ON THE PROPERTY P_{-1}

TAO Liqun
Department of Mathematics, Nanjing University, Nanjing 210093, P. R. China
and
Department of Mathematics, University College Dublin, Belfield, Dublin 4, Ireland
lqtao99@tom.com

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Abstract

Mohanty and Ramasamy recently proved an interesting result that says that no other integers can be added to the set $\{1, 5, 10\}$ such that the product of any two numbers from the new set minus one is a perfect square. We give an alternative proof of this result by considering two simultaneous diophantine equations which are equivalent to those considered in [1]. It turns out our method avoids carefully investigating relations between the solutions of Pell's equations. What we do is just solve some simple diophantine equations.

In [1], S. P. Mohanty and A. M .S. Ramasamy defined an interesting concept. Given an integer k, they say two integers α and β have the property P_k if $\alpha\beta + k$ is a perfect square. They found that $1, 5, 10$ share the property P_{-1}, and showed that no other integers can be added to share the property with the three numbers.

Suppose n is another number satisfying the property. Then

$$n - 1 = x^2 \qquad (1)$$
$$5n - 1 = y^2 \qquad (2)$$
$$10n - 1 = z^2. \qquad (3)$$

Together, these yield

$$5x^2 + 4z^2 = 9y^2. \qquad (4)$$

Eliminating n between (2) and (3), we have

$$z^2 - 2y^2 = 1. \qquad (5)$$

We shall consider in the sequel the nonnegative integer solutions of the simultaneous diophantine equations (4) and (5).

Remark. The authors of [1] considered $5x^2 - y^2 = -4$ and (5) by eliminating n from (1), (2), (3). It seems that these simultaneous equations are a little complicated to treat.

From (5), we see z and y are relatively prime, and z is odd, hence y is even since $z^2 \equiv 1($ mod 8). From (4), we have

$$(3y - 2z)(3y + 2z) = 5x^2. \tag{6}$$

Let $d = (3x - 2y, 3x + 2y)$ be the greatest common divisor. Then $d \mid (6y, 4z)$. So we have the following two cases depending on whether z is divisible by 3 or not: (1) if $3 \nmid z$, then $d = 2$ or 4; (2) if $3 \mid z$, then $d = 6$ or 12.

Now from (6), we have $\frac{3y+2z}{d} \cdot \frac{3y-2z}{d} = 5\left(\frac{x}{d}\right)^2$. So

$$\begin{cases} \frac{3y+2z}{d} = 5s^2 \\ \frac{3y-2z}{d} = t^2 \end{cases} \tag{7}$$

or

$$\begin{cases} \frac{3y+2z}{d} = t^2 \\ \frac{3y-2z}{d} = 5s^2 \end{cases}, \tag{8}$$

where $\frac{x}{d} = st$.

From (7) and (8), we have

$$\pm 4z = d(5s^2 - t^2). \tag{9}$$

So when $d = 2$ or 6, we see $s \equiv t($ mod 2), therefore $\pm 2z = \frac{d}{2}(5s^2 - t^2) \equiv 0($ mod 4). So $2 \mid z$, which is impossible. Therefore $d = 4$ or 12.

Also from (7) and (8), we have

$$6y = d(5s^2 + t^2). \tag{10}$$

Substituting (9) and (10) into (5) yields $9d^2(5s^2 - t^2)^2 - 8d^2(5s^2 + t^2)^2 = 144$ or, equivalently,

$$25s^4 - 170s^2t^2 + t^4 = \frac{144}{d^2}, \tag{11}$$

i.e.,

$$(5s^2 - 17t^2)^2 - 288t^4 = \frac{144}{d^2}. \tag{12}$$

Now let

$$w = \frac{d}{12}(5s^2 - 17t^2). \tag{13}$$

Then we get

$$w^2 - 2d^2t^4 = 1 \tag{14}$$

So, when $d = 4$, we have

$$w^2 - 32t^4 = 1, \tag{15}$$

where $w = \frac{1}{3}(5s^2 - 17t^2)$; while, when $d = 12$, we have

$$w^2 - 288t^4 = 1, \tag{16}$$

where $w = 5s^2 - 17t^2$.

Note. We must be aware that w above is not necessarily nonnegative.

Before solving equations (15) and (16), we recall the following well-known facts (see Mordell [2], pp. 18, 207).

Lemma 1. The equation $x^4 - 2y^4 = 1$ has only one nonnegative integer solution $(x, y) = (1, 0)$.

Lemma 2. The equation $2x^4 - y^4 = 1$ has only one positive integer solution $(x, y) = (1, 1)$.

Now we start to give a complete answer to equations (15) and (16).

Theorem 1. $w^2 - 32t^4 = 1$ has only one nonnegative integer solution $(w, t) = (1, 0)$.

Remark. So, from (15), we have $5s^2 = 3$, which is impossible. Therefore, when $d = 4$, there is no integer n satisfying (1), (2), and (3) simultaneously.

