INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 7 (2007), #A58

FLOOR AND ROOF FUNCTION ANALOGS OF THE BELL NUMBERS

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Received: 9/1/07, Revised: 11/12/07, Accepted: 12/6/07, Published: 12/20/07

Abstract

Define $\{f(n)\}_{n=1}^{\infty}$, the *floor sequence*, by the linear recurrence

$$f(n+1) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor f(k), \qquad f(1) = 1.$$

Similarly, define $\{g(n)\}_{n=1}^{\infty}$, the *roof sequence*, by the linear recurrence

$$g(n+1) = \sum_{k=1}^{n} \left\lceil \frac{n}{k} \right\rceil g(k), \qquad g(1) = 1.$$

This paper studies various properties of these two sequences, including prime criteria, asymptotic approximations of $\left\{\frac{f(n+1)}{f(n)}\right\}_{n=1}^{\infty}$ and $\left\{\frac{g(n+1)}{g(n)}\right\}_{n=1}^{\infty}$, and the iteration coefficients associated with $f(n+r)$ and $g(n+r)$, for any $r \geq 1$.

1. Introduction

The Bell numbers may be defined by the linear recurrence

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k), \tag{1.1}$$

with the initial condition that $B(0) = 1$. These numbers, $1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, ...$ have been extensively studied in [2]. They may also be defined by the exponential generating function

$$\sum_{n=0}^{\infty} B(n)\frac{t^n}{n!} = \exp(e^t - 1). \tag{1.2}$$

In [7], we studied the following generalization of Equation (1.1). Let $\{h(n)\}_{n=0}^{\infty}$ be the sequence defined by the linear difference equation

$$h(n+1) = \sum_{k=0}^{n} \binom{n}{k} h(k), \tag{1.3}$$

where $h(0) = a$ and $h(1) = b$. Note, if $h(0) = 1$ and $h(1) = 1$, Equation (1.3) becomes (1.1). Through successive iterations of (1.3), we were able to show that

$$h(n+r) = \sum_{k=1}^{n} h(k) \sum_{j=0}^{r-1} A_j^r(n) \binom{n+j}{k}, \qquad r \geq 1, \qquad n \geq 1, \tag{1.4}$$

where the $A_j^r(n)$ are polynomials in n that satisfy

$$A_j^{r+1}(n) = \sum_{i=0}^{r-j-1} \binom{n+r}{i} A_j^{r-i}(n). \tag{1.5}$$

One reason the $A_j^r(n)$ are important is that they provide a new partition of the Bell numbers which is reminiscent of the formula

$$B(n) = \sum_{k=0}^{n} S(n,k). \tag{1.6}$$

In Equation (1.6), $S(n,k)$ is the appropriate Stirling number of the second kind.
In particular, we have shown that [7]

$$B(n) = \sum_{j=0}^{n-1} A_j^n(0) \tag{1.7}$$

$$B(n) = A_0^{n+1}(-1). \tag{1.8}$$

When analyzing the proof of Equation (1.4), we realized that the $\binom{n}{k}$ in (1.3) can be replaced by $A(n,k)$, where $A(n,k)$ is an arbitrary function of n [3]. Because $\binom{n}{k}$ is closely related to $\lfloor \frac{n}{k} \rfloor$ by the relation [1]

$$\binom{n}{k} \equiv \left\lfloor \frac{n}{k} \right\rfloor \qquad mod\ k, \tag{1.9}$$

we thought it would be natural to let $A(n,k) = \lfloor \frac{n}{k} \rfloor$.

Thus, we decided to study two particular functions. The first function, a floor function analog of (1.1) , is defined by

$$f(n+1) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor f(k) = \sum_{j=1}^{n} \sum_{d|j} f(d), \tag{1.10}$$

with the initial condition of $f(1) = 1$. The $f(n)$ so generated, $1, 1, 3, 7, 16, 33, 71, 143, 295, 594,$ $1206, 2413, 4871, 9743, 19559, ...$, evidently have not been studied in the literature and were not found in the Online Encyclopedia of Integer Sequences (OEIS). We will call the $\{f(n)\}_{n=1}^{\infty}$ the *floor sequence*.

We also define a roof function analog of (1.1), namely

$$g(n+1) = \sum_{k=1}^{n} \left\lceil \frac{n}{k} \right\rceil g(k), \tag{1.11}$$

with the initial condition that $g(1) = 1$. Recall that $\left\lceil \frac{n}{k} \right\rceil$ denotes the least integer greater than or equal to $\frac{n}{k}$. The $g(n)$ so generated by $1, 1, 3, 8, 20, 50, 121, 297, 716, 1739, 4198, 10157, ...$ behave somewhat differently than the $f(n)$. They also were not found in the OEIS. We call $\{g(n)\}_{n=1}^{\infty}$ the *roof sequence*. Because $\lceil x \rceil = -\lfloor -x \rfloor$, there are interesting relationships between floor and roof.
Note that given positive integers n and k, it is easy to verify that

$$\left\lceil \frac{n}{k} \right\rceil = \left\lfloor \frac{n+k-1}{k} \right\rfloor = \left\lfloor \frac{n-1}{k} \right\rfloor + 1. \tag{1.12}$$

Equation (1.12) allows us to form an alternative recurrence formula for the roof sequence, namely

$$g(n+1) = \sum_{k=1}^{n} g(k) + \sum_{k=1}^{n-1} \left\lfloor \frac{n-1}{k} \right\rfloor g(k), \qquad n \geq 2. \tag{1.13}$$

If we adopt the convention that the second sum on the right is vacuous when $n = 1$, Relation (1.13) is true for $n \geq 1$.

This paper has four main sections. In Section 2, we prove prime criteria for the floor sequence and the roof sequence. These criteria are reminiscent of the prime criteria discussed in [4] and [5]. In Section 3, we discuss the asymptotic nature of $f(n)$ and $g(n)$. In Section 4, we analyze the ordinary generating functions associated with $f(n)$ and $g(n)$. Finally, in Section 5, we give formulas that relate $f(n+r)$ and $g(n+r)$ back to $f(n)$ and $g(n)$. These iteration formulas are similar to Equations (1.3) and (1.4).

