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THE AKIYAMA-TANIGAWA ALGORITHM FOR CARLITZ'S q-BERNOULLI NUMBERS

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Abstract

We show that the Akiyama-Tanigawa algorithm and Chen's variant for computing Bernoulli numbers can be generalized to Carlitz's q-Bernoulli numbers. We also put these algorithms in the larger context of generalized Euler-Seidel matrices.

1. Introduction

Carlitz [?] introduced the q-Bernoulli numbers β_n $(n \geq 1)$ by the recurrence:

$$q(q\beta+1)^n - \beta_n = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n > 1; \end{cases} \tag{1}$$

where $\beta_0 = 1$ and $\beta_k = \beta^k$ after expansion. The first few values of β_n are

$$\beta_0 = 1, \quad \beta_1 = -\frac{1}{[2]}, \quad \beta_2 = \frac{q}{[2][3]}, \quad \beta_3 = -\frac{(q-1)q}{[3][4]},$$

where $[n] = (1-q^n)/(1-q)$ and $[n]! = [1][2]\dots[n]$ for $n \geq 0$. More generally we have the following explicit formula (see [?]):

$$(q-1)^n \beta_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{i+1}{[i+1]}.$$

Recently, Akiyama and Tanigawa's amazing algorithm for computing Bernoulli numbers [?] has attracted the attention of several authors [?, ?, ?]. One of our aims is to show that there is an analogue algorithm for Carlitz's q-Bernoulli numbers as follows: start with the 0-th row $1, \frac{1}{[2]}, \frac{1}{[3]}, \frac{1}{[4]}, \frac{1}{[5]}, \dots$ and define the first row by

$$1 \cdot \left(1 - \frac{1}{[2]}\right), \quad [2] \cdot \left(\frac{1}{[2]} - \frac{1}{[3]}\right), \quad [3] \cdot \left(\frac{1}{[3]} - \frac{1}{[4]}\right), \dots$$

which produces the sequence $\frac{q}{[2]}, \frac{q^2}{[3]}, \frac{q^3}{[4]}, \ldots$. Then define the next row by

$$1 \cdot \left(\frac{q}{[2]} - \frac{q^2}{[3]}\right), \quad [2] \cdot \left(\frac{q^2}{[3]} - \frac{q^3}{[4]}\right), \quad [3] \cdot \left(\frac{q^3}{[4]} - \frac{q^4}{[5]}\right), \ldots$$

thus giving $\frac{q}{[2][3]}, \frac{[2]q^2}{[3][4]}, \frac{[3]q^3}{[4][5]}, \ldots$ as the second row. In general, denoting the m-th ($m = 0, 1, 2, \ldots$) coefficient in the n-th ($n = 0, 1, 2, \ldots$) row by $a_{n,m}$, then the following recurrence relation holds:

$$a_{n,m} = [m+1] \cdot (a_{n-1,m} - a_{n-1,m+1}) \qquad (m \geq 0, n \geq 1). \tag{2}$$

We claim that the 0-th component $a_{n,0}$ of each row is just the n-th q-Bernoulli number β_n for $n \geq 2$.

Chen [?] gave a variant of the Akiyama and Tanigawa algorithm, which generates the Bernoulli numbers starting from $n = 1$. We have also a q-version of Chen's algorithm for q-Bernoulli numbers as follows: if we replace (??) by the following equation

$$a_{n,m} = [m]a_{n-1,m} - [m+1]a_{n-1,m+1} \qquad (m \geq 0, \quad n \geq 1), \tag{3}$$

then the 0-th component $a_{n,0}$ of each row is just the n-th q-Bernoulli number β_n for $n \geq 1$.

The validity of these algorithms is based on two facts: the first one (Theorem 1) relates the 0-th component $a_{n,0}$ of each row to the initial sequence $a_{0,m}$ by means of *q-Stirling numbers of second kind*, and the second one gives two explicit formulae (Theorem 2) of the q-Bernoulli numbers in terms of *q-Stirling numbers of second kind*.

