INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 7 (2007), #A40

AN INVARIANT SUM RELATED TO RECORD STATISTICS

Tamás Lengyel
Department of Mathematics, Occidental College, Los Angeles, California
lengyel@oxy.edu

Received: 1/13/07, Revised: 8/8/07, Accepted: 8/13/07, Published: 8/31/07

Abstract

We use four different methods involving recurrence relations for polynomials, orthogonal polynomials, symbolic summations and generating functions to determine a sum that originates in a calculation involving record statistics.

1. Introduction

In [1] the following quantities were considered for $m \geq 1$

$$R_m(x) = \sum_{k=m+1}^{\infty} \frac{m}{k(k-1)} \sum_{j=0}^{m} \binom{k}{j} x^{k-j}(1-x)^j, 0 \leq x \leq 1.$$

It was proven that $R_m(x)$ does not depend on m and, in fact,

$$R_m(x) = x \text{ for every } m \geq 1. \tag{1}$$

The background comes from the theory of records. Let $X_1, X_2, \dots$ be an infinite sequence of independent identically distributed random variables with some continuous distribution function, and let $N(m)$ be the random variable defined so that $X_{N(m)}, m \geq 0$, is the first variable that is less than exactly m of all its predecessors. Clearly, $X_{N(0)} = X_1$. We note that $X_{N(1)}$ is referred to as the first *near-record* or 2-record, and for a general $m \geq 0$, as the first $m+1$-record. For any c, we set $x = P(X_1 \leq c)$. Notice that for $k > m \geq 1$,

$$P(X_{N(m)} \leq c \mid X_{N(m)} = X_k) = \sum_{j=0}^{m} \binom{k}{j} x^{k-j}(1-x)^j$$

by using the binomial model to count the indices i so that $X_i \leq c$. We also have

$$P(X_{N(m)} = X_k) = \left(\prod_{j=1}^{k-m-1} (1 - \frac{1}{m+j}) \right) \frac{1}{k} = \frac{m}{k(k-1)},$$

since the probability that X_k is less than exactly m of its predecessors is $1/k$, [1]. It follows that $P(X_{N(m)} \leq c) = R_m(x) = x$, and perhaps surprisingly, $X_{N(m)}$ and $X_i, i \geq 1$, have the same distribution. (Of course, $X_{N(0)} = X_1$, so the statement is true for all first $m+1$-records, $m \geq 0$.) In fact, more can be said as $X_{N(m)}, m = 0, 1, 2, \ldots$, are independent and identically distributed random variables and even the *sets* of $m+1$-records, formed by the $m+1$-records in the sequence $X_i, i = 1, 2, \ldots$, are independent and identically distributed random sets (cf. [3]).

After some experimentation we noticed that the sum $R_m(x)$ could be generalized. Here we extend identity (1) for the modified sum, $a_{m,n}(x)$, given any integers m and n with $0 \leq n-1 \leq m$, and set

$$a_{m,n}(x) = \binom{m}{n-1} \sum_{k=m+1}^{\infty} \frac{1}{\binom{k}{n}} \sum_{j=0}^{m} \binom{k}{j} x^{k-j}(1-x)^j. \tag{2}$$

Often, for notational convenience, we will simply leave out the argument x and write, for example, $a_{m,n}$ instead of $a_{m,n}(x)$. We prove that $a_{m,n}(x)$ does not depend on m if $m \geq n-1$.

Theorem 1. For integers $n > 0$ and $m \geq n-1$, we have

$$a_{m,n}(x) = \begin{cases} \frac{n}{n-1}\big(1-(1-x)^{n-1}\big), & \text{if } n \geq 2 \text{ and } 0 \leq x \leq 1 \\ -\ln(1-x), & \text{if } n = 1 \text{ and } 0 \leq x < 1. \end{cases}$$

Remark. Note that $R_m(x) = a_{m,2}(x)/2 = x$ by [1].