2. Prime Number Criteria for the Floor and Roof Sequences

For the floor sequence $f(n)$ defined by (1.10), we state a useful prime number criterion. The proof of this criterion uses the following lemma.

Lemma 2.1 *Let $f(n)$ be as defined by (1.10). Then, for $n \geq 2$,*

$$f(n+1) - f(n) = \sum_{d|n} f(d) \tag{2.1}$$

Proof of Lemma 2.1. We have

$$\begin{aligned} f(n+1) - f(n) &= \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor f(k) - \sum_{k=1}^{n-1} \left\lfloor \frac{n-1}{k} \right\rfloor f(k) \\ &= \sum_{k=1}^{n} \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor \right) f(k) \\ &= \sum_{d|n} f(d). \end{aligned}$$

The last equality follows because

$$\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor = \begin{cases} 1, & \text{if } k|n; \\ 0, & \text{if } k \nmid n. \end{cases}$$

□

Remark 2.1 *Lemma 2.1 is an alternative recurrence that shows how to calculate $f(n+1)$ from $f(n)$.*

Theorem 2.1 (Prime number criterion for the floor sequence) *Let $f(n)$ be the function defined by Equation (1.10). Then n is prime if and only if*

$$f(n+1) = 2f(n) + 1. \tag{2.2}$$

Proof of Theorem 2.1. When n is prime, the only divisors of n are itself and 1. Thus, Equation (2.1) implies

$$f(n+1) - f(n) = \sum_{d|n} f(d) = f(1) + f(n),$$

which is simply Equation (2.2) restated.

On the other hand, if n is not prime then at least one of the positive summands on the right side of Equation (2.1), other than $d = 1$ and $d = n$, is non-zero. This means that Equation (2.2) could not hold. □

Remark 2.2 *Relations (2.1) and (2.2) show that the sequence defined by $f(n)$ increases somewhat faster than 2^{n-1} for $n \geq 6$.*

We now turn to the roof sequence, namely $g(n)$ defined by (1.11), and find the corresponding versions of Lemma 2.1 and Theorem 2.2.

Lemma 2.2 *Let $g(n)$ be as defined by (1.11). Then, for $n \geq 2$,*

$$g(n+1) = 2g(n) + \sum_{d|(n-1)} g(d) \tag{2.3}$$

Proof of Lemma 2.2. We have

$$\begin{aligned} g(n+1) - g(n) &= \sum_{k=1}^{n} \left\lceil \frac{n}{k} \right\rceil g(k) - \sum_{k=1}^{n-1} \left\lceil \frac{n-1}{k} \right\rceil g(k) \\ &= g(n) + \sum_{k=1}^{n-1} \left(\left\lceil \frac{n}{k} \right\rceil - \left\lceil \frac{n-1}{k} \right\rceil \right) g(k) \\ &= g(n) + \sum_{d|(n-1)} g(d) \end{aligned}$$

The last equality follows because

$$\left\lceil \frac{n}{k} \right\rceil - \left\lceil \frac{n-1}{k} \right\rceil = \begin{cases} 1, & \text{if } k|(n-1); \\ 0, & \text{if } k \nmid (n-1). \end{cases}$$

□

We no longer detect a simple prime criterion for the roof sequence. We shall be content with just the following theorem, whose proof follows directly from Lemma 2.2.

Theorem 2.2 *(Prime number criterion for roof sequence) Let $g(n)$ be the function defined by Equation (1.11). Then $n-1$ is prime if and only if*

$$g(n+1) = 2g(n) + g(n-1) + 1. \tag{2.4}$$

Remark 2.3 *We see from Relations (2.3) and (2.4) that $g(n)$ increases considerably faster than 2^{n-1} for $n \geq 5$.*

3. Growth Estimates for $f(n)$ and $g(n)$ Using a Third Sequence

We define a fraction analog $h(n)$ of (1.10) and (1.11) by

$$h(n+1) = \sum_{k=1}^{n} \frac{n}{k} h(k), \tag{3.1}$$

with the initial condition that $h(1) = 1$. Since we are working with all positive numbers, then

$$\left\lfloor \frac{n}{k} \right\rfloor \leq \frac{n}{k} \leq \left\lceil \frac{n}{k} \right\rceil, \tag{3.2}$$

The first few values of the h sequence are $1, 1, 3, 7.5, 17.5, 39.375, 86.625, 187.8675$, and 402.1875.

Theorem 3.1 (Recurrence relation for $h(n)$) *For all $n \geq 2$,*

$$h(n+1) = \frac{2n-1}{n-1} h(n). \tag{3.3}$$

Proof of Theorem 3.1. By (3.1), we have

$$\begin{aligned} h(n+1) = \sum_{k=1}^{n} \frac{n}{k} \cdot h(k) &= h(n) + \sum_{k=1}^{n-1} \frac{n}{k} h(k) \\ &= h(n) + \frac{n}{n-1} \sum_{k=1}^{n-1} \frac{n-1}{k} h(k) \\ &= h(n) + \frac{n}{n-1} h(n) \\ &= \frac{2n-1}{n-1} h(n), \end{aligned}$$

which is precisely (3.3). □

From (3.3), we have immediately obtain $\frac{h(n+1)}{h(n)} = \frac{2n-1}{n-1}$, giving us the following result.

Theorem 3.2 (Limit of $\frac{h(n+1)}{h(n)}$) *The sequence $\left\{\frac{h(n+1)}{h(n)}\right\}_{n=3}^{\infty}$ is a decreasing sequence that approaches 2 as $n \to \infty$.*

Applying (3.3) iteratively leads to the explicit formula given in the next theorem.