Recall [?] that the *q-Stirling numbers of second kind* $\left\{ {n \atop k} \right\}_q$ are defined by the recurrence:

$$\left\{ {n \atop k} \right\}_q = \left\{ {n-1 \atop k-1} \right\}_q + [k] \left\{ {n-1 \atop k} \right\}_q \qquad \text{for} \quad n \geq k \geq 1, \tag{4}$$

where $\left\{ {n \atop 0} \right\}_q = \left\{ {0 \atop k} \right\}_q = 0$ except $\left\{ {0 \atop 0} \right\}_q = 1$.

Theorem 1. *Let $(a_n)_{n \geq 0}$ be a sequence in a commutative ring. If we define the array $(a_{n,m})_{m,n \geq 0}$ by $a_{0,m} = a_m$ for $m \geq 0$ and the recurrence (??), then*

$$a_{n,0} = \sum_{k=0}^{n} (-1)^k [k]! \left\{ {n+1 \atop k+1} \right\}_q a_{0,k} \qquad (n \geq 0), \tag{5}$$

and if we use the recurrence (??) instead of (??), then

$$a_{n,0} = \sum_{k=0}^{n} (-1)^k [k]! \left\{ {n \atop k} \right\}_q a_{0,k} \qquad (n \geq 0). \tag{6}$$

We shall give the first proof of Theorem 1 in Section 2 by using q-differential operator and generating functions, and the second one in Section 4 by applying Theorem 3 in Section 4.

Theorem 2. *We have*

$$\beta_n = \sum_{k=0}^{n} (-1)^k \left\{ {n \atop k} \right\}_q \frac{[k]!}{[k+1]} \qquad (n \geq 1), \tag{7}$$

and

$$\beta_n = \sum_{k=0}^{n} (-1)^k \left\{ {n+1 \atop k+1} \right\}_q \frac{[k]!}{[k+1]} \qquad (n \geq 2). \tag{8}$$

Note that Eq. (**??**) was already given by Carlitz [**?**]. For the sake of completeness we shall include a proof of Theorem 2 in Section 3, which is essentially due to Carlitz [**?**].

The definition of the Akiyama-Tanigawa algorithm is reminiscent of the so-called Euler-Seidel matrix, a term coined by Dumont [**?**]. Recall that the Euler-Seidel matrix associated to a sequence (a_n) is an infinite matrix $(a_{n,m})$ $(n \geq 0, m \geq 0)$ given by the recurrence $a_{n,0} = a_n$ $(n \geq 0)$ and

$$a_{n,m} \quad = \quad a_{n-1,m} + a_{n-1,m+1} \quad (m \geq 0, n \geq 1).$$

The sequence $(a_{0,m})$, first row of the matrix, is called initial sequence, while the sequence $(a_{n,0})$, first column of the matrix, is called the final sequence. Note that the Euler-Seidel matrix may be used as a simple device for computing its initial and final sequences quickly, see Arnold [**?**, **?**] and Dumont [**?**].

In the following theorem we shall unify the Akiyama-Tanigawa type algorithms and the classical Euler-Seidel matrices and prove a general formula about the corresponding coefficients.

Theorem 3. *Let $(x_m), (y_m)$ and (z_m) $(m \geq 0)$ be three sequences in a commutative ring. The generalized Euler-Seidel matrix $(a_{n,m})$ $(n, m \geq 0)$ associated to (x_m) is defined by $a_{0,m} = x_m$ $(m \geq 0)$ and*

$$a_{n,m} = y_m a_{n-1,m} + z_m a_{n-1,m+1} \qquad (m \geq 0,\ n \geq 1). \tag{9}$$

Then

$$a_{n,m} = \sum_{k=0}^{n} x_{m+k} \left(\prod_{j=0}^{k-1} z_{m+j} \right) h_{n-k}(y_m, y_{m+1}, \ldots, y_{m+k}), \tag{10}$$

where $h_n(z_1, \ldots, z_r)$ is the n-th complete symmetric function of $z_1, \ldots, z_r$ defined by

$$\sum_{n \geq 0} h_n(z_1, \ldots, z_r) t^n = \frac{1}{(1 - z_1 t)(1 - z_2 t) \cdots (1 - z_r t)}.$$

In particular, we have

$$a_{n,0} = \sum_{k=0}^{n} x_k \left(\prod_{j=0}^{k-1} z_j \right) h_{n-k}(y_0, y_1, \ldots, y_k). \tag{11}$$

We will prove Theorem 3 and give some applications in Section 4. In particular, we shall derive an alternative proof of Theorem 1.