In Section 2, we present a recurrence based proof. Section 3 offers an alternative proof via differentiation. Section 4 is devoted mainly to a proof of the theorem by symbolic summations provided by the *Mathematica* package *MultiSum* [9], with alternative summation methods mentioned including the more traditional approaches based on the manipulation of binomial sums and hypergeometric series. The last section is based on the use of generating functions. We note that the methods presented or mentioned in Sections 3, 4.3, and 5 do not require the a priori knowledge of the suggested closed form of $a_{m,n}(x)$.

2. Proof by Recurrence Relations

We will eventually need a basic fact. It is well known that for $n \geq 2$

$$\sum_{k \geq n} \frac{1}{\binom{k}{n}} = \frac{n}{n-1}.$$

(In fact, it is a WZ companion identity of the Vandermonde identity [7].) We can easily generalize this and obtain

Lemma 2. For any integer T with $2 \leq n \leq T$, we have

$$\binom{T}{n} \sum_{k \geq T} \frac{1}{\binom{k}{n}} = \frac{T}{n-1}.$$

Proof. By using hypergeometric series, it is easy to see that

$$\sum_{k \geq T} \frac{1}{\binom{k}{n}} = \frac{1}{\binom{T}{n}} \, {}_2F_1(T-n+1, 1; T+1; 1) = \frac{1}{\binom{T}{n}} \frac{T}{n-1}$$

by Gauss' identity as $n \geq 2$. □

Proof of Theorem 1. In fact, by accounting for the changes in (2) as we move from $a_{m-1,n}$ to $a_{m,n}$, we get the recurrence

$$\begin{aligned} a_{m,n} &= \tfrac{m}{m-n+1} a_{m-1,n} - \tfrac{n}{m-n+1} + \tfrac{n}{m-n+1}(1-x)^{n-1} \\ &= \tfrac{m}{m-n+1} a_{m-1,n} - \tfrac{n}{m-n+1}\left(1-(1-x)^{n-1}\right) \end{aligned} \tag{3}$$

for $m > n-1 \geq 0$. The second term on the right-hand side of the first identity removes the extra terms with $k = m$, and the third term adds the new terms due to $j = m$.

The case $n = 1$ follows immediately. First observe that if $m = n-1 = 0$ then we have $a_{0,1} = \sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x)$ by (2), and for $n = 1$, equation (3) implies that $a_{m,1} = a_{m-1,1}$ for all $m \geq 1$.

A similar direct use of (3) fails for $n \geq 2$ since we do not readily have the proof of Theorem 1 for any particular $a_{m,n}(x), m \geq n-1$. (The case $n = 2$ is proven in [1] but here we look for a self contained proof.) If $n \geq 2$ then we set

$$b_{m,n}(x) = a_{m,n}(x) - \frac{n}{n-1}\left(1-(1-x)^{n-1}\right) \tag{4}$$

and prove that

$$b_{m,n} = \binom{m}{n-1} b_{n-1,n}. \tag{5}$$

To see this, we observe that (3) can be rewritten as

$$\begin{aligned} b_{m,n} + \tfrac{n}{n-1}\left(1-(1-x)^{n-1}\right) = \; & \tfrac{m}{m-n+1}\left(b_{m-1,n} + \tfrac{n}{n-1}\left(1-(1-x)^{n-1}\right)\right) \\ & - \tfrac{n}{m-n+1}\left(1-(1-x)^{n-1}\right) \end{aligned}$$

which can be simplified to

$$b_{m,n} = \frac{m}{m-n+1} b_{m-1,n},$$

and thus implies equation (5). Now we use an argument on the asymptotic order of magnitude of $b_{m,n}$. If $m \to \infty$ then $b_{m,n} \sim \frac{m^{n-1}}{(n-1)!} b_{n-1,n}$, and thus either $b_{m,n} = 0$ for all $m \geq n-1$ or $b_{m,n} \to \infty$.

