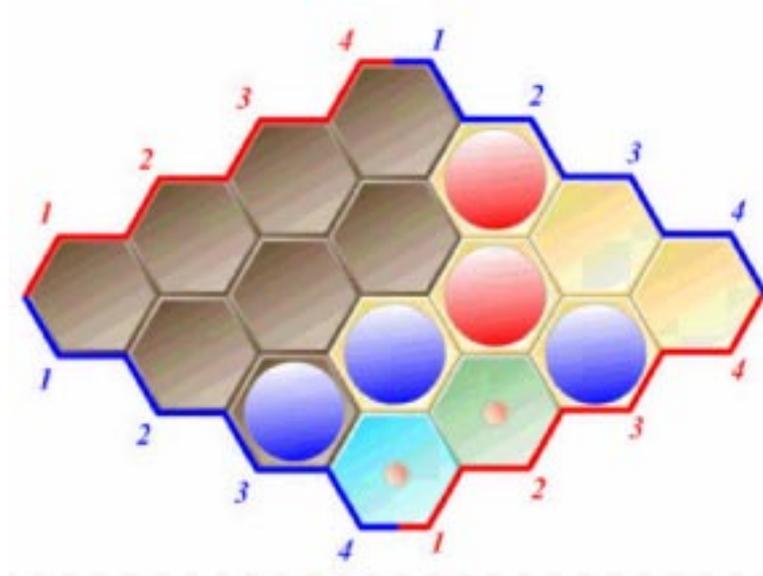


Figure 20. Blue's first move, $[4, 3]$ in the set of group II tiles and red's response, $[3, 3]$.

But now, the only way blue can create a winning path is through paths containing both tiles $[4, 1]$ and $[4, 2]$. After, blue's next turn, one of these must be left unoccupied and red can simply play in this unoccupied tile. Therefore, blue cannot create a winning path through the group II tiles and hence cannot create a winning path that contains the tiles $[3, 4]$ or $[4, 4]$. This completes the proof that an opening play in tile $[3, 2]$ does not lead to a win for blue and so completes the proof that the first player has no winning strategy in $Hex(4, 4)$.

□

8. Appendix: an alternate play for a category II move.

The figure below illustrates that if blue's first move is in tile $[4, 2]$, then red can win at $Hex(4)$. This can be used as an alternate proof of the fact that blue's first move being in category II results in a loss in $Hex(4, 4)$.

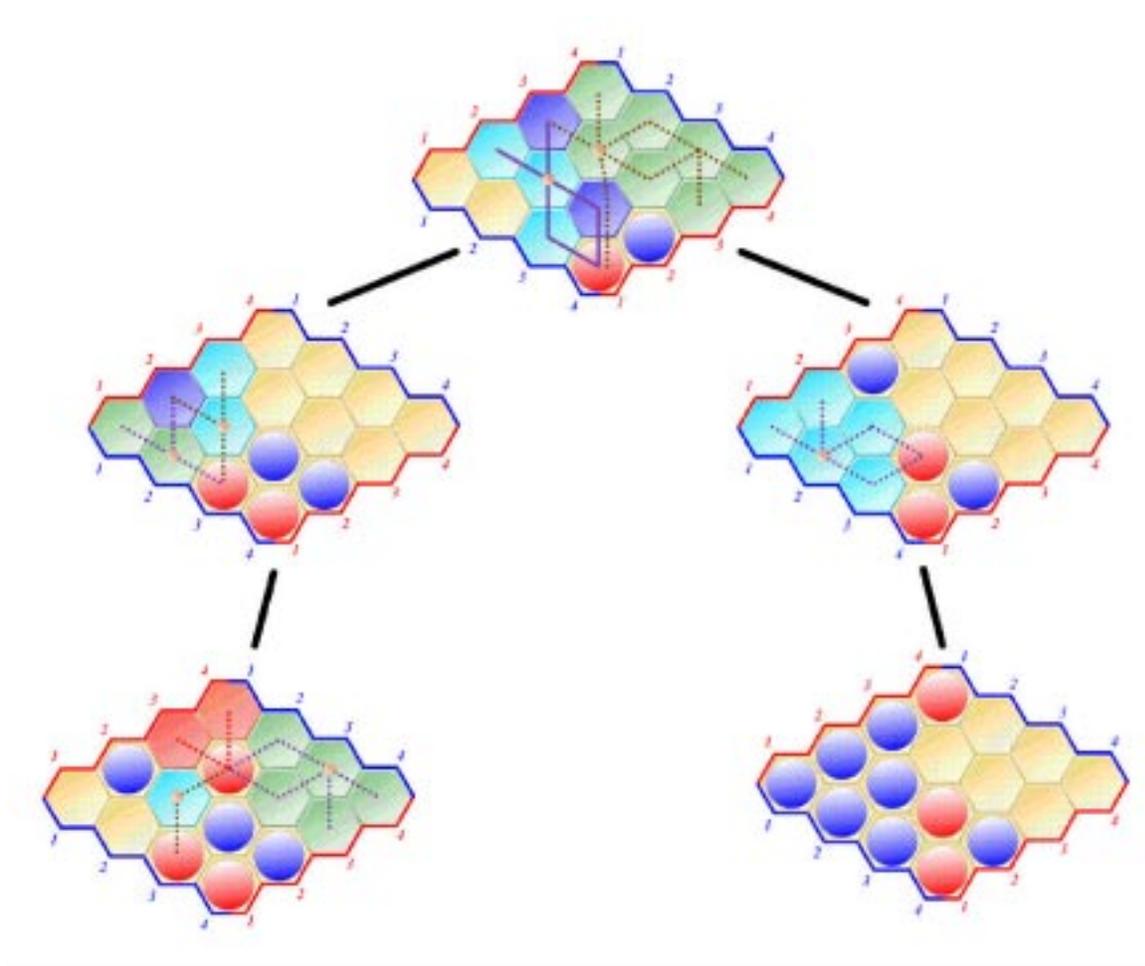


Figure 21. Partial game tree illustrating that red can win at $Hex(4)$ if blue opens with a move in category II, namely $[4, 2]$.

If blue first moves in tile $[4, 2]$, red can respond by placing a counter in tile $[4, 1]$ (just as in the previous argument). Now, if red places a counter in tile $[2, 2]$ this guarantees a winning path through one of the paths shown by the solid line in the diagram at the top of the tree. Perhaps less clear is the fact that placement of a red counter in tile $[2, 3]$ also guarantees a win for red, this time through one of the dotted paths. This is because the connection from $[2, 3]$ to the top red edge cannot be prevented and connection from $[2, 3]$ to the bottom red edge cannot be prevented since red has the choice of playing in tile $[3, 2]$ or tile $[3, 4]$ to guarantee this connection. Therefore, by proposition 8, the only way blue can prevent a loss is by placing a counter in tile $[1, 3]$ or tile $[3, 2]$.

If blue plays in tile $[3, 2]$, red can respond by playing in tile $[3, 1]$. Then if blue must play in tile $[1, 2]$ to prevent a loss through one of the paths illustrated in the second level, left node of the game tree. But then, red can guarantee a win with play in tile $[2, 3]$ as shown in the bottom left node of the figure.

If blue plays in tile $[1, 3]$, then red can respond by playing in tile $[3, 2]$. If red could then play in tile $[2, 1]$ and guarantee a win through one of the paths shown in the second level, right node of the game tree. Therefore, blue must play in one of the colored tiles. We allow, blue to play in all of these tiles. Red is forced to play in tile $[4, 1]$ and from here on, the remainder of the game (with the exception of the last play) is forced ending in a win for red.

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