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COMBINATORIAL PROOFS OF SOME MORIARTY-TYPE BINOMIAL COEFFICIENT IDENTITIES

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Abstract

In this note, we present combinatorial proofs of some Moriarty-type binomial coefficient identities using linear and circular domino arrangements.

1. Introduction

The following identities appear, respectively, as entries 3.121, 3.120, 3.179, and 3.180 of Gould's *Combinatorial Identities* [2]:

$$\sum_{k=j}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}\binom{k}{j}=2^{n-2j}\binom{n-j}{j}, \tag{1.1}$$

$$\sum_{k=j}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{k}{j}=2^{n-2j-1}\binom{n-j}{j}\frac{n}{n-j}, \tag{1.2}$$

$$\sum_{k=j}^{\lfloor n/2\rfloor}(-1)^k\binom{n-k}{k}\binom{k}{j}2^{n-2k}=(-1)^j\binom{n+1}{2j+1}, \tag{1.3}$$

and

$$\sum_{k=j}^{\lfloor n/2\rfloor}(-1)^k\frac{n}{n-k}\binom{n-k}{k}\binom{k}{j}2^{n-2k-1}=(-1)^j\binom{n}{2j}, \tag{1.4}$$

where $n \geqslant 1$ and $0 \leqslant j \leqslant \lfloor n/2\rfloor$. Gould [3] attributes close variants of (1.1) and (1.2) to Moriarty, while (1.3) and (1.4) seem to have originated with Gould, who terms them "inverse Moriarty formulas."

Identity (1.1) is closely allied with the Fibonacci polynomials $f_n(x)$ (see, e.g., [5] or [6]) defined by the recurrence $f_n(x) = f_{n-1}(x) + xf_{n-2}(x)$ with initial values $f_0(x) = f_1(x) = 1$. Expanding the Binet formula

$$f_n(x) = \frac{\left(\frac{1+\sqrt{1+4x}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{1+4x}}{2}\right)^{n+1}}{\sqrt{1+4x}}, \qquad n \geqslant 0, \tag{1.5}$$

by the binomial theorem and comparing coefficients with the explicit formula

$$f_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^k, \qquad n \geqslant 0, \tag{1.6}$$

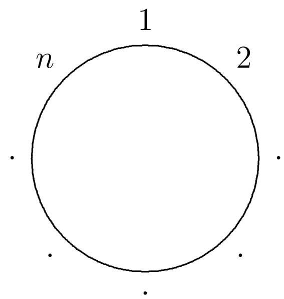
yields (1.1). Similarly, identity (1.2) arises in connection with the Lucas polynomials $l_n(x)$ defined by the recurrence $l_n(x) = l_{n-1}(x) + xl_{n-2}(x)$ with initial values $l_0(x) = 2$, $l_1(x) = 1$. Additional algebraic proofs have been given for (1.1) and (1.2), including induction [4], generating functions [3], and matrix expansion [3]. Inversion [3] of formulas (1.1) and (1.2) yields (1.3) and (1.4), which can also be established directly by induction [1]. In this note, we present combinatorial proofs of identities (1.1)–(1.4) using linear and circular domino arrangements.

2. Combinatorial Proofs

We first provide combinatorial interpretations for identities (1.1) and (1.2). Given $n \geqslant 1$ and $0 \leqslant j \leqslant \lfloor n/2 \rfloor$, let $\mathcal{R}_{n,j}$ denote the set of coverings of the numbers $1, 2, ..., n$, arranged in a row, by j indistinguishable dominos and $n - 2j$ indistinguishable squares, where pieces do not overlap, a *domino* is a rectangular piece covering two consecutive numbers, and a *square* is a piece covering a single number. Each such covering corresponds uniquely to a word in the alphabet $\{d, s\}$ comprising j d's and $n - 2j$ s's so that

$$|\mathcal{R}_{n,j}| = \binom{n-j}{j}, \qquad 0 \leqslant j \leqslant \lfloor n/2 \rfloor. \tag{2.1}$$

Similarly, let $\mathcal{C}_{n,j}$ denote the set of coverings of the numbers $1, 2, ..., n$, arranged clockwise around a circle, by j dominos and $n - 2j$ squares, where the numbers 1 and n are now considered consecutive:



Classifying members of $\mathcal{C}_{n,j}$ according to whether 1 is covered by a domino or a square, and applying (2.1), yields

$$|\mathcal{C}_{n,j}| = \frac{n}{n-j}\binom{n-j}{j}, \qquad 0 \leqslant j \leqslant \lfloor n/2 \rfloor. \tag{2.2}$$

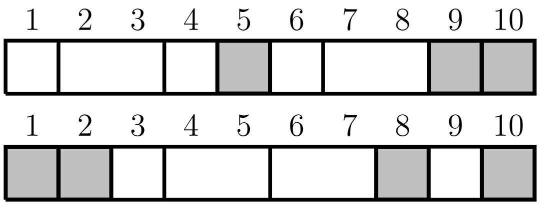
See, e.g., [7, p. 73] or [5].

Let $\mathcal{R}'_{n,j}$ and $\mathcal{C}'_{n,j}$ denote colored versions of $\mathcal{R}_{n,j}$ and $\mathcal{C}_{n,j}$, respectively, wherein the squares are each colored black or white (the dominos are not colored). From (2.1) and (2.2), we see that the right-hand sides of (1.1) and of twice (1.2) give the cardinality of $\mathcal{R}'_{n,j}$ and $\mathcal{C}'_{n,j}$, respectively.