2. Proof of Theorem 1

For any formal power series $f(x)$ denote by D_q the q-derivative operator:

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

We will need the two associated operators:

$$\Delta_q = 1 - (1 - xq)D_q,$$
$$\delta_q = (x-1)D_q.$$

For $n \geq 0$ define $(x;q)_0 = 1$ and $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$. The following formulas are easy to prove once discovered.

Lemma 1. *For $n \geq 1$ we have*

$$\Delta_q^n = \sum_{k=0}^{n} (-1)^k \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\}_q (xq;q)_k D_q^k, \qquad (12)$$

$$\delta_q^n = \sum_{k=0}^{n} (-1)^k \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q (x;q)_k D_q^k. \qquad (13)$$

Proof. We proceed by induction on $n \geq 1$. It is easy to check (**??**) and (**??**) for $n = 1$. For example we have

$$\Delta_q = 1 - (1-xq)D_q = \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\}_q - \left\{ \begin{matrix} 2 \\ 2 \end{matrix} \right\}_q (1-xq)D_q = 1 - (1-xq)D_q.$$

Note that $D_q(x^n) = [n]x^{n-1}$ and

$$D_q\left((xq;q)_n\right) = -q[n]_q(xq^2;q)_{n-1},$$
$$D_q\left((x;q)_n\right) = -[n]_q(xq;q)_{n-1}.$$

Suppose the formulas are true for n. Then, using the rule

$$D_q(f(x)g(x)) = f(x)D_q(g(x)) + g(qx)D_q(f(x)),$$

and the induction hypothesis we have

$$\begin{aligned}
\Delta_q^{n+1} &= (1-(1-xq)D_q)) \sum_{k=0}^{n} (-1)^k \left\{ {n+1 \atop k+1} \right\}_q (xq;q)_k D_q^k \\
&= \sum_{k=0}^{n} (-1)^k \left\{ {n+1 \atop k+1} \right\}_q [k+1]_q (xq;q)_k D_q^k \\
&\quad - \sum_{k=0}^{n} (-1)^k \left\{ {n+1 \atop k+1} \right\}_q (xq;q)_{k+1} D_q^{k+1} \\
&= \left\{ {n+1 \atop 1} \right\}_q + \sum_{k=1}^{n} (-1)^k \left([k+1] \left\{ {n+1 \atop k+1} \right\}_q + \left\{ {n+1 \atop k} \right\}_q \right) (xq;q)_k D_q^k \\
&\quad +(-1)^{n+1} \left\{ {n+1 \atop n+1} \right\}_q (xq;q)_{n+1} D_q^{n+1} \\
&= \sum_{k=1}^{n+1} (-1)^k \left\{ {n+2 \atop k+1} \right\}_q (xq;q)_k D_q^k,
\end{aligned}$$

and

$$\begin{aligned}
\delta_q^{n+1} &= (x-1)D_q \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q (x-1)(xq-1)\dots(xq^{k-1}-1) D_q^k \\
&= \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q [k](x-1)(xq-1)\dots(xq^{k-1}-1) D_q^k \\
&\quad +(x-1) \sum_{k=1}^{n} \left\{ {n \atop k} \right\}_q (xq-1)(xq^2-1)\dots(xq^k-1) D_q^{k+1} \\
&= \sum_{k=0}^{n+1} \left\{ {n+1 \atop k} \right\}_q (x-1)(xq-1)\dots(xq^{k-1}-1) D_q^k.
\end{aligned}$$

This completes the proof. □

Remark: A q-analogue of $(x\frac{d}{dx})^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k \left(\frac{d}{dx}\right)^k$ is the following formula:

$$(xD_q)^n = \sum_{k=0}^{n} q^{\binom{k}{2}} \left\{ {n \atop k} \right\}_q x^k D_q^k,$$

which can be verified easily by induction on n.

