THE DINNER TABLE PROBLEM: THE RECTANGULAR CASE

Roberto Tauraso
Dipartimento di Matematica, Università di Roma "Tor Vergata", 00133 Roma, Italy
tauraso@mat.uniroma2.it

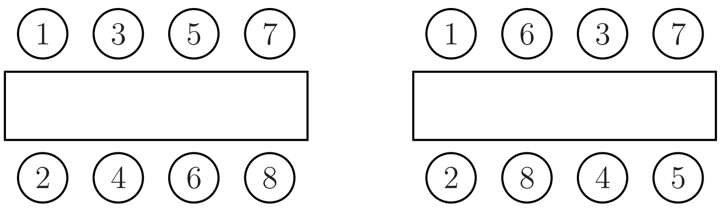
Received: 9/3/05, Revised: 4/5/06, Accepted: 4/7/06, Published: 4/12/06

Abstract

Consider n people who are seated randomly at a rectangular table with $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ seats along the two opposite sides, for two dinners. What is the probability that neighbors at the first dinner are no longer neighbors at the second one? We give an explicit formula and show that its asymptotic behavior as n goes to infinity is $e^{-2}(1+4/n)$ (it is known that it is $e^{-2}(1-4/n)$ for a round table). A more general permutation problem is also considered.

1. Introduction

Assume that 8 people are seated around a table and we want to enumerate the number of ways that they can be permuted such that neighbors are no longer neighbors after the rearrangement. Of course the answer depends on the *topology* of the table: if the table is a circle, then it is easy to check by a simple computer program that the number of permutations that satisfy this property are 2832. If it is a long bar and all people sit along one side, then there are 5242. Furthermore, if it is a rectangular table with two sides then the rearrangements number 9512. The first two cases are respectively described by sequences *A089222* and *A002464* of the On-Line Encyclopedia of Integer Sequences [8]. On the other hand, the rectangular case does not appear in the literature and recently the corresponding sequence has been labeled as *A110128*. Here is a valid rearrangement for $n = 8$:



First dinner Second dinner

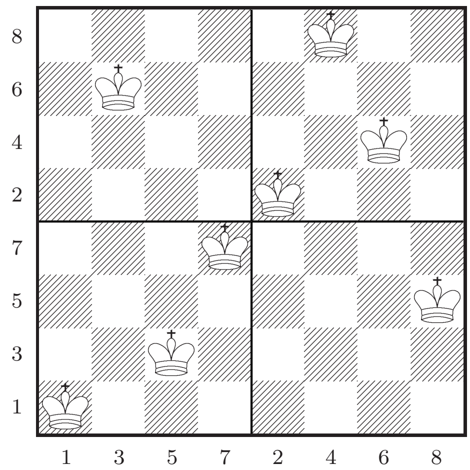
whose associated permutation is

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 6 & 8 & 3 & 4 & 7 & 5 \end{pmatrix}.$$

For a generic number of persons n the required property can be established more formally in this way:

$$|\pi(i+2) - \pi(i)| \neq 2 \quad \text{for } 1 \leq i \leq n-2.$$

It is interesting to note that this rearrangement problem around a table also has another remarkable interpretation. Consider n kings to be placed on a $n \times n$ board, one in each row and column, in such a way that they are non-attacking with respect to these different *topologies* of the board: if we enumerate the ways on a toroidal board we find the sequence *A089222*, for a regular board we have *A002464*, and finally if we divide the board in the main four quadrants we are considering the new sequence. Here is the 8 kings displacement that corresponds to the permutation π introduced before:



Closed formulas for the first two sequences are known: for *A089222* it is (see [2])

$$a_{n,0} = \sum_{r=0}^{n-1} (-1)^r \left(\frac{n}{n-r}\right)^2 (n-r)! \sum_{c=0}^{r} 2^c \binom{r-1}{c-1}\binom{n-r}{c} + (-1)^n 2n$$

and for *A002464* it is (see [1],[4],[5],[6],[7])

$$a_{n,1} = \sum_{r=0}^{n-1} (-1)^r (n-r)! \sum_{c=0}^{r} 2^c \binom{r-1}{c-1}\binom{n-r}{c}$$

(where $\binom{k}{-1} = 0$ if $k \neq -1$ and $\binom{-1}{-1} = 1$).

In this paper we study the sequence $a_{n,d}$ defined for $1 \leq d \leq n-1$ as follows: $a_{n,d}$ denotes the total number of permutations π of $\{1, 2, \ldots, n\}$ such that

$$|\pi(i+d) - \pi(i)| \neq d \quad \text{for } 1 \leq i \leq n-d.$$

Note that if a permutation π has this property then π^{-1} also has the same property.