Theorem 3.3 (Explicit formula for $h(n)$) *For all $n \geq 0$, $h(n+2) = \frac{(2n+2)!}{n!(n+1)!2^{n+1}}$.*

Remark 3.1 *Substituting (3.4) back into (3.1) gives the binomial identity*

$$n + \sum_{k=2}^{n} \frac{n}{k} \frac{(2k-2)!}{(k-2)!(k-1)!2^{k-1}} = \frac{(2n)!}{(n-1)!n!2^n}$$

for $n \geq 2$. This may be restated in binomial coefficient form as

$$\sum_{k=2}^{n} \binom{2k-2}{k} 2^{n+1-k} = \binom{2n}{n} - 2^n \tag{3.4}$$

for $n \geq 2$, and does not appear in [8]. The identity may be proved quickly by induction.

It is a routine calculation with the binomial series to use (3.4) to establish the following generating function result.

Theorem 3.4 (Generating function for $h(n)$)

$$\sum_{n=1}^{\infty} h(n+1)x^n = \frac{x}{(1-2x)^{\frac{3}{2}}}. \tag{3.5}$$

Remark 3.2 *In analogy to (3.1), (3.2), and (3.3), it is not difficult to use the binomial expansion (4.1) to obtain a generating function for $\frac{n}{k}$.*

$$\sum_{n=k}^{\infty} \frac{n}{k} x^{n-k} = \frac{1}{1-x} + \frac{x}{k(1-x)^2}. \tag{3.6}$$

From these results, and numerical tables, we are led to the following result.

Theorem 3.5 (Bounds for ratios of successive terms) *For all $n \geq 4$, the sequences f, h, and g satisfy*

$$f(n) < h(n) < g(n) \tag{3.7}$$

Moreover, for all for $n \geq 4$, we have

$$2 < \frac{f(n+1)}{f(n)} < \frac{h(n+1)}{h(n)} < \frac{g(n+1)}{g(n)}. \tag{3.8}$$

Furthermore, $\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 2$ and $\lim_{n\to\infty} \frac{h(n+1)}{h(n)} = 2$.

Table 2 exhibits values of the ratios in Equation (3.8) for $n = 1, 2, \ldots, 24$.

The following 4 lemmas will prove Theorem 3.5.

Lemma 3.1 *Let $f(n), g(n)$, and $h(n)$ be as previously defined. Then, for $n \geq 4$,*

$$f(n) < h(n) < g(n).$$

Proof of Lemma 3.1. The proof will use mathematical induction. For example, in order to show that $f(n) < h(n)$, we note that $f(4) = 7 < 7.5 = h(4)$. We now assume the induction hypothesis, i.e., for all integer values greater than 4 and less then or equal to n, $f(n) < h(n)$. Thus,

$$\begin{aligned} f(n+1) &= \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor f(k) = nf(1) + \sum_{k=2}^{n-1} \left\lfloor \frac{n}{k} \right\rfloor f(k) + f(n) \\ &\leq nf(1) + \sum_{k=2}^{n-1} \frac{n}{k} f(k) + f(n) \\ &< nh(1) + \sum_{k=2}^{n-1} \frac{n}{k} f(k) + h(n) \\ &< nh(1) + \sum_{k=2}^{n-1} \frac{n}{k} h(k) + h(n) = h(n+1). \end{aligned}$$

The inequality between the first and second lines comes from (3.2), while the other two inequalities are a result of the induction hypothesis. The proof of $h(n) < g(n)$ is similar and will be omitted. □

Lemma 3.2 *For $n \geq 4$, we have $2 < \frac{f(n+1)}{f(n)}$.*

Proof of Lemma 3.2. We want to show $\frac{f(n+1)}{f(n)} > \frac{2n-2}{n-1} = \frac{2n-1}{n-1} - \frac{1}{n-1}$; that is, we want to show

$$f(n+1) - \frac{2n-1}{n-1} f(n) > \frac{-f(n)}{n-1}. \tag{3.9}$$

However, the calculations of Theorem 2.1 imply, for $n \geq 4$, that $f(n+1) - 2f(n) > 0$. Thus,

$$f(n+1) - 2f(n) = f(n+1) + \frac{-2n+2}{n-1} f(n) = f(n+1) + \frac{-2n+1}{n-1} f(n) + \frac{f(n)}{n-1} > 0.$$

The right hand inequality is simply a restatement of (3.10), which proves our claim. □

Lemma 3.3 *For $n \geq 4$, we have $\frac{f(n+1)}{f(n)} < \frac{h(n+1)}{h(n)}$.*

Proof of Lemma 3.3. Table 2 shows, for $4 \le n < 15$, that $\frac{f(n+1)}{f(n)} < \frac{h(n+1)}{h(n)}$. So we now assume $n \ge 15$. By Equation (3.3), it is sufficient to show $\frac{f(n+1)}{f(n)} - \frac{2n-1}{n-1} < 0$, i.e., it is sufficient to show

$$f(n+1) - 2f(n) < \frac{f(n)}{n-1}. \tag{3.10}$$

By Equation (2.1), we know $f(n+1) - 2f(n) = 1 + \sum_{\substack{d|n \\ d\neq 1,n}} f(d)$. Thus, (3.11) becomes

$$(n-1)\left[1 + \sum_{\substack{d|n \\ d\neq 1,n}} f(d)\right] < f(n) = \sum_{k=1}^{n-1} \left\lfloor \frac{n-1}{k} \right\rfloor f(k). \tag{3.11}$$

Since the largest possible divisor in $\left[1 + \sum_{\substack{d|n \\ d\neq 1,n}} f(d)\right]$ is $\left\lfloor \frac{n}{2} \right\rfloor$, we have

$$1 + \sum_{\substack{d|n \\ d\neq 1,n}} f(d) = \sum_{\substack{d|n \\ d\neq n}} f(d) \le f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \sum_{\substack{d|n \\ d\neq n}} 1 \le f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} 1 = \left\lfloor \frac{n}{2} \right\rfloor f\left(\left\lfloor \frac{n}{2} \right\rfloor\right).$$

We claim that

$$(n-1)\left\lfloor \frac{n}{2} \right\rfloor f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) < f(n-1), \qquad n \ge 15 \tag{3.12}$$

The justification for (3.12) is as follows. First, rewrite Equation (3.12) as

$$(n-1)\left\lfloor \frac{n}{2} \right\rfloor < \frac{f(n-1)}{f\left(\left\lfloor \frac{n}{2} \right\rfloor\right)}. \tag{3.13}$$