We are now ready to prove Theorem 1. For fixed $n \geq 0$, consider the generating function of $a_{n,k}$ $(k \geq 0)$ defined by (**??**):

$$\begin{aligned} g_n(x) = \sum_{k=0}^{\infty} a_{n,k}x^k &= \sum_{k=0}^{\infty}[k+1](a_{n-1,k} - a_{n-1,k+1})x^k \\ &= D_q\left(\sum_{k=0}^{\infty} a_{n-1,k}x^{k+1}\right) - D_q\left(\sum_{k=0}^{\infty} a_{n-1,k+1}x^{k+1}\right) \\ &= D_q(xg_{n-1}(x)) - D_q(g_{n-1}(x) - a_{n-1,0}) \\ &= g_{n-1}(x) + (xq-1)D_q(g_{n-1}(x)) \\ &= \Delta_q(g_{n-1}(x)). \end{aligned}$$

By iteration $g_n(x) = \Delta_q^n(g_0(x))$ and Lemma 1 implies that

$$g_n(x) = \sum_{k=0}^{n}(-1)^k \left\{ {n+1 \atop k+1} \right\}_q (1-xq)(1-xq^2)\dots(1-xq^k)D_q^k(g_0(x)).$$

Putting $x = 0$ in the above equation yields (**??**).

Similarly, we consider the generating function of $a_{n,k}$ $(k \geq 0)$ defined by (**??**):

$$\begin{aligned} h_n(x) = \sum_{k=0}^{\infty} a_{n,k}x^k &= \sum_{k=0}^{\infty}([k]a_{n-1,k} - [k+1]a_{n-1,k+1})x^k \\ &= xD_q\left(\sum_{k=0}^{\infty} a_{n-1,k}x^{k}\right) - D_q\left(\sum_{k=0}^{\infty} a_{n-1,k+1}x^{k+1}\right) \\ &= xD_q(h_{n-1}(x)) - D_q(h_{n-1}(x) - a_{n-1,0}) \\ &= \delta_q(h_{n-1}(x)). \end{aligned}$$

Thus $h_n(x) = \delta_q^n(h_0(x))$ and Lemma 1 implies that

$$h_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q (x-1)(xq-1)\dots(xq^{k-1}-1)D_q^k(h_0(x)).$$

Putting $x = 0$ in the last equation yields (**??**).

Now, if we take $a_{0,m} = 1/[m+1]$ in Theorem 1 and apply algorithm (**??**), then it follows from (**??**) and (**??**) that $a_{n,0} = \beta_n$ for $n \geq 2$; while applying algorithm (**??**) will yield that $a_{n,0} = \beta_n$ for $n \geq 1$ by (**??**) and (**??**).

3. Proof of Theorem 2

Let $[x] = (q^x - 1)/(q-1)$. For integer $s \geq 0$ define $[x]_s = [x][x-1]\cdots[x-s+1]$ and the q-binomial coefficient $\left[{x \atop s} \right] = [x]_s/[s]!$.

Let $\eta_0, \eta_1, \eta_2, \ldots$ be a sequence such that by $\eta_0 = 1$, $\eta_1 = 0$ and

$$\sum_{i=0}^{m} \binom{m}{i} q^i \eta_i = \eta_m \qquad (m > 1). \tag{14}$$

We define the polynomials $\eta_m(x)$ $(m \geq 0)$ in q^x by

$$\eta_m(x) = \sum_{i=0}^{m} \binom{m}{i} \eta_i [x]^{m-i} q^{ix}. \tag{15}$$

Then $\eta_m(0) = \eta_m$ and

$$\eta_m(x+y) = \sum_{i=0}^{m} \binom{m}{i} \eta_i(y) [x]^{m-i} q^{ix}. \tag{16}$$

Indeed, substituting $\eta_i(y)$ by (**??**) and exchanging the order of summations in the right-hand side of (**??**) yields

$$\begin{aligned}
\sum_{j=0}^{m} \sum_{i=j}^{m} \binom{m}{i} \binom{i}{j} [x]^{m-i} [y]^{i-j} q^{ix+jy} &= \sum_{j=0}^{m} \binom{m}{j} \eta_j q^{j(x+y)} \sum_{i=0}^{m-j} \binom{m-j}{i} [x]^{m-j-i} (q^x [y])^i \\
&= \sum_{j=0}^{m} \binom{m}{j} \eta_j q^{j(x+y)} \left([x] + q^x [y]\right)^{m-j} \\
&= \sum_{j=0}^{m} \binom{m}{j} \eta_j [x+y]^{m-j} q^{j(x+y)},
\end{aligned}$$

which is equal to $\eta_m(x+y)$ by (**??**).