The table below provides some numerical values:

n	$a_{n,0}$	$a_{n,1}$	$a_{n,2}$	$a_{n,3}$
1	1	1	1	1
2	0	0	2	2
3	0	0	4	6
4	0	2	16	20
5	10	14	44	80
6	36	90	200	384
7	322	646	1288	2240
8	2832	5242	9512	15424
9	27954	47622	78652	123456
10	299260	479306	744360	1110928
11	3474482	5296790	7867148	11287232
12	43546872	63779034	91310696	127016304
13	586722162	831283558	1154292796	1565107248
14	8463487844	11661506218	15784573160	20935873872
15	130214368530	175203184374	232050062524	301974271248
16	2129319003680	2806878055610	3648471927912	4669727780624

We will show that the following formula holds for $d \geq 2$:

$$a_{n,d} = \sum_{r_1=0}^{n_1-1} \cdots \sum_{r_d=0}^{n_d-1} (-1)^r \sum_{c_1=0}^{r_1 \wedge (n_1-r_1)} \cdots \sum_{c_d=0}^{r_d \wedge (n_d-r_d)} 2^c (n-r-c)! \prod_{k=1}^{d} \binom{n_k - r_k}{c_k} \sum_{\substack{\sum_{i=1}^{c_1} l_{i,1}=r_1 \\ l_{i,1}\geq 1}} \cdots \sum_{\substack{\sum_{i=1}^{c_d} l_{i,d}=r_d \\ l_{i,d}\geq 1}} q_{n,d}(L)$$

where $n_k = |N_k| = |\{1 \leq i \leq n : i \equiv k \mod d\}|$, $r_k \wedge (n_k - r_k)$ is the minimum of r_k and $n_k - r_k$, $r = \sum_{k=1}^{d} r_k$, $c = \sum_{k=1}^{d} c_k$, $L = [l_{1,1}, \cdots, l_{c_d,d}]$ or $[l_1, \cdots, l_c]$ after reindexing, and

$$q_{n,d}(L) \;=\; \sum_{\substack{J_1 \mathring{\cup} \cdots \mathring{\cup} J_d = \{1,\ldots,c\} \\ n_k \geq \sum_{i\in J_k} l_i}} \prod_{k=1}^{d} \binom{n_k - \sum_{i\in J_k} l_i}{|J_k|} \cdot |J_k|!\,.$$

Even if the above formula seems very complicated, it is quite manageable to attempt an asymptotic analysis. In the last section, we prove that the probability that a permutation belongs to the set enumerated by $a_{n,d}$ always tends to e^{-2} as n goes to infinity. A more precise expansion will reveal how the limiting probability depends on d:

$$\frac{a_{n,d}}{n!} = e^{-2}\left(1 + \frac{4(d-1)}{n} + O\left(\frac{1}{n^2}\right)\right).$$

I would like to warmly thank Alessandro Nicolosi and Giorgio Minenkov for drawing my attention to this problem.

2. Asymptotic Analysis: Cases $d = 0$ and $d = 1$

Proposition 1 *The following asymptotic expansions hold:*

$$\frac{a_{n,0}}{n!} \sim e^{-2}\left(1 - \frac{4}{n} + \frac{20}{3n^3} + \frac{58}{3n^4} + \frac{736}{15n^5} + O\left(\frac{1}{n^6}\right)\right),$$

and

$$\frac{a_{n,1}}{n!} \sim e^{-2}\left(1 - \frac{2}{n^2} - \frac{10}{3n^3} - \frac{6}{n^4} - \frac{154}{15n^5} + O\left(\frac{1}{n^6}\right)\right).$$

Proof. The expansion of $a_{n,0}/n!$ is contained in [2] and it was obtained from a recurrence relation by the method of undetermined coefficients.

With regard to $a_{n,1}/n!$, we give the detailed proof only for the coefficients of $1/n$ and $1/n^2$ (the others can be computed in a similar way). Since

$$\frac{a_{n,1}}{n!} = \sum_{r=0}^{n-1}(-1)^r \cdot \frac{(n-r)!}{n!}\sum_{c=0}^{r} 2^c \binom{r-1}{c-1}\binom{n-r}{c}$$

and, by the Chu-Vandermonde identity (see, for example, p.169 in [3]) ,

$$\begin{aligned} 0 \le \frac{(n-r)!}{n!}\sum_{c=0}^{r} 2^c \binom{r-1}{c-1}\binom{n-r}{c} &\le 2^r \frac{(n-r)!}{n!}\sum_{c=0}^{r}\binom{r-1}{c-1}\binom{n-r}{n-r-c} \\ &\le \frac{2^r}{r!}\binom{n}{r}^{-1}\binom{n-1}{r} \le \frac{2^r}{r!}, \end{aligned}$$

the alternating sum of $a_{n,1}/n!$ is dominated for any $n \ge 1$ by the convergent series $\sum_{r=0}^{+\infty} 2^r/r! = e^2$. Therefore, by uniform convergence, we can study the asymptotics of $a_{n,1}/n!$ term by term. Moreover

$$\frac{(n-r)!}{n!}\sum_{c=0}^{r} 2^c \binom{r-1}{c-1}\binom{n-r}{c} = \sum_{c=0}^{r} \frac{2^c}{c!}\binom{r-1}{c-1}\frac{n^{\underline{r+c}}}{(n^{\underline{r}})^2}$$