By Remark 2.2, we know

$$2^{n-1-\left\lfloor \frac{n}{2} \right\rfloor} < \frac{f(n-1)}{f\left(\left\lfloor \frac{n}{2} \right\rfloor\right)}, \qquad n \ge 12. \tag{3.14}$$

Then by a simple induction argument, it is easy to show, for $n \ge 15$, that

$$(n-1)\left\lfloor \frac{n}{2} \right\rfloor < 2^{n-1-\left\lfloor \frac{n}{2} \right\rfloor} < \frac{f(n-1)}{f\left(\left\lfloor \frac{n}{2} \right\rfloor\right)}.$$

By combining the previous calculations, we have, for $n \ge 15$,

$$(n-1)\left[1 + \sum_{\substack{d|n \\ d\neq 1,n}} f(d)\right] \le (n-1)\left\lfloor \frac{n}{2} \right\rfloor f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) < f(n-1) \le \sum_{k=1}^{n-1} \left\lfloor \frac{n-1}{k} \right\rfloor = f(n),$$

which is Equation (3.12). $\square$

Remark 3.3 *In the proof of Lemma 3.3, we showed that* $\{f(n)\}_{n=1}^{\infty}$ *is an increasing sequence.*

Remark 3.4 *Using Theorem 3.2, Lemma 3.2, Lemma 3.3, and The Squeeze Theorem, we* $\frac{f(n+1)}{f(n)}$ *tends to the limit 2 as* n *increases indefinitely.*

Lemma 3.4 *For* $n \geq 4$*, we have*

$$\frac{h(n+1)}{h(n)} < \frac{g(n+1)}{g(n)}.$$

Proof of Lemma 3.4. The result is clearly true when $n = 4$. We now assume $n > 4$. Using Theorem 3.2, it suffices to show that $\frac{g(n+1)}{g(n)} > \frac{2n-1}{n-1}$ i.e., it suffices to show

$$g(n+1) - g(n) > \frac{g(n)}{n-1}. \tag{3.15}$$

By (2.4), we can rewrite (3.15) as

$$(n-1) \sum_{d|(n-1)} g(d) > g(n). \tag{3.16}$$

Clearly,

$$(n-1) \sum_{d|(n-1)} g(d) \geq (n-1)g(n-1). \tag{3.17}$$

Thus, we now want to show

$$(n-1)g(n-1) > g(n). \tag{3.18}$$

In order to prove (3.18), and thus finish the proof of Lemma 3.4, we first note that (3.18) is equivalent to

$$ng(n-1) - g(n-1) > g(n-1) + \sum_{k=1}^{n-2} \left\lceil \frac{n-1}{k} \right\rceil g(k),$$

i.e.,

$$ng(n-1) > 2g(n-1) + \sum_{k=1}^{n-2} \left\lceil \frac{n-1}{k} \right\rceil g(k).$$

Thus, proving (3.18) is equivalent to proving

$$\sum_{k=1}^{n-2} \left\lceil \frac{n-1}{k} \right\rceil g(k) < (n-2) \sum_{k=1}^{n-2} \left\lceil \frac{n-2}{k} \right\rceil g(k). \tag{3.19}$$

A term-by-term comparison of the sums in (3.19) implies that we must show

$$\left\lceil \frac{n-1}{k} \right\rceil < (n-2)\left\lceil \frac{n-2}{k} \right\rceil. \tag{3.20}$$

Note that

$$\frac{n-1}{k} = \frac{n-2}{k} + \frac{1}{k}.$$

Using properties of the floor and the fact that $\lfloor x \rfloor = -\lceil -x \rceil$, it is easy to show that for all real numbers a and b, $\lceil a+b \rceil \leq \lceil a \rceil + \lceil b \rceil$. Thus, the previous line implies, since $1 \leq k \leq n-2$,

$$\left\lceil \frac{n-1}{k} \right\rceil \leq 1 + \left\lceil \frac{n-2}{k} \right\rceil \leq \left\lceil \frac{n-2}{k} \right\rceil + \left\lceil \frac{n-2}{k} \right\rceil < (n-2)\left\lceil \frac{n-2}{k} \right\rceil,$$

which is a restatement of (3.20). □

3.1. Open Questions

By analyzing the ratios in Table 2, we form the following conjectures, whose proofs remain open questions.

Conjecture 3.1 *The sequence* $\left\{\frac{f(n+1)}{f(n)}\right\}_{n=4}^{\infty}$ *alternately increases then decreases to its limit.*

Conjecture 3.2 *The sequence* $\left\{\frac{g(n+1)}{g(n)}\right\}_{n=4}^{\infty}$ *alternately increases then decreases to the limit of* $1+\sqrt{2}$.

A plausibility argument for the limit $\frac{g(n+1)}{g(n)}$ may run as follows. From Equation (2.4), with $n = p$ being a prime, we find

$$\frac{g(p+2)}{g(p+1)} = 2 + \frac{1}{\frac{g(p+1)}{g(p)}} + \frac{1}{g(p+1)}. \tag{3.21}$$

It is easy to show that $g(p)$ is an increasing sequence. Thus, if $\frac{g(n+1)}{g(n)}$ has limit L, we are led to the equation $L = 2 + \frac{1}{L}$, from which we deduce that $L = 1+\sqrt{2}$.

One possible way to compute the limit of $\left\{\frac{g(n+1)}{g(n)}\right\}_{n=4}^{\infty}$ is to use (1.13) to obtain a bounding sequence for $g(n)$. In particular

$$\begin{aligned} g(n+1) &= \sum_{k=1}^{n} \left\lceil \frac{n}{k} \right\rceil g(k) = \sum_{k=1}^{n-1} \left\lfloor \frac{n-1}{k} \right\rfloor g(k) + \sum_{k=1}^{n} g(k) \\ &< \sum_{k=1}^{n-1} \frac{n-1}{k} g(k) + \sum_{k=1}^{n} g(k) = \sum_{k=1}^{n-1} \frac{n-1+k}{k} g(k) + g(n). \end{aligned}$$

We may then define a bounding sequence $M(n)$ as follows:

$$M(n+1) = \sum_{k=1}^{n-1} \frac{n-1+k}{k} M(k) + M(n), \tag{3.22}$$

with initial condition that $M(1) = 1$. Then, $g(n) < M(n)$ for all $n \geq 5$. This sequence has the values $1, 1, 3, 8, 20.5, 51.5, 128, 316.1, \ldots$.