Proof. From $w^2 - 32t^4 = 1$, we have $\frac{w+1}{2}\frac{w-1}{2} = 8t^4$. Hence

$$\left\{ \begin{array}{l} \frac{w+1}{2} = 8u^4 \\ \frac{w-1}{2} = v^4 \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \frac{w+1}{2} = v^4 \\ \frac{w-1}{2} = 8u^4 \end{array} \right. .$$

So $8u^4 - v^4 = \pm 1$. Since v is odd, hence $v^4 + 1 \equiv 2(\bmod 8)$, we see $8u^4 - v^4 = 1$ is impossible. Now $8u^4 - v^4 = -1$, so $\frac{v^2+1}{2}\frac{v^2-1}{2} = 2u^4$. Since $v^2 \equiv 1(\bmod 8)$, $\frac{v^2-1}{2}$ is even, and we obtain

$$\left\{ \begin{array}{l} \frac{v^2+1}{2} = \alpha^4 \\ \frac{v^2-1}{2} = 2\beta^4 \end{array} \right. ,$$

where $u = \alpha\beta$.

Thus, $\alpha^4 - 2\beta^4 = 1$. By Lemma 1, $\beta = 0$, so that $u = t = 0$. Hence, $(w, t) = (1, 0)$ is the only nonnegative integer solution of (15).

Theorem 2. $w^2 - 288t^4 = 1$ has two nonnegative integer solutions: $(w, t) = (1, 0)$ and $(17, 1)$.

Remark. So from (16) and (13), we get only one nonnegative solution $(s,t) = (0,1)$ with $(w,t) = (-17,1)$. Then from (9) and (10), we see $y = 2, z = 3$, hence $n = 1$ in (1).

Proof. From $w^2 - 288t^4 = 1$, we have $\frac{w+1}{2}\frac{w-1}{2} = 72t^4$. Then

$$\begin{cases} \frac{w+1}{2} = 8u^4 \\ \frac{w-1}{2} = 9v^4 \end{cases} \quad\text{or}\quad \begin{cases} \frac{w+1}{2} = 9v^4 \\ \frac{w-1}{2} = 8u^4 \end{cases} \quad\text{or}\quad \begin{cases} \frac{w+1}{2} = 72u^4 \\ \frac{w-1}{2} = v^4 \end{cases} \quad\text{or}\quad \begin{cases} \frac{w+1}{2} = v^4 \\ \frac{w-1}{2} = 72u^4 \end{cases},$$

where $uv = t$.

So $8u^4 - 9v^4 = \pm 1$ or $72u^4 - v^4 = \pm 1$. Since v is odd, hence $v^4 + 1 \equiv 2 (\bmod 8)$, we see $8u^4 - 9v^4 = 1$ and $72u^4 - v^4 = 1$ are impossible. We solve the remaining two equations $8u^4 - 9v^4 = -1$ or $72u^4 - v^4 = -1$ in the following separately.

First we consider the equation $8u^4 - 9v^4 = -1$. Since $\frac{3v^2+1}{2}\frac{3v^2-1}{2} = 2u^4$, observing that $\frac{3v^2+1}{2}$ is even, we have $\begin{cases} \frac{3v^2+1}{2} = 2\alpha^4 \\ \frac{3v^2-1}{2} = \beta^4 \end{cases}$. So $2\alpha^4 - \beta^4 = 1$. By Lemma 2, we get $(\alpha, \beta) = (1,1)$. Thus $(u,v) = (1,1)$. Hence $(w,t) = (17,1)$.

Now we consider the nonnegative integer solutions of the equation $72u^4 - v^4 = -1$. Suppose $uv \neq 0$. Then $u, v > 0$. So $v > 1$. Let (u,v) be the solution of the equation such that u (hence v, since $72u^4 + 1 = v^4$) is the smallest positive integer. Since $\frac{v^2+1}{2}\frac{v^2-1}{2} = 18u^4$, noticing that $v^2 \equiv 1 (\bmod 24)$ (since $v^4 = 72u^4 + 1 \equiv 1 (\bmod 6)$, v is prime to 6), we have

$$\begin{cases} \frac{v^2+1}{2} = \alpha^4 \\ \frac{v^2-1}{2} = 18\beta^4 \end{cases}, \tag{17}$$

where $\alpha, \beta > 0$.

So $\alpha^4 - 18\beta^4 = 1$. Then from $\frac{\alpha^2+1}{2}\frac{\alpha^2-1}{2} = 72(\frac{\beta}{2})^4$, we have

$$\begin{cases} \frac{\alpha^2+1}{2} = \gamma^4 \\ \frac{\alpha^2-1}{2} = 72\delta^4 \end{cases} \tag{18}$$

Therefore $\gamma^4 - 72\delta^4 = 1$. But since $v > 1$, from (17) we have $\alpha > 1$, $v^2 = 2\alpha^4 - 1 > \alpha^4$, hence $v > \alpha > 1$. Similarly from (18), we get $\alpha > \gamma > 1$. So $v > \gamma$, and $\delta, \gamma > 0$. This contradicts the minimality of (u,v). So $uv = 0$. Thus $(u,v) = (0,1)$. Hence $(w,t) = (1,0)$. This completes the proof of the theorem.

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References

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[2] L. J. Mordell, Diophantine Equations, Academic Press, London, 1969.