We will see that the former case applies here. In view of the equation (2), for all x with $0 \leq x \leq 1$ we have

$$\begin{aligned} |b_{m,n}(x)| &= \left| a_{m,n}(x) - \tfrac{n}{n-1}\big(1-(1-x)^{n-1}\big) \right| \leq \tbinom{m}{n-1} \textstyle\sum_{k=m+1}^{\infty} \frac{1}{\binom{k}{n}} \;+\; \frac{n}{n-1} \\ &= \tfrac{n}{m+1}\tbinom{m+1}{n} \textstyle\sum_{k=m+1}^{\infty} \frac{1}{\binom{k}{n}} \;+\; \frac{n}{n-1} = \frac{n}{m+1}\frac{m+1}{n-1} + \frac{n}{n-1} \leq \frac{2n}{n-1} \end{aligned}$$

by Lemma 2. This prevents $b_{m,n}(x)$ from growing indefinitely as $m \to \infty$. Since $b_{m,n}(x)$ is zero if $1 \leq n-1 \leq m$ and $0 \leq x \leq 1$, we have $a_{m,n}(x) = \frac{n}{n-1}\big(1-(1-x)^{n-1}\big)$. □

3. Proof by Differentiation

Proof of Theorem 1. The statement of Theorem 1 is true for $x = 0$, as we get $a_{m,n}(0) = 0$, and it is also true if $n \geq 2$ and $x = 1$, since

$$a_{m,n}(1) = \binom{m}{n-1} \sum_{k=m+1}^{\infty} \frac{1}{\binom{k}{n}} = \frac{n}{n-1}$$

by Lemma 2.

We assume that $0 \leq x < 1$ in the remainder of this paper.

Inspired by the fact that $L_n(x) = 1-(1-x)^n$, $n \geq 1$, satisfies the familiar recurrence

$$x\,L_n'(x) = n\,L_n(x) - n\,L_{n-1}(x)$$

for the Laguerre polynomials (although here $L_1(x) = x$ rather than $1-x$), we take the derivative of $a_{m,n}(x)$ in hopes of finding some recurrence relation that does not contain m anymore. Indeed, the derivative of $a_{m,n}(x)$, $0 \leq n-1 \leq m$, is $n(1-x)^{n-2}$ since

$$\begin{aligned} a_{m,n}'(x) &= \tbinom{m}{n-1} \textstyle\sum_{k=m+1}^{\infty} \frac{k}{\binom{k}{n}} \Big(\sum_{j=0}^{m} \binom{k-1}{j} x^{k-1-j}(1-x)^j \;-\; \sum_{j=1}^{m} \binom{k-1}{j-1} x^{k-j}(1-x)^{j-1} \Big) \\ &= n\tbinom{m}{n-1} \textstyle\sum_{k=m}^{\infty} \frac{\binom{k}{m}}{\binom{k}{n-1}} x^{k-m}(1-x)^m \\ &= n(1-x)^m\, {}_2F_1(1, m-n+2; 1; x) = n(1-x)^{n-2}, \end{aligned}$$

by a standard hypergeometric identity. This implies Theorem 1. □

We note that this seems to be the shortest way to prove Theorem 1.

4. Proof by Symbolic Summations

As in Section 2, we find a recurrence relation for $a_{m,n}(x)$ but this time without any "manual effort." After checking the initial cases with $n = 1$ and 2, we prove the statement inductively for $n \geq 3$.

Here is the sketch of the steps. The *Mathematica* program package *MultiSum* uses the notation $\texttt{SUM}[m, n] = a_{m,n}(x)$. As above, we assume that $m \geq n - 1 \geq 0$. We rewrite the summand in (2) in proper hypergeometric terms [9, Definition 2.1] below. We face a summation with nonstandard boundary conditions [9, Sections 2.7.3 and 3.4; and Chapter 5] such as $k \geq m+1$ and $j \leq m$. We are able to transform this problem into one with standard boundary conditions with the help of the so called "limit argument." In fact, the role of the extra parameter ε in (6) is that the summand vanishes as $\varepsilon \to 0$ for all indices outside the summation region (and the condition $j \geq 0$ is taken care of by the denominator without the limit argument).