Lemma 3.5 (Limit of $\frac{M(n+1)}{M(n)}$) *If*

$$\lim_{n\to\infty} \frac{1}{M(n+2)} \sum_{k=1}^{n} \frac{M(k)}{k} = 0 \tag{3.23}$$

and

$$\lim_{n\to\infty} \frac{M(n+1)}{M(n)} = L \tag{3.24}$$

exist, then $L = 1 + \sqrt{2}$.

Proof of Lemma 3.5. From (3.23), we have

$$M(n+1) = \sum_{k=1}^{n-1} \frac{n-1}{k} M(k) + \sum_{k=1}^{n} M(k), \tag{3.25}$$

$$M(n) = \sum_{k=1}^{n-2} \frac{n-2}{k} M(k) + \sum_{k=1}^{n-1} M(k). \tag{3.26}$$

Subtracting (3.25) from (3.26), we find

$$M(n+1) - M(n) = \sum_{k=1}^{n-1} \frac{n-1}{k} M(k) - \sum_{k=1}^{n-2} \frac{n-2}{k} M(k) + M(n),$$

which then gives

$$M(n+1) = 2M(n) + M(n-1) + \sum_{k=1}^{n-2} \frac{M(k)}{k}.$$

This then yields the relation

$$\frac{M(n+1)}{M(n)} = 2 + \frac{1}{\frac{M(n)}{M(n-1)}} + \frac{1}{M(n)} \sum_{k=1}^{n-2} \frac{M(k)}{k}. \tag{3.27}$$

Therefore, if we assume the limit (3.24) exists and that $\frac{M(n+1)}{M(n)}$ approaches a limit L, we find that $L = 2 + 1/L$, which gives $L = 1 + \sqrt{2}$. $\square$

Remark 3.5 (The converse of Lemma 3.5) *We find from (3.28) that if $M(n+1)/M(n)$ approaches a limit L, and if the limit in (3.24) is R, then $R = L - 2 - 1/L$, from which we could compute L if we knew R.*

Thus, from Lemma 3.5, we see that the $M(n)$ sequence would be useful if we could show that

$$1 + \sqrt{2} < \frac{g(n+1)}{g(n)} < \frac{M(n+1)}{M(n)}.$$

Then, by the Squeeze Theorem, we would a have a proof of Conjecture 2.

3.2. Asymptotic Tables

Table 1 below gives values for the four sequences. Table 2 gives values of the ratios in (3.9) and $\frac{M(n+1)}{M(n)}$.

n	f	h	g	$M(n)$
1	1	1	1	1
2	1	1	1	1
3	3	3	3	3
4	7	7.5	8	8
5	16	17.5	20	20.5
6	33	39.375	50	51.5
7	71	86.625	121	128
8	143	187.6875	297	316.1
9	295	402.1875	716	777.3833333
10	594	854.6484375	1739	1906.335714
11	1206	1804.257812	4198	4665.036310
12	2413	3788.941406	10157	11397.76581
13	4871	7922.332031	24513	27812.55897
14	9743	16504.85840	59246	67798.969
15	19559	34279.32129	143006	1.651363960×10^5
16	39138	71007.16553	345381	4.019370878×10^5
17	78428	1.467481421×10^5	833792	9.777186817×10^5
18	156857	3.026680431×10^5	2013272	2.377091654×10^6
19	314047	6.231400887×10^5	4860337	5.776740262×10^6
20	628095	1.280899071×10^6	11734717	1.403292331×10^7
21	1256809	2.629213883×10^6	28329772	3.407699867×10^7
22	2513693	5.389888460×10^6	68396030	8.272537140×10^7
23	5028594	1.103643827×10^7	165121957	2.007678384×10^8
24	10057189	2.257453283×10^7	398644144	4.871238593×10^8

Table 1: The Sequences $f(n)$, $h(n)$, $g(n)$, and $M(n)$ for $n = 1, 2, 3, ...24$

n	$\frac{f(n+1)}{f(n)}$	$\frac{h(n+1)}{h(n)}$	$\frac{g(n+1)}{g(n)}$	$\frac{M(n+1)}{M(n)}$
1	1.000000000	1.000000000	1.000000000	1.000000000
2	1.000000000	1.000000000	1.000000000	1.000000000
3	3.000000000	3.000000000	3.000000000	3.000000000
4	2.333333333	2.500000000	2.666666666	2.666666667
5	2.285714286	2.333333333	2.500000000	2.562500000
6	2.062500000	2.250000000	2.500000000	2.512195122
7	2.151515152	2.200000000	2.420000000	2.485436893
8	2.014084507	2.166666666	2.454545455	2.469531250
9	2.062937063	2.142857143	2.410774411	2.459295582
10	2.013559322	2.125000000	2.428770950	2.452246701
11	2.030303030	2.111111111	2.414031052	2.447122128
12	2.000829187	2.100000000	2.419485469	2.443231960
13	2.018648985	2.090909091	2.413409471	2.440176385
14	2.000205297	2.083333333	2.416921633	2.437710571
15	2.007492559	2.076923077	2.413766330	2.435677098
16	2.001022547	2.071428571	2.415150413	2.433970326
17	2.003883694	2.066666667	2.414122375	2.432516708
18	2.000012751	2.062500000	2.414597406	2.431263408
19	2.002122953	2.058823529	2.414148212	2.430171445
20	2.000003184	2.055555556	2.414383406	2.429211402
21	2.000985520	2.052631579	2.414184509	2.428360642
22	2.000059675	2.050000000	2.414280990	2.427601450
23	2.000480568	2.047619048	2.414203821	2.426919759
24	2.000000199	2.045454545	2.414240669	2.426304249
25	2.000258621	2.043478261	2.414208914	2.425745711
26	2.000000845	2.041666667	2.414225288	2.425236570