(where $n^{\underline{s}} = n(n-1)\cdots(n-s+1)$ is the falling factorial), and it suffices to analyze the cases when c is equal to r, $r-1$ and $r-2$ because for $r+c \le n$ the rational function $n^{\underline{r+c}}/(n^{\underline{r}})^2 \sim 1/n^{r-c}$. Since the falling factorial $n^{\underline{s}}$ is the generating function for the Stirling numbers of the first kind $\left[{s \atop k}\right]$ (see, for example, p.249 in [3])

$$n^{\underline{s}} = \sum_{k=0}^{s} \left[{s \atop k}\right](-1)^{n-k} n^k,$$

for $c = r$ we obtain

$$\binom{r-1}{r-1}\frac{n^{\underline{2r}}}{(n^{\underline{r}})^2} \sim \frac{1 - \left[{2r \atop 2r-1}\right]\frac{1}{n} + \left[{2r \atop 2r-2}\right]\frac{1}{n^2}}{\left(1 - \left[{r \atop r-1}\right]\frac{1}{n} + \left[{r \atop r-2}\right]\frac{1}{n^2}\right)^2} \sim 1 - \frac{r^{\underline{2}} + r}{n} + \frac{r^{\underline{4}} + 4r^{\underline{3}} + 2r^{\underline{2}}}{2n^2}.$$

In a similar way, for $c = r - 1$,

$$\binom{r-1}{r-2}\frac{n^{\underline{2r-1}}}{(n^{\underline{r}})^2} \sim \frac{(r-1)}{n} - \frac{(r-1)^{\underline{3}} + 3(r-1)^{\underline{2}} + (r-1)}{n^2},$$

and for $c = r - 2$,

$$\binom{r-1}{r-3}\frac{n^{\underline{2r-2}}}{(n^{\underline{r}})^2} \sim \frac{(r-2)^{\underline{2}} + 2(r-2)}{2n^2}.$$

Hence,

$$\begin{aligned}\frac{a_{n,1}}{n!} \sim & \sum_{r=0}^{n-1}\frac{(-2)^r}{r!}\left(1 - \frac{r^{\underline{2}}+r}{n} + \frac{r^{\underline{4}} + 4r^{\underline{3}} + 2r^{\underline{2}}}{2n^2}\right) + \\ & -\sum_{r=1}^{n-1}\frac{(-2)^{r-1}}{(r-1)!}\left(\frac{(r-1)}{n} - \frac{(r-1)^{\underline{3}} + 3(r-1)^{\underline{2}} + (r-1)}{n^2}\right) + \\ & +\sum_{r=2}^{n-1}\frac{(-2)^{r-2}}{(r-2)!}\left(\frac{(r-2)^{\underline{2}} + 2(r-2)}{2n^2}\right).\end{aligned}$$

Since, for $s \geq 0$,

$$\sum_{r=k}^{+\infty}\frac{(-2)^{r-k}}{(r-k)!}\cdot(r-k)^{\underline{s}} = \sum_{r=k+s}^{+\infty}\frac{(-2)^{r-k}}{(r-k-s)!} = (-2)^s e^{-2},$$

taking the sums we find that

$$\begin{aligned}\frac{a_{n,1}}{n!} & \sim e^{-2}\left(1 - \frac{(2^2-2)-2}{n} + \frac{(2^4 - 4\cdot 2^3 + 2\cdot 2^2) + 2\cdot(-2^3 + 3\cdot 2^2 - 2) + (2^2 - 2\cdot 2)}{2n^2}\right) \\ & \sim e^{-2}\left(1 - \frac{2}{n^2}\right).\end{aligned}$$

Note that the same strategy can also be applied to $a_{n,0}/n!$. For example, the coefficient of $1/n$ can be easily calculated in this way: for $c = r$ the formula gives

$$\binom{r-1}{r-1}\frac{n^{\underline{2r}}}{(n^{\underline{r}})^2}\cdot\left(\frac{n}{n-r}\right)^2 \sim \left(1 - \frac{r^{\underline{2}}+r}{n}\right)\cdot\left(1 + \frac{2r}{n}\right) \sim 1 - \frac{r^{\underline{2}} - r}{n}$$

and therefore

$$\begin{aligned}\frac{a_{n,0}}{n!} & \sim \sum_{r=0}^{n-1}\frac{(-2)^r}{r!}\left(1 - \frac{r^{\underline{2}} - r}{n}\right) - \sum_{r=1}^{n-1}\frac{(-2)^{r-1}}{(r-1)!}\left(\frac{r-1}{n}\right) \\ & \sim e^{-2}\left(1 - \frac{(2^2+2)-2}{n}\right) \sim e^{-2}\left(1 - \frac{4}{n}\right).\end{aligned}$$

□

3. The Formula for $d \geq 2$

Theorem 1 *For $d \geq 2$*

$$a_{n,d} = \sum_{r_1=0}^{n_1-1} \cdots \sum_{r_d=0}^{n_d-1} (-1)^r \sum_{c_1=0}^{r_1 \wedge (n_1-r_1)} \cdots \sum_{c_d=0}^{r_d \wedge (n_d-r_d)} 2^c (n-r-c)! \prod_{k=1}^{d} \binom{n_k - r_k}{c_k} \sum_{\substack{\sum_{i=1}^{c_1} l_{i,1} = r_1 \\ l_{i,1} \geq 1}} \cdots \sum_{\substack{\sum_{i=1}^{c_d} l_{i,d} = r_d \\ l_{i,d} \geq 1}} q_{n,d}(L)$$

where $n_k = |N_k| = |\{1 \leq i \leq n : i \equiv k \bmod d\}|$, $r = \sum_{k=1}^{d} r_k$, $c = \sum_{k=1}^{d} c_k$, $L = [l_{1,1}, \cdots, l_{c_d,d}]$ or $[l_1, \cdots, l_c]$ after reindexing, and

$$q_{n,d}(L) \;=\; \sum_{\substack{J_1 \mathring{\cup} \cdots \mathring{\cup} J_d = \{1,\ldots,c\} \\ n_k \geq \sum_{i \in J_k} l_i}} \prod_{k=1}^{d} \binom{n_k - \sum_{i \in J_k} l_i}{|J_k|} \cdot |J_k|!\,.$$