4.1. General Case

We execute the following four *Mathematica* commands.

$$\begin{gathered}\texttt{FindRecurrence}[\tfrac{n\,(1-x)^j\,x^{-j+k}\,(-1+\varepsilon+k-m)!\,m!\,(\varepsilon-j+m)!\,(k-n)!}{j!\,(-j+k)!\,(-1+k-m)!\,(-j+m)!\,(1+m-n)!},\\ \{m, n\}, \{0, 0\}, \{k, j\}, \{0, 2\}]\end{gathered} \tag{6}$$

$$(\%//\texttt{SumCertificate})/.\ \varepsilon \to 0$$

$$\texttt{Solve}[\%, \texttt{SUM}[2 + m, 2 + n]]$$

$$\texttt{ShiftRecurrence}[\%, \{m, -2\}, \{n, -1\}]$$

We obtain that

$$\begin{aligned}\texttt{SUM}[m, 1 + n] \to \quad & \tfrac{1+n}{(-1+n)n^2}\big((-1 + m)\ n\ (-1 + x)\ \texttt{SUM}[-2 + m, -1 + n] \\ & - (1 + m - n)\ n\ (-1 + x)\ \texttt{SUM}[-1 + m, -1 + n] \\ & - (-1 + n)\ (1 - m - n - x + nx)\ \texttt{SUM}[-1 + m, n] \\ & + (-1 + n)\ (-1 - m + n)\ \texttt{SUM}[m, n]\big)\end{aligned} \tag{7}$$

The case of $\texttt{SUM}[m,3]$ follows immediately if we substitute $\texttt{SUM}[\_,n\_]$ by $f[n]$ defined as

$$f[n\_] := \texttt{If}[n == 1, -\texttt{Log}[1-x], \frac{n}{n-1}\left(1-(1-x)^{n-1}\right)]$$

For $\texttt{SUM}[m,n+1]$ with $n \geq 3$, we can proceed by the substitution

$$\%/.\texttt{SUM}[\_,n\_] \to \frac{n\left(1-(1-x)^{-1+n}\right)}{-1+n}$$

in (7). In fact, after simplification we get that $\texttt{SUM}[m,n+1]$ is equal to

$$-\frac{(1+n)\left(-1+(1-x)^n\right)}{n}.$$

□

4.2. Initial Cases

We are left with checking the initial cases $n = 1$ and 2. Can we derive these initial cases within the framework of symbolic summation?

In fact, we can. We extend the range of summation for k in the definition (8) to simplify the structure of summation and find a recurrence. The difference between $c_{m,n}(x)$ and $a_{m,n}(x)$ can be easily calculated, without any symbolic manipulation, as it will become apparent in identity (10). We set

$$c_{m,n}(x) = \binom{m}{n-1}\sum_{k=n}^{\infty}\frac{1}{\binom{k}{n}}\sum_{j=0}^{m}\binom{k}{j}x^{k-j}(1-x)^j, \tag{8}$$

with $c_{n-1,n}(x) = a_{n-1,n}(x)$.