Table 2: The Ratios $\frac{f(n+1)}{f(n)}, \frac{h(n+1)}{h(n)}, \frac{g(n+1)}{g(n)}, \frac{M(n+1)}{M(n)}$ for $n = 1, 2, 3, ...26$

4. Generating Functions for the Floor and Roof Sequences

Recall that a variation of the binomial theorem ([1]) is

$$\sum_{n=k}^{\infty} \binom{n}{k} x^{n-k} = \frac{1}{(1-x)^{k+1}}. \tag{4.1}$$

The floor function analog of Equation (4.1) ([1]) is

$$\sum_{n=k}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor x^{n-k} = \frac{1}{(1-x)(1-x^k)}, \tag{4.2}$$

while the roof function analog of (4.1) is

$$\sum_{n=k}^{\infty} \left\lceil \frac{n}{k} \right\rceil x^{n-k} = \frac{1+x-x^k}{(1-x)(1-x^k)}. \tag{4.3}$$

Remark 4.1 *The idea of studying the floor sum in Equation (1.10) is not entirely new. Relation (4.2) was used in [6] to study the series transform*

$$G(n) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor F(k), \qquad n \geq 1, \tag{4.4}$$

which has the inverse

$$F(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) [G(d) - G(d-1)], \qquad n \geq 1. \tag{4.5}$$

What is novel now is when, in Equation (4.4), we make $G(n) = F(n)$, Equation (4.5) does not hold.

For the $f(n)$ defined by (1.10), we define the ordinary generating function

$$F(t) = \sum_{n=1}^{\infty} f(n) t^n. \tag{4.6}$$

Using (4.2), we find

$$\begin{aligned} \sum_{n=1}^{\infty} t^n f(n+1) &= \sum_{n=1}^{\infty} t^n \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor f(k) = \sum_{k=1}^{\infty} f(k) t^k \sum_{n=k}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor t^{n-k} \\ &= \frac{1}{1-t} \sum_{k=1}^{\infty} \frac{f(k) t^k}{1-t^k} = \frac{-1}{1-t} \sum_{k=1}^{\infty} f(k) + \frac{1}{1-t} \sum_{k=1}^{\infty} \frac{f(k)}{1-t^k} \\ &= \frac{1}{1-t} \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} f(k) (t^r)^k = \frac{1}{1-t} \sum_{r=1}^{\infty} F(t^r). \end{aligned}$$

On the other hand

$$\sum_{n=1}^{\infty} t^n f(n+1) = \sum_{n=2}^{\infty} t^{n-1} f(n) = \frac{1}{t} \sum_{n=1}^{\infty} t^n f(n) - 1 = \frac{1}{t} F(t) - 1,$$

so that the generating function for the floor sequence must satisfy the functional equation

$$\frac{1}{t} F(t) - 1 = \frac{1}{1-t} \sum_{r=1}^{\infty} F(t^r). \tag{4.7}$$

For the $g(n)$ defined by (1.11), we define the ordinary generating function

$$G(t) = \sum_{n=1}^{\infty} g(n)t^n. \tag{4.8}$$

Equation (4.3) implies

$$\begin{aligned}
\sum_{n=1}^{\infty} t^n g(n+1) &= \sum_{n=1}^{\infty} t^n \sum_{k=1}^{n} \left\lceil \frac{n}{k} \right\rceil g(k) = \sum_{k=1}^{\infty} g(k)t^k \sum_{n=k}^{\infty} \left\lceil \frac{n}{k} \right\rceil t^{n-k} \\
&= \frac{1}{1-t} \sum_{k=1}^{\infty} \frac{t^k + t^{k+1} - t^{2k}}{1-t^k} g(k) \\
&= \frac{1}{1-t} \sum_{k=1}^{\infty} g(k)t^k - \frac{t}{1-t} \sum_{k=1}^{\infty} g(k) + \frac{t}{1-t} \sum_{k=1}^{\infty} \frac{g(k)}{1-t^k} \\
&= \frac{G(t)}{1-t} + \frac{t}{1-t} \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} g(k)(t^r)^k \\
&= \frac{G(t)}{1-t} + \frac{t}{1-t} \sum_{r=1}^{\infty} G(t^r).
\end{aligned}$$

On the other hand,

$$\sum_{n=1}^{\infty} t^n g(n+1) = \sum_{n=2}^{\infty} t^{n-1} g(n) = \frac{1}{t} \sum_{n=1}^{\infty} t^n g(n) - 1 = \frac{1}{t} G(t) - 1.$$

Thus, the generating function for the roof sequence must satisfy

$$\frac{1-2t}{t(1-t)} G(t) - 1 = \frac{t}{1-t} \sum_{r=1}^{\infty} G(t^r). \tag{4.9}$$

5. Expansions Involving $f(n+r)$ and $g(n+r)$

In [3], we studied a class of functions, H, defined by the relationship

$$H(n+1) = \sum_{k=0}^{n} A(n,k)H(k), \tag{5.1}$$

where $A(n,k)$ is an arbitrary function of n. In this situation, we can let $A(n,k) = \left\lfloor \frac{n}{k} \right\rfloor$, $H(0) = 0$, and $H(1) = 1$ or $A(n,k) = \left\lceil \frac{n}{k} \right\rceil$, $H(0) = 0$, and $H(1) = 1$. It is an easy exercise to restate Theorems 2.1 to 2.4 of [3] in the context of the floor function and the roof function. These restatements are recorded below as Corollaries 5.1 to 5.5 respectively.