Setting $x = 1$ in (**??**) we see that $\eta_m(1) = \eta_m$ for $m > 1$. It follows from (**??**) with $y = 1$ that for $m \geq 0$,

$$\eta_m(x+1) - \eta_m(x) = \sum_{i=0}^{m} \binom{m}{i} (\eta_i(1) - \eta_i) [x]^{m-i} q^{ix} = m q^x [x]^{m-1}. \tag{17}$$

It follows that for $k \geq 1$,

$$\begin{aligned}
\sum_{i=0}^{k-1} q^i [i]^m &= \frac{1}{m+1} \sum_{i=0}^{k-1} (\eta_m(i+1) - \eta_m(i)) \\
&= \frac{1}{m+1} (\eta_{m+1}(k) - \eta_{m+1}) \\
&= \frac{1}{m+1} \sum_{i=1}^{m+1} \binom{m+1}{i} [k]^i q^{(m+1-i)k} \eta_{m+1-i} + (q^{(m+1)k} - 1) \frac{\eta_{m+1}}{m+1}.
\end{aligned} \tag{18}$$

On the other hand, it is readily seen by induction on $n \geq 1$ that

$$x^n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}_q x(x-[1])\ldots(x-[k-1]), \tag{19}$$

which, by substitution $x \to [x]$, yields

$$[x]^n = \sum_{k=1}^{n} q^{k(k-1)/2} \left\{ {n \atop k} \right\}_q [x]_k.$$

Therefore

$$\sum_{i=0}^{k-1} q^i [i]^m = \sum_{i=0}^{k-1} q^i \sum_{s=1}^{m} q^{s(s-1)/2} \left\{ {m \atop s} \right\}_q [i]_s = \sum_{s=0}^{m} q^{s(s+1)/2} \left\{ {m \atop s} \right\}_q \frac{[k]_{s+1}}{[s+1]}, \tag{20}$$

where we used the identity for q-binomial coefficients:

$$\sum_{i=s}^{k-1} q^{i-s} \begin{bmatrix} i \\ s \end{bmatrix} = \begin{bmatrix} k \\ s+1 \end{bmatrix}.$$

Combining (**??**) and (**??**) we obtain a polynomial identity on q^k. Dividing both sides by $[k]$ and setting $k = 0$ leads to

$$\eta_m + (q-1)\eta_{m+1} = \sum_{s=0}^{m} q^{s(s+1)/2} \left\{ {n \atop s} \right\}_q \frac{[-1]_s}{[s+1]}.$$

Now, it remains to prove $\beta_m = \eta_m + (q-1)\eta_{m+1}$. Indeed, the sequence $\eta_n + (q-1)\eta_{n+1}$ satisfies the recurrence (**??**) for $n > 1$:

$$\begin{aligned} q\sum_{i=0}^{n} \binom{n}{i} q^i(\eta_i + (q-1)\eta_{i+1}) &= q\sum_{i=0}^{n} \binom{n}{i} q^i \eta_i + (q-1)\sum_{i=1}^{n+1} \binom{n}{i-1} q^i \eta_i \\ &= q\eta_n + (q-1)\sum_{i=1}^{n+1} \left(\binom{n+1}{i} - \binom{n}{i} \right) q^i \eta_i \\ &= q\eta_n + (q-1)\left(\eta_{n+1} - \eta_n\right) \\ &= \eta_n + (q-1)\eta_{n+1}. \end{aligned}$$

This completes the proof of (**??**).