Remark 1 Note that for $d = 1$ the above formula coincides with the one we gave in the introduction:

$$\begin{aligned}
a_{n,1} &= \sum_{r=0}^{n-1} (-1)^r \sum_{c=0}^{r \wedge (n-r)} 2^c (n-r-c)! \binom{n-r}{c} \sum_{\substack{l_1 + \cdots + l_c = r \\ l_i \geq 1}} q_{n,1}([l_1, \cdots, l_c]) \\
&= \sum_{r=0}^{n-1} (-1)^r \sum_{c=0}^{r \wedge (n-r)} 2^c (n-r-c)! \binom{n-r}{c} \binom{r-1}{c-1} \binom{n-r}{c} c! \\
&= \sum_{r=0}^{n-1} (-1)^r (n-r)! \sum_{c=0}^{r} 2^c \binom{r-1}{c-1} \binom{n-r}{c}
\end{aligned}$$

because

$$\begin{aligned}
\sum_{\substack{l_1 + \cdots + l_c = r \\ l_i \geq 1}} q_{n,1}([l_1, \cdots, l_c]) &= \sum_{\substack{l_1 + \cdots + l_c = r \\ l_i \geq 1}} \sum_{\substack{J = \{1,\ldots,c\} \\ n \geq r}} \binom{n - \sum_{i \in J} l_i}{|J|} |J|! \\
&= \sum_{\substack{l_1 + \cdots + l_c = r \\ l_i \geq 1}} \binom{n-r}{c} c! = \binom{r-1}{c-1} \binom{n-r}{c} c!\,.
\end{aligned}$$

It is interesting to note that the generating function of the sequence $a_{n,1}$,

$$f(x) = \sum_{s=0}^{+\infty} s! \left(\frac{x(1-x)}{1+x} \right)^s,$$

which is due to to L. Carlitz (see [6]), can be easily proved by a very similar argument:

$$\begin{aligned} f(x) &= \sum_{s=0}^{+\infty} s! x^s \left(1 + \frac{2(-x)}{1-(-x)}\right)^s \\ &= \sum_{s=0}^{+\infty} s!\, x^s \sum_{c=0}^{s} \binom{s}{c} 2^c \left(\sum_{l=1}^{+\infty} (-x)^l\right)^c . \end{aligned}$$

Hence, for $n \geq 1$, the coefficient of x^n is equal to

$$\begin{aligned} [x^n] f(x) &= \sum_{r=0}^{n-1} (n-r)! \sum_{c=0}^{n-r} \binom{n-r}{c} 2^c \sum_{\substack{l_1+\cdots+l_c=r \\ l_i \geq 1}} (-1)^{l_1+\cdots+l_c} \\ &= \sum_{r=0}^{n-1} (-1)^r (n-r)! \sum_{c=0}^{n-r} 2^c \binom{n-r}{c} \binom{r-1}{c-1} = a_{n,1}. \end{aligned}$$

Proof of Theorem 1 (1st part). For $i = 1, \ldots, n-d$, let $T_{i,d}$ be the set of permutations of $\{1, 2, \ldots, n\}$ such that i and $i+d$ are *d-consecutive*, that is, the distance between i and $i+d$ in the list $\pi(1), \pi(2), \ldots, \pi(n)$ is equal to d:

$$T_{i,d} = \left\{\pi \in S_n \,:\, |\pi^{-1}(i+d) - \pi^{-1}(i)| = d\right\}$$

(for $i = n-d+1, \ldots, n$ we consider $T_{i,d}$ as an empty set). Then, by the Inclusion-Exclusion Principle,

$$a_{n,d} = \sum_{I \subset \{1,2,\ldots,n\}} (-1)^{|I|} \Big| \bigcap_{i \in I} T_{i,d} \Big|,$$

assuming the convention that when the intersection is made over an empty set of indices then it is the whole set of permutations S_n. A *d-component* of a set of indices I is a maximal subset of d-consecutive integers, and we denote by $\sharp I$ the number of d-components of I. So the above formula can be rewritten as

$$\begin{aligned} a_{n,d} &= \sum_{r_1=0}^{n_1-1} \cdots \sum_{r_d=0}^{n_d-1} (-1)^r \Big| \bigcap_{\substack{i \in I_1 \cup \cdots \cup I_d \\ I_k \subset N_k,\, |I_k| = r_k}} T_{i,d} \Big| \\ &= \sum_{r_1=0}^{n_1-1} \cdots \sum_{r_d=0}^{n_d-1} (-1)^r \sum_{c_1=0}^{r_1 \wedge (n_1-r_1)} \cdots \sum_{c_d=0}^{r_d \wedge (n_d-r_d)} \Big| \bigcap_{\substack{i \in I_1 \cup \cdots \cup I_d \\ I_k \subset N_k,\, |I_k|=r_k,\, \sharp I_k = c_k}} T_{i,d} \Big| \end{aligned}$$