Corollary 5.1 *Let f be the function defined by Equation (1.10). Let g be the function defined by Equation (1.11). Let r be a positive integer. There exist functions of n, namely $A_j^r(n)$ and $C_j^r(n)$, such that,*

$$f(n+r) = \sum_{k=1}^{n} f(k) \sum_{j=0}^{r-1} A_j^r(n) \left\lfloor \frac{n+j}{k} \right\rfloor, \qquad r \geq 1, \qquad n \geq 1, \tag{5.2}$$

$$g(n+r) = \sum_{k=1}^{n} g(k) \sum_{j=0}^{r-1} C_j^r(n) \left\lceil \frac{n+j}{k} \right\rceil, \qquad r \geq 1, \qquad n \geq 1, \tag{5.3}$$

where $A_j^r(n)$ and $C_j^r(n)$ satisfy the recurrence relations

$$A_j^{r+1}(n) = \sum_{i=0}^{r-j-1} A_j^{r-i}(n) \left\lfloor \frac{n+r}{n+r-i} \right\rfloor, \qquad 0 \leq j \leq r-1, \tag{5.4}$$

$$C_j^{r+1}(n) = \sum_{i=0}^{r-j-1} C_j^{r-i}(n) \left\lceil \frac{n+r}{n+r-i} \right\rceil, \qquad 0 \leq j \leq r-1. \tag{5.5}$$

Note that $A_{r-1}^r(n) = 1$ and $A_j^r(n) = 0$ if $j < 0$ or $j > r-1$. A similar statement holds for the $C_j^r(n)$.

Corollary 5.2 (The Shift Theorem of [3]) *The $A_j^r(n)$ (and $C_j^r(n)$) coefficients satisfy the relation*

$$A_{j+1}^{r+1}(n) = A_j^r(n+1), \qquad j \geq 1, \qquad r \geq 0. \tag{5.6}$$

Corollary 5.3

$$A_{k-1}^r(0) = \sum_{j=k}^{r-1} A_j^r(0) \left\lfloor \frac{j}{k} \right\rfloor, \qquad 0 < k < r-1 \tag{5.7}$$

$$C_{k-1}^r(0) = \sum_{j=k}^{r-1} C_j^r(0) \left\lceil \frac{j}{k} \right\rceil, \qquad 0 < k < r-1. \tag{5.8}$$

Corollary 5.4 (The Inversion Theorem for the Floor Function) *We have*

$$F(r) = \sum_{j=0}^{r-1} A_j^r(0) G(j), \qquad r \geq 1 \tag{5.9}$$

if and only if

$$G(r) = F(r+1) - \sum_{j=1}^{r} \left\lfloor \frac{r}{j} \right\rfloor F(j), \qquad \textit{with} \qquad G(0) = F(1). \tag{5.10}$$

Corollary 5.5 (The Inversion Theorem for the Roof Function) *We have*

$$F(r) = \sum_{j=0}^{r-1} C_j^r(0) G(j), \qquad r \geq 1 \tag{5.11}$$

if and only if

$$G(r) = F(r+1) - \sum_{j=1}^{r} \left\lceil \frac{r}{j} \right\rceil F(j), \qquad \textit{with} \qquad G(0) = F(1). \tag{5.12}$$

Below we provide a table of the $A_j^r(0)$ and a table of the $C_j^r(0)$. Using Equations (5.4) and (5.5) it is easy to show that $f(n) = A_0^n(0)$ and $g(n) = C_0^n(0)$. In other words, note that the left most diagonal of the table is our sequence $f(n)$ or $g(n)$.

1							
1	1						
3	1	1					
7	2	1	1				
16	5	2	1	1			
33	10	4	2	1	1		
71	22	9	4	2	1	1	
143	44	18	8	4	2	1	1

Table 3: Values of $A_j^r(0)$.

1							
1	1						
3	1	1					
8	3	1	1				
20	7	3	1	1			
50	18	7	3	1	1		
121	43	17	7	3	1	1	
297	106	42	17	7	3	1	1

Table 4: Values of $C_j^r(0)$

For Tables 3 and 4, rows correspond to $r = 1, 2, 3, ...$ and diagonals to $j = 0, 1, ..., r-1$.

Inspection of Tables 3 and 4 suggests that as $n \to \infty$, the rows tend to stabilize to a fixed sequence. In the case of Table 3, the fixed sequence is $b_n = 2^n$. Thus, we have the following theorem.

Theorem 5.1 *Let $n \geq 0$. Then, $A_n^{2n+2}(0) = 2^n$.*

For Table 4, the stabilization sequence is more complicated. In particular, we have the following theorem.

Theorem 5.2 *Let a_n be the sequence defined by the recursion $a_n = 2a_{n-1} + a_{n-2}, a_0 = 1, a_1 = 1$. Then, for $n \geq 2, C_{n-2}^{2n-1}(0) = a_n$.*

We will prove Theorem 5.2. Since the proof of Theorem 5.1 is similar, its proof is omitted.

In order to prove Theorem 5.2, we will need the following two lemmas.

Lemma 5.1 *Let $n \geq 2$. Let $i \geq 0$ and $0 \leq k \leq n$. Then,*

$$\left\lceil \frac{2n+i}{2n+i-k} \right\rceil = \left\lceil \frac{2n}{2n-k} \right\rceil \tag{5.13}$$

Proof of Lemma 5.1. Clearly Equation (5.13) is true when $k = 0$. Now we assume $1 \leq k \leq n$, or equivalently, $-1 \geq -k \geq -n$. By adding $2n$ to each term of the inequality, we obtain $2n > 2n-1 \geq 2n-k \geq n$. In other words,

$$\frac{1}{2n} < \frac{1}{2n-1} \leq \frac{1}{2n-k} \leq \frac{1}{n}.$$

Multiply each term in the above inequality by $2n$ to obtain

$$1 < \frac{2n}{2n-1} \leq \frac{2n}{2n-k} \leq 2. \tag{5.14}$$

Thus, Equation (5.14) implies that if $1 \leq k \leq n$, $\lceil \frac{2n}{2n-k} \rceil = 2$.