The *MultiSum* commands $\texttt{FindRecurrence}[\frac{(1-x)^j\,x^{-j+k}\,(-1+\varepsilon+k)!\,(\varepsilon-j+m)!}{j!\,(-j+k)!\,(-j+m)!}, m, \{k,j\}]$ and $\texttt{FindRecurrence}[\frac{2\,(1-x)^j\,x^{-j+k}\,(-2+\varepsilon+k)!\,m!\,(\varepsilon-j+m)!}{j!\,(-j+k)!\,(-1+m)!\,(-j+m)!}, m, \{k,j\}]$, for $n = 1$ and $n = 2$, respectively find the following recurrences for $c_{m,1}(x)$ and $c_{m,2}(x)$ in the single variable m

$$\texttt{SUM}[m] = \begin{cases} \frac{1}{m}\left((1-m)\texttt{SUM}[-2+m] + (-1+2m)\texttt{SUM}[-1+m]\right), & \text{if } n = 1 \text{ and } m \geq 2; \\ -\texttt{SUM}[-2+m] + 2\texttt{SUM}[-1+m], & \text{if } n = 2 \text{ and } m \geq 3, \end{cases} \tag{9}$$

with the notation $\texttt{SUM}[m] = c_{m,n}(x)$.

We set

$$\begin{aligned} g_{m,n}(x) = c_{m,n}(x) - a_{m,n}(x) &= \tbinom{m}{n-1}\textstyle\sum_{k=n}^{m}\frac{1}{\binom{k}{n}}\sum_{j=0}^{m}\binom{k}{j}x^{i-j}(1-x)^j \\ &= \begin{cases} \sum_{i=1}^{m}\frac{1}{i}, & \text{if } n = 1; \\ \frac{n}{n-1}\left(\binom{m}{n-1} - 1\right), & \text{if } n \geq 2, \end{cases} \end{aligned} \tag{10}$$

by Lemma 2 since $\sum_{j=0}^{m}\binom{k}{j}x^{i-j}(1-x)^j = 1$ as $k \le m$. We note that $g_{m,1}$ is the mth harmonic number while $g_{m,2} = 2(m-1)$, and in general, $g_{m,n}$ is a constant polynomial, so we can drop its argument.

It is easy to check that $c_{0,1}(x) = -\texttt{Log}[1-x]$ and $c_{1,1}(x) = -\texttt{Log}[1-x]+1$, and in general, $c_{m,1} = -\texttt{Log}[1-x] + g_{m,1}, m \ge 0$, satisfies (9). Similarly, $c_{1,2}(x) = 2x$ and $c_{2,2}(x) = 2x+2$, and $c_{m,2} = 2x + g_{m,2}, m \ge 1$, satisfies (9). Now it follows that $a_{m,1}(x) = -\texttt{Log}[1-x]$ and $a_{m,2}(x) = 2x$. $\square$

4.3. Other Options for Symbolic Summation

Paule [5] obtained another "automatic" proof of Theorem 1 using only single summations and the RISC implementation of the Gosper's and Zeilberger's algorithms.

Of course in hindsight, hand calculations can replace the symbolic techniques, if one knows the lucky sequence of manipulations in advance. We try to avoid these calculations although we note that they can be accomplished by repeated and somewhat dull applications of identities for binomial summations [2, identities (5.24) and (5.25) on p. 169] Another non-automatic option is to use the *HYP* package for handling hypergeometric series and identities in a systematic way, e.g., by repeatedly applying the Chu-Vandermonde identity, cf. [4]. Paule [5] and Schneider [8] also obtained recurrence relations for the truncated sum

$$a_{m,n}^{(K)} = \binom{m}{n-1}\sum_{k=m+1}^{K}\frac{1}{\binom{k}{n}}\sum_{j=0}^{m}\binom{k}{j}x^{k-j}(1-x)^j$$

using the package *Sigma*, cf. [6]. We took another approach as we need the modified sum $c_{m,n}(x)$ in order to determine $a_{m,n}(x)$ via the generating function $\sum_{m=n-1}^{\infty} c_{m,n}(x)t^m$ in the next section.

5. Proof by Generating Function

Finally, we use the generating function of the polynomials $c_{m,n}(x)$ defined in (8). More precisely, we define

$$C(x,t,n) = \sum_{m=n-1}^{\infty} c_{m,n}(x)t^m$$

for $-1 < t < 1$ which will guarantee the absolute convergence of our sums for $0 \le x < 1$.