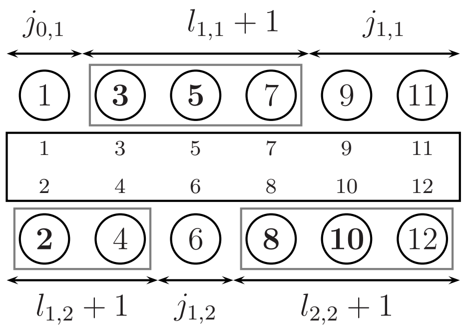
where $N_k = \{1 \leq i \leq n : i \equiv k \mod d\}$ and $n_k = |N_k|$.

Remark 2 In order to better illustrate the idea of the proof, which is inspired by the one of Robbins in [7], we give an example of how a permutation π that belongs to the above intersection of $T_{i,d}$'s can be selected. Assume that $n = 12$, $d = 2$, $r_1 = 2$, $r_2 = 3$, $c_1 = 1$, $c_2 = 2$. N_1 and N_2 are respectively the odd numbers and the even numbers between 1 and 12. Now we choose I_1 and I_2: let $l_{i,k}$ be the size of the ith-component in N_k (each component fixes $l_{i,k} + 1$ numbers) and let $j_{i,k}$ be the size of the gap between the ith-component and the $(i+1)$st-component in N_k. Then the choice of I_1 and I_2 is equivalent to selecting an integral solution of

$$\begin{cases} l_{1,1} = r_1 = 2 & , \quad l_{i,1} \geq 1 \\ l_{1,2} + l_{2,2} = r_2 = 3 & , \quad l_{i,2} \geq 1 \\ j_{0,1} + j_{1,1} = n_1 - r_1 - c_1 = 3 & , \quad j_{i,1} \geq 0 \\ j_{0,2} + j_{1,2} + j_{2,2} = n_2 - r_2 - c_2 = 1 & , \quad j_{i,2} \geq 0 \end{cases}.$$

For example, taking $l_{1,1} = 2$, $l_{1,2} = 1$, $l_{2,2} = 2$, $j_{0,1} = 1$, $j_{1,1} = 2$, $j_{0,2} = 0$, $j_{1,2} = 1$ and $j_{2,2} = 0$, we select the set of indices $I_1 = \{3, 5\}$ and $I_2 = \{2, 8, 10\}$.

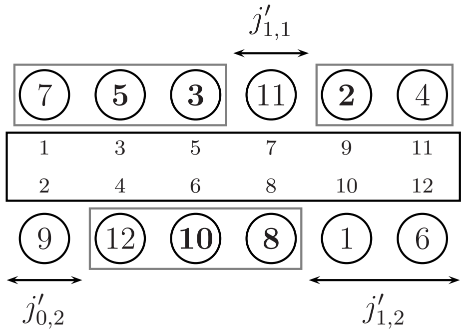
Here is the corresponding table arrangement:



Now we redistribute the three components selecting a partition $J_1 \mathring{\cup} J_2 = \{1, 2, 3\}$, say $J_1 = \{1, 2\}$ and $J_2 = \{3\}$. This means that the first two components $\{2\}$ and $\{3, 5\}$ will go to the odd seats and the third component $\{8, 10\}$ will go to the even seats. Then we decide the component displacements and orientations: for example, in the odd seats we place first $\{3, 5\}$ reversed and then $\{2\}$, and in the even seats we place $\{8, 10\}$ reversed. To determine the component positions we need the new gap sizes, and therefore we solve the two equations

$$\begin{cases} j'_{0,1} + j'_{1,1} + j'_{2,1} = n_1 - l_{1,1} - l_{1,2} - |J_1| = 1 & , \quad j'_{i,1} \geq 0 \\ j'_{0,2} + j'_{1,2} = n_2 - l_{2,2} - |J_2| = 3 & , \quad j'_{i,2} \geq 0 \end{cases}$$

where $j'_{i,k}$ is the size of the new gap between the ith-component and the $(i+1)$st-component in N_k. If we take $j'_{0,1} = 0$, $j'_{1,1} = 1$, $j'_{2,1} = 0$, $j'_{0,2} = 1$, $j'_{1,2} = 2$ and we fill the empty places with the remaining numbers 1, 6, 9 and 11, we obtain the following table rearragement:



Proof of Theorem 1 (2nd part). Following the notations introduced in the previous remark, each I_k, for $1 \leq k \leq d$, is determined by an integral solution of

$$\begin{cases} l_{1,k} + l_{2,k} + \cdots + l_{c_k,k} = r_k & , \quad l_{i,k} \geq 1 \\ j_{0,k} + j_{1,k} + \cdots + j_{c_k,k} = n_k - r_k - c_k & , \quad j_{i,k} \geq 0 \end{cases}$$

where $l_{1,k}, \ldots, l_{c_k,k}$ are the component sizes and $j_{0,k}, \ldots, j_{c_k,k}$ are the gap sizes in N_k. The counting of the integral solutions of these d systems yields the following factor in the formula

$$\prod_{k=1}^{d} \binom{n_k - r_k}{c_k} \sum_{\substack{\sum_{i=1}^{c_1} l_{i,1} = r_1 \\ l_{i,1} \geq 1}} \cdots \sum_{\substack{\sum_{i=1}^{c_d} l_{i,d} = r_d \\ l_{i,d} \geq 1}} .$$