We now want to obtain the left side of (5.13). Once again, assume $1 \leq k \leq n$, or equivalently. $-1 \geq -k \geq -n$. Add $2n+i$ to each term and obtain $2n+i > 2n+i-1 \geq 2n+i-k \geq n+i$. Thus, $\frac{1}{2n+i} < \frac{1}{2n+i-1} \leq \frac{1}{2n+i-k} \leq \frac{1}{n+i}$. Multiply each term in the above inequality by $2n+i$ to obtain

$$1 < \frac{2n+i}{2n+i-1} \leq \frac{2n+i}{2n+i-k} \leq \frac{2n+i}{n+i} = 2 - \frac{i}{n+i} \leq 2. \tag{5.15}$$

Thus, Equation (5.15) implies that if $1 \leq k \leq n$, $\lceil \frac{2n+i}{2n+i-k} \rceil = 2$. Combining (5.14) and (5.15) proves the lemma. □

Lemma 5.2 (Stabilization of the left to right diagonal) *Let $n \geq 2$ and $i \geq 0$. Then,*

$$C_{n-2}^{2n-1}(0) = C_{n-2+i}^{2n-1+i}(0). \tag{5.16}$$

Proof of Lemma 5.2. We use induction on n. If $n = 2$, it easy to show, via Equation (5.5), that $C_0^3(0) = 3 = C_i^{3+i}(0)$. We now assume (5.16) is true for all integer values less than or equal to n. In otherwords, we assume the first n left to right diagonals of Table 4 stabilize to a fixed number. By Equation (5.5),

$$\begin{aligned} C_{n-1+i}^{2n+1+i}(0) &= \sum_{k=0}^{n} C_{n-1+i}^{2n+i-k}(0) \left\lceil \frac{2n+i}{2n+i-k} \right\rceil \\ &= \sum_{k=0}^{n} C_{n-1}^{2n-k}(0) \left\lceil \frac{2n+i}{2n+i-k} \right\rceil, \qquad \textit{inductive hypothesis} \\ &= \sum_{k=0}^{n} C_{n-1}^{2n-k}(0) \left\lceil \frac{2n}{2n-k} \right\rceil, \qquad \textit{by, Lemma } 4.1 \\ &= C_{n-1}^{2n+1}(0) \qquad \textit{by } (5.5). \end{aligned}$$

The above calculations prove the lemma. □

Remark 5.1 *Lemma 5.2 implies that for $n \geq 4$, and $i \geq 0$, $C_{n-4}^{2n-5}(0) = C_{n-4+i}^{2n-5+i}(0)$.*

Proof of Theorem 5.2. Note that $C_0^3(0) = 3 = a_2$ and $C_1^5(0) = 7 = a_3$. If we can show, for $n \geq 4$, that

$$C_{n-2}^{2n-1}(0) = 2C_{n-3}^{2n-3}(0) + C_{n-4}^{2n-5}(0), \tag{5.17}$$

we will prove the theorem, since both the $C_{n-2}^{2n-1}(0)$ and a_n obey the same recursion relation and have the same initial conditions. By Equation (5.8),

$$C_{n-2}^{2n-1}(0) = \sum_{j=n-1}^{2n-2} C_j^{2n-1}(0) \left\lceil \frac{j}{n-1} \right\rceil = C_{n-1}^{2n-1}(0) + 2C_n^{2n-1}(0) + 2\sum_{j=n+1}^{2n-2} C_j^{2n-1}(0). \tag{5.18}$$

The last equality is a simple adaptation of Lemma 5.1. Also, by (5.8), we have

$$\begin{aligned} C_{n-3}^{2n-3}(0) &= \sum_{j=n-2}^{2n-4} C_j^{2n-3}(0) \left\lceil \frac{j}{n-2} \right\rceil \\ &= C_{n-2}^{2n-3}(0) + 2\sum_{j=n-1}^{2n-4} C_j^{2n-3}(0) \\ &= C_n^{2n-1}(0) + 2\sum_{j=n-1}^{2n-4} C_{j+2}^{2n-1}(0) = C_n^{2n-1}(0) + 2\sum_{j=n+1}^{2n-2} C_j^{2n-1}(0). \end{aligned}$$

The third equality comes from letting $i = 2$ in Remark 5.1. Thus, the preceding calculations imply that (5.18) is, in fact, $C_{n-2}^{2n-1}(0) = C_{n-1}^{2n-1}(0) + C_n^{2n-1}(0) + C_{n-3}^{2n-3}(0)$. By repeated applications of Lemma 5.2, the above equation becomes

$$C_{n-2}^{2n-1}(0) = C_{n-3}^{2n-3}(0) + C_n^{2n-1}(0) + C_{n-3}^{2n-3}(0) = 2C_{n-3}^{2n-3}(0) + C_{n-4}^{2n-5}(0).$$

which is exactly Equation (5.17). □.

Remark 5.2 *Lemma 5.2 implies, that for all* $i \geq 0$, $a_n = C_{n-2}^{2n-1}(0) = C_{n-2+i}^{2n-1+i}(0)$. *Thus, the proof of Theorem 5.2 shows that* $a_n = C_{n-2+i}^{2n-1+i}(0)$, *whenever* $i \geq 0$ *and* $n \geq 2$. *Similiarly, we can show that whenever* $i \geq 0$, $A_{n+i}^{2n+2+i}(0) = 2^n$.

We end this section with Corollary 5.6, which is a result of Theorems 5.1 and 5.2. This corollary states that the limiting sequence is the central column of Tables 3 and 4.

Corollary 5.6 *Let* $n \geq 0$. *Let* a_n *and* b_n *be as defined in Theorems 5.1 and 5.2. Then,* $b_n = A_n^{2n+1}(0)$ *and* $a_n = C_n^{2n+1}(0)$.

6. Closing Remarks: A Combinatorial Question

There remains the problem of determining combinatorial structures enumerated by $f(n)$ and $g(n)$. Although $h(n)$ is fractional, removing the powers of 2 in (3.4) and relabeling, the sequence $a(n) = n\binom{2n}{n}$, which has the values 2, 12, 60, 280, 1260, 5544, 24024, 102960,... does have some interest. This sequence is A005430 in the OEIS and the numbers are sometimes called Apery numbers since they occur in Roger Apery's proof of the irrationality of $\zeta(3)$.

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