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FACTORIZATIONS AND PAIGE'S THEOREM ON COMPLETE MAPS

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Abstract

A theorem of L. J. Paige on complete maps is proved using factorizations of abelian groups.

Let G be a finite abelian group written multiplicatively with identity element e. Let $a_1, \ldots, a_n$ be all the elements of G. A permutation $b_1, \ldots, b_n$ of the elements of G is called a complete permutation of G if $a_1b_1, \ldots, a_nb_n$ is also a permutation of the elements of G. In other words a function $f : G \to G$ is called a complete map of G if f is one-to-one and if the function $g : G \to G$ defined by $g(a) = af(a)$, $a \in G$ is also one-to-one. In 1947 L. J. Paige has proved the following result.

Theorem 1. *If a finite abelian group G does not have exactly one element of order two, then G possess a complete map.*

For extensions of Paige's theorem see [1], [2] and for an application to geometry see [4], [5]. Let $A_1, \ldots, A_n$ be subsets of G. If each element a of G is uniquely expressible in the form

$$a = a_1 \cdots a_n, \quad a_1 \in A_1, \ldots, a_n \in A_n,$$

then we say that the equation $G = A_1 \cdots A_n$ is a factorization of G. In this note we give a new proof for Paige's theorem using factorizations. If a finite abelian group G is a direct product of cyclic groups of orders $t_1, \ldots, t_n$ respectively, then we say that G is of type $(t_1, \ldots, t_n)$. The order of an element a of G is denoted by $|a|$ and $\langle a \rangle$ stands for the span of a.

Proof. We divide the proof into smaller steps.

(1) A group of type $(2, 2, 2)$ has a complete map.

In order to prove this claim Let G be a group of type $(2, 2, 2)$ with basis elements x, y, z, where $|x| = |y| = |z| = 2$. Table 1 shows that G has a complete map.

(2) A group of type $(2n, 2)$, where $n \geq 1$ has a complete map.

$a:$	e	x	y	xy	z	xz	yz	xyz
$f(a):$	e	z	xz	x	xyz	xy	y	yz
$af(a):$	e	xz	xyz	y	xy	yz	z	x

Table 1

In order to prove the claim let G be a group of type $(2n, 2)$, $n \geq 1$. Let x, y be basis elements of G such that $|x| = 2n$, $|y| = 2$. Set

$$H = \langle x \rangle, \quad K = \langle y \rangle, \quad M = \langle x^2 \rangle, \quad N = \langle x^n \rangle,$$
$$C = \{e, x, x^2, \ldots, x^{n-1}\}.$$

We use Table 2 to show that G has a complete map.

$a:$	x^{2k}	$x^{2k}y$	x^{2k+1}	$x^{2k+1}y$
$f(a):$	x^{n-k}	$x^{2n-k}y$	$x^{n-k}y$	x^{2n-k}
$af(a):$	x^{n+k}	x^k	$x^{n+k+1}y$	$x^{k+1}y$

Table 2

As k runs from 0 to $n-1$, the elements in the first row run over the elements of the sets M, My, Mx, Mxy respectively. Note that

$$\begin{aligned} G &= HK \\ &= M\{e, x\}\{e, y\} \\ &= M\{e, x, y, xy\} \end{aligned}$$

are factorizations of G. It follows that the sets M, My, Mx, Mxy form a partition of G. Thus a runs over the elements of G. Similarly, the factorizations

$$\begin{aligned} G &= HK \\ &= xHK \\ &= xCNK \\ &= xC\{e, x^n\}\{e, y\} \end{aligned}$$

give that the sets Cx, $Cx^{n+1}y$, Cxy, Cx^{n+1} form a partition of G. Therefore $f(a)$ runs over the elements of G. Finally, the equations

$$\begin{aligned} G &= HK \\ &= H\{e, y\} \\ &= H \cup Hy \\ &= H \cup Hxy \\ &= H\{e, xy\} \\ &= CN\{e, xy\} \\ &= C\{e, x^n\}\{e, xy\} \end{aligned}$$

show that the sets Cx^n, C, $Cx^{n+1}y$, Cxy form a partition of G. It follows that $af(a)$ runs over the elements of G. Therefore G has a complete map.

(3) Let G be a finite abelian group and let H be a subgroup of G. If both H and the factor group G/H have a complete map, then so does G. In particular, if G is the direct product of the groups H and K such that both H and K have complete maps, then so does G.