Now, given $I_1, \ldots, I_d$, we select a permutation $\pi \in \left|\bigcap_{i \in I_1 \cup \cdots \cup I_d} T_{i,d}\right|$ following these steps:

- We redistribute the $c = \sum_{k=1}^{d} c_k$ components selecting a partition $J_1, \ldots, J_d$ of the set of indices $\{1, \ldots, c\}$ (we allow J_k to be empty).
- For each set J_k of the partition, we determine the sizes of the new gaps solving
$$j'_{0,k} + j'_{1,k} + \cdots + j'_{|J_k|,k} = n_k - \sum_{i \in J_k} l_i - |J_k| \quad , \quad j'_{i,k} \geq 0.$$
This can be done in $\binom{n_k - \sum_{i \in J_k} l_i}{|J_k|}$ ways.
- For each set J_k of the partition, we choose the order of the corresponding $|J_k|$ components and their orientation in $|J_k|! \cdot 2^{|J_k|}$ ways.
- We fill the empty places with the remaining numbers in $(n - r - c)!$ ways.

Taking into account all these effects, we obtain

$$\sum_{\substack{J_1 \mathring{\cup} \cdots \mathring{\cup} J_d = \{1, \ldots, c\} \\ n_k \geq \sum_{i \in J_k} l_i}} (n - r - c)! \prod_{k=1}^{d} \binom{n_k - \sum_{i \in J_k} l_i}{|J_k|} \cdot |J_k|! \cdot 2^{|J_k|}.$$

Finally, since $c = \sum_{k=1}^{d} |J_k|$, then $\prod_{k=1}^{d} 2^{|J_k|} = 2^c$ and we get the desired formula. $\square$

4. Asymptotic Analysis: The General Case

Theorem 2 *For $d \geq 0$*

$$\frac{a_{n,d}}{n!} = e^{-2}\left(1 + \frac{4(d-1)}{n} + O\left(\frac{1}{n^2}\right)\right).$$

Proof. The cases $d = 0$ and $d = 1$ have been already discussed, so we assume that $d \geq 2$. From the beginning of the proof of Theorem 1 we have

$$\frac{a_{n,d}}{n!} = \sum_{r_1=0}^{n_1-1} \cdots \sum_{r_d=0}^{n_d-1} (-1)^r \frac{1}{n!} \Big| \bigcap_{\substack{i \in I_1 \cup \cdots \cup I_d \\ I_k \subset N_k,\, |I_k| = r_k}} T_{i,d} \Big|.$$

Moreover, since each I_k can be selected in $\binom{n_k}{r_k}$ ways, each index in I_k can be ordered in 2 ways and the remaining numbers can be arranged in $(n-r)!$ ways, we have

$$0 \leq \frac{1}{n!} \Big| \bigcap_{\substack{i \in I_1 \cup \cdots \cup I_d \\ I_k \subset N_k,\, |I_k| = r_k}} T_{i,d} \Big| \leq \frac{(n-r)!}{n!} \prod_{k=1}^{d} \binom{n_k}{r_k} 2^{r_k}$$

$$\leq \binom{n}{n_1, \ldots, n_d}^{-1} \binom{n-r}{n_1 - r_1, \ldots, n_d - r_d} \prod_{k=1}^{d} \frac{2^{r_k}}{r_k!} \leq \prod_{k=1}^{d} \frac{2^{r_k}}{r_k!}$$

This means that the alternating sum of $a_{n,d}/n!$ is dominated, for any $n \geq 1$, by the convergent series

$$\sum_{r_1=0}^{+\infty} \cdots \sum_{r_d=0}^{+\infty} \prod_{k=1}^{d} \frac{2^{r_k}}{r_k!} = e^{2d}.$$

Therefore, by uniform convergence, we can study the asymptotics of $a_{n,d}/n!$ term by term.

By Theorem 1, and since $c = \sum_{k=1}^{d} c_k = \sum_{k=1}^{d} |J_k|$, each term has the following form:

$$(-1)^r 2^c \frac{(n-r-c)!}{n!} \prod_{k=1}^{d} \binom{n_k - r_k}{c_k} \prod_{k=1}^{d} \binom{n_k - \sum_{i \in J_k} l_i}{|J_k|} \cdot |J_k|! \sim const \cdot \frac{n^{2c}}{n^{r+c}}$$

and therefore goes to zero faster than $1/n$ (as n goes to infinity), unless either $2c = r + c$ or $2c = r + c - 1$. In the first case, $c_k = r_k$ for any $k = 1, \ldots, d$ (remember that $c_k \leq r_k$) and all component sizes are equal to 1. In the second case, the same situation holds with two exceptions: $c_{k_0} = r_{k_0} - 1$ for some index k_0 and one of the components in N_{k_0} has size equal to 2. By symmetry, we can assume that this particular index k_0 is equal to d and multiply the corresponding term by d:

$$\frac{a_{n,d}}{n!} \sim \sum_{r_1=0}^{n/d-1}\cdots\sum_{r_d=0}^{n/d-1}\left((-2)^r\frac{(n-2r)!}{n!}\prod_{k=1}^{d}\binom{n/d-r_k}{r_k} q_{n,d}([\overbrace{1,\dots,1}^{r}])+\right.$$
$$\left.+d\cdot(-2)^{r-1}\frac{(n-2r+1)!}{n!}\prod_{k=1}^{d}\binom{n/d-r_k}{r_k}\binom{n/d-r_d}{r_d-1}(r_d-1)q_{n,d}([\overbrace{1,\dots,1}^{r-2},2])\right),$$

that is

$$\frac{a_{n,d}}{n!} \sim \sum_{r_1=0}^{n/d-1}\cdots\sum_{r_d=0}^{n/d-1}\frac{q_{n,d}([\overbrace{1,\dots,1}^{r}])}{n^{\underline{2r}}}\prod_{k=1}^{d}\frac{(-2)^{r_k}}{r_k!}\prod_{k=1}^{d}\frac{(n/d)^{\underline{2r_k}}}{(n/d)^{\underline{r_k}}}+$$
$$+2d\cdot\sum_{r_1=0}^{n/d-1}\cdots\sum_{r_{d-1}=0}^{n/d-1}\sum_{r_d=2}^{n/d-1}\frac{q_{n,d}([\overbrace{1,\dots,1}^{r-2},2])}{n^{\underline{2r-1}}}\prod_{k=1}^{d-1}\frac{(-2)^{r_k}}{r_k!}\cdot\frac{(-2)^{r_d-2}}{(r_d-2)!}\prod_{k=1}^{d-1}\frac{(n/d)^{\underline{2r_k}}}{(n/d)^{\underline{r_k}}}\cdot\frac{(n/d)^{\underline{2r_d-1}}}{(n/d)^{\underline{r_d}}}.$$

We start by considering the second term. Since

$$\begin{aligned}
q_{n,d}([\overbrace{1,\dots,1}^{r-2},2]) &\sim d\cdot\sum_{\substack{\sum_{k=1}^{d} r'_k=r-2\\ r'_k\geq 0}}\binom{r-2}{r'_1,\dots,r'_d}\prod_{k=1}^{d-1}\binom{n/d-r'_k}{r'_k}\cdot r'_k!\cdot\binom{n/d-r'_d-2}{r'_d}\cdot(r'_d+1)!\\
&\sim d\cdot\sum_{\substack{\sum_{k=1}^{d} r'_k=r-2\\ r'_k\geq 0}}\binom{r-2}{r'_1,\dots,r'_d}\prod_{k=1}^{d-1}\frac{(n/d)^{\underline{2r'_k}}}{(n/d)^{\underline{r'_k}}}\cdot\frac{(n/d)^{\underline{2r'_d+3}}}{(n/d)^{\underline{r'_d+2}}}\\
&\sim d\cdot\left(\frac{n}{d}\right)^{r-1}\sum_{\substack{\sum_{k=1}^{d} r'_k=r-2\\ r'_k\geq 0}}\binom{r-2}{r'_1,\dots,r'_d}=d\cdot\left(\frac{n}{d}\right)^{r-1}d^{r-2}=n^{r-1},
\end{aligned}$$

the second term is

$$2\cdot\sum_{r_1=0}^{n/d-1}\cdots\sum_{r_{d-1}=0}^{n/d-1}\sum_{r_d=2}^{n/d-1}\frac{n^{r-1}}{n^{2r-1}}\prod_{k=1}^{d-1}\frac{(-2/d)^{r_k}}{r_k!}\cdot\frac{(-2/d)^{r_d-2}}{(r_d-2)!}\cdot n^{r-1}\sim 2\left(e^{-2/d}\right)^d\frac{1}{n}=\frac{2e^{-2}}{n}.$$

Now we consider the first term. Since

$$
\begin{aligned}
q_{n,d}([\overbrace{1,\ldots,1}^{r}]) &\sim \sum_{\substack{\sum_{k=1}^{d} r'_k = r \\ r'_k \geq 0}} \binom{r}{r'_1,\ldots,r'_d} \prod_{k=1}^{d} \binom{n/d - r'_k}{r'_k} \cdot r'_k! \\
&\sim \sum_{\substack{\sum_{k=1}^{d} r'_k = r \\ r'_k \geq 0}} \binom{r}{r'_1,\ldots,r'_d} \prod_{k=1}^{d} \frac{(n/d)^{\underline{2r_k}}}{(n/d)^{\underline{r_k}}} \\
&\sim \left(\frac{n}{d}\right)^r \sum_{\substack{\sum_{k=1}^{d} r'_k = r \\ r'_k \geq 0}} \binom{r}{r'_1,\ldots,r'_d} \prod_{k=1}^{d} \frac{1 - \frac{1}{2}(2r'_k)(2r'_k - 1)\cdot\frac{d}{n}}{1 - \frac{1}{2}r'_k(r'_k - 1)\cdot\frac{d}{n}} \\
&\sim \left(\frac{n}{d}\right)^r \sum_{\substack{\sum_{k=1}^{d} r'_k = r \\ r'_k \geq 0}} \binom{r}{r'_1,\ldots,r'_d} \left(1 - \left(\frac{3}{2}\sum_{k=1}^{d} r'_k(r'_k - 1) + \sum_{k=1}^{d} r'_k\right)\cdot\frac{d}{n}\right) \\
&\sim \left(\frac{n}{d}\right)^r \left(d^r - \left(\frac{3}{2}dr(r-1)d^{r-2} + drd^{r-1}\right)\cdot\frac{d}{n}\right) \\
&\sim n^r \left(1 - \left(\frac{3}{2}r(r-1) + dr\right)\frac{1}{n}\right),
\end{aligned}
$$