We now show how the left-hand side of (1.1) also counts $\mathcal{R}'_{n,j}$. First select a subset $S = \{a_1 < a_2 < \cdots < a_{2k+1}\}$ of $[n+1]$ with cardinality at least $2j+1$ (where $[m] := \{1, 2, ..., m\}$ for $m \geqslant 1$). If $a_{2k+1} = n+1$, then pick j members of S of odd index not exceeding $2k-1$. If a_{2i-1} was chosen, then cover the numbers a_{2i-1}, $a_{2i-1}+1$ with a domino and any remaining numbers in the interval $[a_{2i-1}, a_{2i}]$ with black squares. If a_{2i-1} was not chosen, then cover a_{2i-1} with a white square, $a_{2i-1}+1$ with a black square, and any remaining numbers in $[a_{2i-1}, a_{2i}]$ with black squares. Cover any remaining members of $[n]$ with white squares.

If $a_{2k+1} < n+1$, then pick j members of S of even index and cover numbers in the intervals $[a_{2i}, a_{2i+1}]$, $1 \leqslant i \leqslant k$, in the same manner we covered the intervals $[a_{2i-1}, a_{2i}]$ in the prior case. In addition, cover all members of $[a_1]$ with black squares. As before, cover any remaining members of $[n]$ with white squares. Note that in this case, members of $\mathcal{R}'_{n,j}$ start with a black square, while in the prior case, members start with a domino or white square.

We then see that the left-hand side of (1.1) gives the cardinality of $\mathcal{R}'_{n,j}$ according to the value of the sum of the number of dominos and the number of occurrences of a white square directly preceding a black one. We illustrate the correspondence below, where $n = 10$ and $j = 2$:



$\leftrightarrow S = \{2, 3, 4, 5, 7, 10, 11\}$ with $\{2, 7\}$ chosen

$\leftrightarrow S = \{2, 4, 5, 6, 8, 9, 10\}$ with $\{4, 6\}$ chosen

That twice the left-hand side of (1.2) also gives $|\mathcal{C}'_{n,j}|$ follows similarly. Select a subset $S = \{a_1 < a_2 < \cdots < a_{2k}\}$ of $[n]$ with cardinality at least $2j$. Either pick j members of S of odd index or pick j members of even index. In the first case, cover the j chosen members with initial segments of dominos and the other members of S of odd index with white squares directly followed by black ones. Then cover any remaining numbers in the intervals $[a_{2i-1}, a_{2i}]$ with black squares. In the second case, we cover the intervals $[a_{2i}, a_{2i+1}]$, $1 \leqslant i \leqslant k$, using S in the same manner (where $[a_{2k}, a_{2k+1}]$ represents all the numbers along the circle between a_{2k} and a_1, inclusive, going clockwise). In both cases, cover any remaining members of $[n]$ with white squares. Note that in the latter case, the number 1 is covered by either a black square or by a domino with initial segment n.

When $j = 0$, identities (1.1) and (1.2) reduce to the familiar

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} = 2^{n-1}, \qquad n \geqslant 1. \tag{2.3}$$

Summing (1.1) and (1.2) over j and interchanging summation yields, respectively,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^{n-2k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} 2^k \tag{2.4}$$

and

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} 2^{n-2k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{k+1}. \tag{2.5}$$

The argument above for (1.1) and (1.2) applies to (2.4) and (2.5) as well. Note that the sums in (2.4) and (2.5) have familiar closed forms in terms of radicals.

We now turn to formulas (1.3) and (1.4), which we'll prove by sieving out special subsets of $\mathcal{R}'_{n,j}$ and $\mathcal{C}'_{n,j}$, respectively. Let $\mathcal{R} \subseteq \mathcal{R}'_{n,j}$ consist of those arrangements in which no white square directly precedes a black one and $\mathcal{C} \subseteq \mathcal{C}'_{n,j}$ consist of those arrangements in which no white square directly precedes a black one when the circle is traversed clockwise. Then $|\mathcal{R}| = \binom{n+1}{2j+1}$, upon reasoning as we did for the left-hand side of (1.1) (when $k = j$ and $|S| = 2j+1$); similarly, $|\mathcal{C}| = 2\binom{n}{2j}$, as in (1.2). If $k := i + j$, then the terms $\binom{n-k}{k}\binom{k}{j}2^{n-2k}$ and $\frac{n}{n-k}\binom{n-k}{k}\binom{k}{j}2^{n-2k}$ account for members of $\mathcal{R}'_{n,j}$ and $\mathcal{C}'_{n,j}$, respectively, with at least i occurrences of a white square directly preceding a black one, upon treating these occurrences as additional dominos. Formulas (1.3) and (1.4) now follow by the inclusion-exclusion principle.

When $j = 0$, identities (1.3) and (1.4) reduce, respectively, to

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} 2^{n-2k} = n+1, \tag{2.6}$$

and

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} 2^{n-2k} = 2, \tag{2.7}$$

which are entries (1.72) and (1.65) in Gould [2]. Summing (1.3) and (1.4) over j yields, respectively,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} 2^{n-k} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n+1}{2k+1} \tag{2.8}$$

and

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} 2^{n-k-1} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k}. \tag{2.9}$$

A slight modification of the argument above for (1.3) and (1.4) applies to (2.8) and (2.9). Note that the sums in (2.8) and (2.9) have familiar closed forms in terms of imaginary numbers.

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