A similar generating function, in the context of DNA matching, was discussed in [10] by using the method of characteristics. In fact, with the notation $Q_m(x,k) = 1-\sum_{j=0}^{m}\binom{k}{j}x^{k-j}(1-$

$x)^j$, $m = 0, 1, \ldots$, Wilf found that

$$\sum_{m=0}^{\infty} Q_m(x,k)t^m = \frac{1-(1-(1-t)(1-x))^k}{1-t}.$$

Below we will need

$$f(x,t,k) = \sum_{m=0}^{\infty}(1-Q_m(x,k))t^m = \frac{(1-(1-t)(1-x))^k}{1-t},$$

which implies that

$$\frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial t^{n-1}} f(x,t,k) = \sum_{m=n-1}^{\infty}\binom{m}{n-1}(1-Q_m(x,k))t^{m-(n-1)}. \tag{11}$$

On the other hand,

$$\begin{aligned}\frac{\partial^{n-1}}{\partial t^{n-1}}\frac{(1+(1-t)(x-1))^k}{1-t} &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left(\frac{1}{1-t}+\sum_{i=1}^{k}\binom{k}{i}(1-t)^{i-1}(x-1)^i\right)\\ &= \frac{1}{(1-t)^n}\left((n-1)!+(-1)^{n-1}(n-1)!\sum_{i=n}^{k}\binom{k}{i}\binom{i-1}{n-1}\left((1-t)(x-1)\right)^i\right).\end{aligned} \tag{12}$$

By changing the order of summation below, we derive that

$$\begin{aligned}C(x,t,n) &= \sum_{m=n-1}^{\infty}\left(\binom{m}{n-1}\sum_{k=n}^{\infty}\frac{1}{\binom{k}{n}}\sum_{j=0}^{m}\binom{k}{j}x^{k-j}(1-x)^j\right)t^m\\ &= \sum_{m=n-1}^{\infty}\binom{m}{n-1}\sum_{k=n}^{\infty}\frac{1}{\binom{k}{n}}(1-Q_m(x,k))t^m\\ &= \sum_{k=n}^{\infty}\frac{1}{\binom{k}{n}}\sum_{m=n-1}^{\infty}\binom{m}{n-1}(1-Q_m(x,k))t^m\\ &= \frac{t^{n-1}}{(n-1)!}\sum_{k=n}^{\infty}\frac{1}{\binom{k}{n}}\frac{\partial^{n-1}}{\partial t^{n-1}}f(x,t,k)\end{aligned} \tag{13}$$

by (11). Observe that if $n = 1$ then

$$C(x,t,1) = \sum_{k=1}^{\infty}\frac{(1-(1-t)(1-x))^k}{k(1-t)} = -\frac{\ln((1-t)(1-x))}{1-t} = \sum_{m=0}^{\infty}\left(-\ln(1-x)+\sum_{i=1}^{m}\frac{1}{i}\right)t^m,$$

and we are done according to (10).

From now on we assume that $n \geq 2$. By another change in the order of summation, we can rewrite the last sum in (13) by (12), without the first term $\frac{t^{n-1}}{(1-t)^n}$, and we get

$$\begin{aligned}\sum_{k=n}^{\infty}\frac{1}{\binom{k}{n}}\left(\sum_{i=n}^{k}\binom{k}{i}\binom{i-1}{n-1}\left((1-t)(x-1)\right)^i\right) &= \sum_{k=n}^{\infty}\sum_{i=n}^{k}\frac{\binom{k}{i}\binom{i}{n}\frac{n}{i}}{\binom{k}{n}}\left((1-t)(x-1)\right)^i\\ &= \sum_{k=n}^{\infty}\sum_{i=n}^{k}\frac{n}{i}\frac{\binom{k}{n}\binom{k-n}{i-n}}{\binom{k}{n}}\left((1-t)(x-1)\right)^i\\ &= n\sum_{k=n}^{\infty}\sum_{i=n}^{k}\frac{1}{i}\binom{k-n}{i-n}\left((1-t)(x-1)\right)^i.\end{aligned} \tag{14}$$