the first term is

$$
\sum_{r_1=0}^{n/d-1} \cdots \sum_{r_d=0}^{n/d-1} \frac{1 - \left(\frac{3}{2}r(r-1) + dr\right)\cdot\frac{1}{n}}{1 - r(2r-1)\cdot\frac{1}{n}} \prod_{k=1}^{d} \frac{(-2/d)^{r_k}}{r_k!} \prod_{k=1}^{d} \left(\frac{1 - r_k(2r_k - 1)\cdot\frac{d}{n}}{1 - \frac{1}{2}r_k(r_k - 1)\cdot\frac{d}{n}}\right);
$$

that is,

$$
\sum_{r_1=0}^{n/d-1} \cdots \sum_{r_d=0}^{n/d-1} \left(1 + \left(\frac{r^2 + (1-2d)r}{2}\right)\cdot\frac{1}{n}\right) \prod_{k=1}^{d} \frac{(-2/d)^{r_k}}{r_k!} \prod_{k=1}^{d} \left(1 - \left(\frac{3r_k(r_k - 1)}{2} + r_k\right)\cdot\frac{d}{n}\right).
$$

Recalling that $r = \sum_{k=1}^{d} r_k$, we have

$$
1 + \left(\frac{r^2 + (1-2d)r}{2}\right)\cdot\frac{1}{n} = 1 + \left(\frac{1}{2}\sum_{k=1}^{d} r_k(r_k - 1) + (1-d)\sum_{k=1}^{d} r_k + \sum_{k=1}^{d}\sum_{k'=k+1}^{d} r_k r_{k'}\right)\cdot\frac{1}{n}.
$$

Moreover

$$
\prod_{k=1}^{d} \left(1 - \left(\frac{3r_k(r_k - 1)}{2} + r_k\right)\cdot\frac{d}{n}\right) \sim 1 + \left(-\frac{3d}{2}\sum_{k=1}^{d} r_k(r_k - 1) - d\sum_{k=1}^{d} r_k\right)\cdot\frac{1}{n}.
$$

Therefore, the first term is equivalent to

$$e^{-2}+\frac{1}{n}\cdot\sum_{r_1=0}^{n/d-1}\cdots\sum_{r_d=0}^{n/d-1}\left(\frac{1-3d}{2}\sum_{k=1}^{d}r_k(r_k-1)\prod_{k=1}^{d}\frac{(-2/d)^{r_k}}{r_k!}\ +\right.$$
$$\left.+(1-2d)\sum_{k=1}^{d}r_k\prod_{k=1}^{d}\frac{(-2/d)^{r_k}}{r_k!}+\sum_{k=1}^{d}\sum_{k'=k+1}^{d}r_kr_{k'}\prod_{k=1}^{d}\frac{(-2/d)^{r_k}}{r_k!}\right).$$

Taking the sums we obtain

$$e^{-2}\left(1+\frac{1}{n}\cdot\left(\frac{1-3d}{2}\left(\frac{-2}{d}\right)^2d+(1-2d)\left(\frac{-2}{d}\right)d+\left(\frac{-2}{d}\right)^2\frac{d(d-1)}{2}\right)\right);$$

that is,

$$e^{-2}\left(1+\frac{4d-6}{n}\right).$$

Finally, putting everything together we find that

$$\frac{a_{n,d}}{n!}\sim e^{-2}\left(1+\frac{4d-6}{n}\right)+\frac{2e^{-2}}{n}=e^{-2}\left(1+\frac{4(d-1)}{n}\right).$$

□

References

[1] M. Abramson and W. Moser, Combinations, successions, and the n-kings problem, *Math. Magazine*, **39** (1966), 269–273.

[2] B. Aspvall and F. M. Liang. The dinner table problem. Technical Report STAN-CS-80-829, Computer Science Department, Stanford University, Stanford, California, 1980.

[3] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1990.

[4] I. Kaplansky, Symbolic solution of certain problems in permutations, *Bull. Amer. Math. Soc.*, **50** (1944), 906–914.

[5] I. Kaplansky, The asymptotic distribution of runs of consecutive elements , *Ann. Math. Statist.*, **16** (1945), 200–203.

[6] J. Riordan. A recurrence for permutations without rising or falling successions, *Ann. Math. Statist.*, **36** (1965), 708–710.

[7] D. P. Robbins. The probability that neighbors remain neighbors after random rearrangements, *Amer. Math. Monthly*, **87** (1980), 122–124.

[8] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences, on the web at `http://www.research.att.com/∼njas/sequences/`.