A QUESTION OF SIERPINSKI ON TRIANGULAR NUMBERS

Michael A. Bennett
Department of Mathematics, University of British Columbia, Vancouver BC, Canada
bennett@math.ubc.edu

Received: 8/30/05, Accepted: 11/6/05, Published: 11/22/05

Abstract

We answer a question of Sierpinski by showing that there do not exist four distinct triangular number in geometric progression.

In D23 of [2], it is stated that Sierpinski asked the question of whether or not there exist four (distinct) triangular numbers in geometric progression and, further, that Szymiczek [4] conjectured the answer to this to be a negative one. Recall that a triangular number is one of the form

$$T_n = \frac{n(n+1)}{2}$$

for $n \in \mathbb{Z}$. The problem of finding three such triangular numbers is readily reduced to finding solutions to a Pell equation (whereby, an old result of Gérardin [1] (see also [3]) implies that there are infinitely many such triples, the smallest of which is (T_1, T_3, T_8)). It is easy to show that the answer to Sierpinski's question is in the negative. This is, in fact, an immediate consequence of the following.

Lemma *If a and b are positive integers with $b > 1$ then at least one of $ab+1$ and ab^3+1 is not a perfect square.*

Proof. Suppose that we have

$$ab + 1 = x^2 \text{ and } ab^3 + 1 = y^2$$

for positive integers x and y. From the theory of quadratic fields, since we may assume that ab is not a perfect square, it follows that

$$y + b\sqrt{ab} = \left(x + \sqrt{ab}\right)^k$$

for some positive integer k (which, since $b > 1$, we may assume to be at least 2). If $k \geq 3$, then we would have that

$$y + b\sqrt{ab} \geq \left(x + \sqrt{ab}\right)^3$$

and hence

$$b \geq 3x^2 + ab,$$

a contradiction, since $a, x \geq 1$. It follows that $k = 2$ and hence that $b = 2x$. Since $ab+1 = x^2$, we have $x = 1$, contradicting the fact that a and b are positive integers. This completes the proof of our Lemma.

To connect this to Sierpinski's question, suppose that the four triangular numbers in geometric progression are

$$T_x < T_y < T_u < T_v.$$

Taking $a = 8T_x$ and $b = T_y/T_x$, it follows that one of $8T_y + 1$ or $8T_v + 1$ cannot be a perfect square, contradicting the identity $8T_n + 1 = (2n+1)^2$.

References

[1] A. Gérardin. *Sphinx-Oedipe* 9 (1914), 75, 145–146

[2] R. Guy. Unsolved Problems in Number Theory, 3rd edition. Springer Verlag, New York, 2004.

[3] K. Szimiczek. L' équation $uv = w^2$ en nombres triangulaires. (French) *Publ. Inst. Math. (Beograd) (N.S.)* 3 (17) (1963), 139–141.

[4] K. Szimiczek. The equation $(x^2-1)(y^2-1) = (z^2-1)^2$, *Eureka* 35 (1972), 21–25.