To evaluate the last expression, we set

$$h_k(a) = \sum_{i=n}^{k} \binom{k-n}{i-n} \frac{1}{i} a^i.$$

Observe that we have

$$h_k'(a) = a^{n-1} \sum_{i=n}^{k} \binom{k-n}{i-n} a^{i-n} = a^{n-1}(1+a)^{k-n},$$

and thus, for $-2 < a < 0$,

$$\sum_{k=n}^{\infty} h_k'(a) = a^{n-1} \frac{1}{1-(1+a)} = -a^{n-2}.$$

The result also holds for $a = 0$. Clearly, $h_k(0) = 0$ for $k \geq n$; thus, we can rewrite (14) as $n \sum_{k=n}^{\infty} h_k(a) = -\frac{n}{n-1} a^{n-1}$ by integration.

As $-1 < t < 1$ and $0 \leq x < 1$, for $a = (1-t)(x-1)$ we have $-2 < a < 0$. Putting back the first term and using Lemma 2, we obtain that

$$C(x,t,n) = \frac{n}{n-1} \frac{t^{n-1}}{(1-t)^n} \left(1 - ((1-t)(1-x))^{n-1}\right) = \frac{n}{n-1} \frac{t^{n-1}}{(1-t)^n} - \frac{n}{n-1} \frac{t^{n-1}(1-x)^{n-1}}{1-t}.$$

By expanding this according to the powers of t, we get that $c_{m,n}(x) = \frac{n}{n-1} \left(\binom{m}{n-1} - 1 \right) + \frac{n}{n-1} \left(1 - (1-x)^{n-1}\right) = g_{m,n}(x) + a_{m,n}(x)$, for $m \geq n-1$, and simply apply (10). $\square$

Acknowledgments. I wish to thank Peter Paule and Carsten Schneider for sharing their results regarding the original and the truncated sums, Gregory Tollisen and the referee for careful reading of the manuscript and suggestions.

References

[1] G. Blom, Problem 6522, *Amer. Math. Monthly* **93:485**, Solution by Isreal, R. B., Persistence of a distribution, in **95** (1988), 360-362, 1986.

[2] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete Mathematics*, Addison-Wesley Publishing Company, Reading, MA, 1994.

[3] Z. Ignatov, Point processes generated by order statistics and their applications, in *Point Processes and Queuing Problems (Colloq., Keszthely, 1978)*, vol. 24 of *Colloq. Math. Soc. János Bolyai*, 109-116, North-Holland, Amsterdam, 1981.

[4] C. Krattenthaler, personal communication, 2006. The software is available at `http://www.mat.univie.ac.at/~kratt/hyp_hypq/hyp.html`.

[5] P. Paule, personal communication, 2006. The software is available at http://www.risc.uni-linz.ac.at/research/combinat/software.

[6] P. Paule and C. Schneider, Truncating binomial series with symbolic summation, manuscript, 2006.

[7] M. Petkovsek, H. S. Wilf, and D. Zeilberger, *A=B*, A. K. Peters, Wellesley, MA, 1996.

[8] C. Schneider, personal communication, 2006. The software will be available at http://www.risc.uni-linz.ac.at/research/combinat/software.

[9] K. Wegschaider, Computer generated proofs of binomial multi-sum Kepler University, May 1997. It is available at http://apache.risc.uni-linz.ac.at/internals/ActivityDB/publications/download/risc_2245/diplom.ps. The software is available at http://www.risc.uni-linz.ac.at/research/combinat/software/MultiSum.

[10] H. S. Wilf, The method of characteristics, and 'problem 89' of Graham, Knuth, and Patashnik, available at http://www.math.upenn.edu/∼wilf/website/GKPProblem89.pdf, 2005.