For $n \geq 2$, simplifying x in (**??**) and setting $x = 0$ we get

$$0 = \sum_{m=1}^{n} (-1)^m \left\{ {n \atop m} \right\}_q [m-1]!. \tag{21}$$

Now, using (**??**) we have

$$\sum_{k=0}^{n}(-1)^k \left\{ {n+1 \atop k+1} \right\}_q \frac{[k]!}{[k+1]} = \sum_{k=0}^{n}(-1)^k \left(\left\{ {n \atop k} \right\}_q + [k+1] \left\{ {n \atop k+1} \right\}_q \right) \frac{[k]!}{[k+1]}$$
$$= \sum_{k=0}^{n}(-1)^k \left\{ {n \atop k} \right\}_q \frac{[k]!}{[k+1]} + \sum_{k=0}^{n}(-1)^k \left\{ {n \atop k+1} \right\}_q [k]!.$$

Formula (**??**) follows then from (**??**) and (**??**).

4. Proof of Theorem 3 and applications

By induction on $n \geq 0$. The formula is clear for $n = 0$ and $n = 1$. Suppose that the formula is true until $n \geq 1$. Then

$$\begin{aligned} a_{n+1,m} &= y_m a_{n,m} + z_m a_{n,m+1} \\ &= y_m \sum_{k=0}^{n} x_{m+k} z_m \dots z_{m+k-1} h_{n-k}(y_m, \dots, y_{m+k}) \\ &+ z_m \sum_{k=0}^{n} x_{m+k+1} z_{m+1} \dots z_{m+k} h_{n-k}(y_{m+1}, \dots, y_{m+k+1}) \\ &= x_m y_m^{n+1} + \sum_{k=1}^{n} x_{m+k} z_m \dots z_{m+k-1}(y_m h_{n-k}(y_m, \dots, y_{m+k}) + h_{n-k+1}(y_{m+1}, \dots, y_{m+k})) \\ &+ x_{m+n+1} z_m \dots z_{m+n}. \end{aligned}$$

Since $y_m h_{n-k}(y_m, \dots, y_{m+k}) + h_{n-k+1}(y_{m+1}, \dots, y_{m+k}) = h_{n+k-1}(y_m, \dots, y_{m+k})$, we are done.

The following examples are special cases of Theorem 3:

- if $y_m = z_m = 1$, we recover the so-called Euler-Seidel matrix (see [**?**]) associated to the initial sequence x_m $(m \geq 0)$.
- if $z_m = xq^m$ and $y_m = 1$, then we recover the q-Seidel matrix introduced by Clarke, Han and Zeng [**?**].
- if $z_m = -y_m$, then Theorem 3 reduces to a result of Lascoux [**?**].
- if $y_m = -z_m = [m+1]$ and $x_m = 1/[m+1]$, then this is our q-analogue of the Akiyama-Tanigawa algorithm. Indeed, it is readily seen that

$$\sum_{n \geq k} \left\{ {n \atop k} \right\}_q t^n = \frac{t^k}{(1-[1]t)(1-[2]t)\dots(1-[k]t)},$$

which yields the explicit formula: $\left\{ {n \atop k} \right\}_q = h_{n-k}([1], \dots, [k])$, so

$$\left\{ {n+1 \atop k+1} \right\}_q = h_{n-k}([1], \dots, [k+1]).$$

Eq. (**??**) follows then from (**??**) with the above specializations.

- if $y_m = [m]$ and $z_m = -[m+1]$ and $x_m = 1/[m+1]$, this is our q-analogue of Chen's algorithm and (**??**) follows directly from (**??**).

It may be worth pointing out that it is possible to write explicitly the general coefficients $a_{n,m}$ in Theorem 1, because

$$h_{n-k}([m], \dots, [m+k]) = \sum_{i=0}^{k} (-1)^{k-i} q^{-k(m+i)+\binom{i+1}{2}} \frac{[m+i]^n}{[i]![k-i]!}. \tag{22}$$

Indeed, there holds

$$\frac{1}{(1-z_0t)(1-z_1t)\dots(1-z_kt)} = \sum_{i=0}^{k} \frac{\prod_{j=0, j\neq i}^{k} (1-z_j/z_i)^{-1}}{1-z_it}.$$

Equating the coefficients of t^n $(n \geq 0)$ in the two sides yields

$$h_n(z_0, \dots, z_k) = \sum_{i=0}^{k} \prod_{j=0, j\neq i}^{k} (1-z_j/z_i)^{-1} \, z_i^n,$$

which gives (**??**) by taking $z_i = [m+i]$ for $i = 0, \dots, k$.

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