To prove the claim assume that $h_1, \ldots, h_r$ are all the elements of H and $k_1, \ldots, k_r$ is a complete permutation of H, that is, $h_1k_1, \ldots, h_rk_r$ are all the elements of H. Then assume that $a_1H, \ldots, a_sH$ are all the elements of G/H and $b_1H, \ldots, b_sH$ is a complete permutation of G/H. This means that $a_1b_1H, \ldots, a_sb_sH$ is a rearrangement of the elements of G/H. It follows that these cosets are disjoint and their union is equal to G, that is, $G = \{a_1b_1, \ldots, a_sb_s\}H$ is a factorization of G. In other words

$$\begin{matrix} a_1b_1h_1k_1, & \ldots & , a_1b_1h_rk_r \\ \vdots & \ddots & \vdots \\ a_sb_sh_1k_1, & \ldots & , a_sb_sh_rk_r \end{matrix}$$

are all the elements of G. Therefore G has a complete map.

(4) A group of type $(2^{\alpha(1)}, 2^{\alpha(2)})$, where $\alpha(1) \geq \alpha(2) \geq 1$ has a complete map.

To prove the claim let G be a group of type $(2^{\alpha(1)}, 2^{\alpha(2)})$, with $\alpha(1) \geq \alpha(2) \geq 1$. If $\alpha(1) = 1$, then by step (2), G has a complete map. So we may assume that $\alpha(2) \geq 2$ and start an induction on $\alpha(2)$. Now G has a subgroup H of type $(2, 2)$ such that the factor group G/H is of type $(2^{\alpha(1)-1}, 2^{\alpha(2)-1})$. By step (2), H has a complete map. By the inductive assumption G/H has a complete map. Therefore by step (3), G has a complete map.

(5) A non-cyclic group of type $(2, \ldots, 2)$ has a complete map.

In order to prove this assertion let G be a group of type $(2, \ldots, 2)$, where the number of 2's is n and $n \geq 2$. First let us deal with the case when n is even. The $n = 2$ case has already been settled in step (2). So we may assume that $n \geq 4$. As G is a direct product of subgroups of types $(2, 2), \ldots, (2, 2)$, one can use step (3) repeatedly to show that G has a complete map. Let us turn to the case when n is odd. The $n = 3$ case has already been settled in step (1). We may assume that $n \geq 5$. Now G is a direct product of groups of types $(2, 2, 2), (2, 2), \ldots, (2, 2)$ and we can use step (3) to show that G has a complete map.

(6) A group of type $(2^{\alpha(1)}, \ldots, 2^{\alpha(n)})$, where $n \geq 3$ and $\alpha(1) \geq \cdots \geq \alpha(n) \geq 1$ has a complete map.

In order to verify the claim consider a group G of type $(2^{\alpha(1)}, \ldots, 2^{\alpha(n)})$, where $n \geq 3$ and $\alpha(1) \geq \cdots \geq \alpha(n) \geq 1$. Set $t = \alpha(1) + \cdots + \alpha(n)$. If $\alpha(1) = 1$, that is, if $t = n$, then by step (5) we are done. We may assume that $\alpha(1) \geq 2$, that is, $t \geq n + 1$ and start an induction on t. Clearly G has a subgroup H of type $(2,2)$ such that the factor group G/H is of type $(2^{\alpha(1)-1}, 2^{\alpha(2)-1}, 2^{\alpha(3)}, \ldots, 2^{\alpha(n)})$ or $(2^{\alpha(1)-1}, 2^{\alpha(3)}, \ldots, 2^{\alpha(n)})$ depending on whether $\alpha(2) \geq 2$ or $\alpha(2) = 1$. By step (2), H has a complete map. By the inductive assumption G/H has a complete map. Finally by step (3), G has a complete map.

(7) A finite abelian group of odd order has a complete map.

Indeed, the map $f : G \to G$ defined by $f(a) = a$, $a \in G$ is suitable.

(8) We are ready to finish the proof. Let G be a finite abelian group such that G does not have exactly one element of order two. The group G can be written uniquely as a direct product of the groups H and K such that the order of H is odd and the order of K is a power of 2. Since G does not have exactly one element of order two, K is not a cyclic group, that is, K is not of type (2^α). Therefore, by steps (4), (5), (6), K has a complete map. By step (7), H has a complete map. Hence by step (3), G has a complete map.

This completes the proof.

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