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# EXPLICIT FORMULAS FOR BERNOULLI AND EULER NUMBERS

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**Abstract**

Explicit and recursive formulas for Bernoulli and Euler numbers are derived from the Faá di Bruno formula for the higher derivatives of a composite function. Along the way we prove a result about composite generating functions which can be systematically used to derive such identities.

## 1. Introduction and Review of Partitions

In this note we present explicit and recursive formulas for the sequences of Bernoulli and Euler numbers. The approach taken to derive these formulas is based on viewing the generating functions of these sequences as composites of other functions. In order to profit from this point of view, we require an explicit way to relate the coefficients of the powers of $x$ in the composed function to the coefficients of the 'factors'. Since the coefficients in question may be computed by Taylor's theorem, a natural way to phrase our approach is to ask if there is a way to compute the $n^{\text{th}}$ Taylor coefficient of a composite function in terms of the Taylor coefficients of the factors. The answer to this is yes and is remarkably simple to derive from Faá di Bruno's generalization of the chain rule of calculus to higher derivatives (see Corollary 4). Despite the fact that Faá di Bruno first published his formula 1855, we are unaware of any instances in the literature of Faá di Bruno's formula being applied in this particular way.

While we focus in this note on the Bernoulli and Euler numbers, our methods can be used to systematically derive a great many identities of a combinatorial nature, providing new proofs of many known results, as well as, we hope, leading to new identities. For example, the exponential generating function of the Bell numbers, originally obtained by E.T. Bell in the 1930's in [1], can be derived using our approach in a single short paragraph. We explore this and many other results in a future paper.

Since Faá di Bruno's formula is expressed as a sum over the partitions of an integer $n$, we begin with a quick review of partitions to set up our notation. If $n$ is a positive integer, a *partition* $\pi$ of $n$ is a way of writing $n$ as a sum of positive integers: $n = p_1+p_2+...+p_m$, where the order of the summands, called the *parts* of $\pi$, is irrelevant. We could write the partition by simply omitting the addition signs and list the parts as a multiset: $\pi = \{p_1, p_2, ..., p_m\}$, where some of the parts $p_i$ could be repeated in the list. The number of parts of $\pi$ we'll call the *length* of $\pi$, and denote it by $\ell(\pi) = m$. Let $\mathcal{P}_n$ denote the set of all partitions of $n$ and $\mathcal{P}_{n,m}$ the set of all partitions of $n$ of given length $m$.

For each $i$ $(1 \leq i \leq n)$, the number of times that $i$ appears as a part of $\pi$ is called the *multiplicity* of $i$ in $\pi$, denoted $\pi_i$. An alternate way to write partitions down then is by the standard notation $\pi = [1^{\pi_1}, 2^{\pi_2}, ..., n^{\pi_n}]$ with parts of multiplicity 0 omitted. For example the partition $\pi = \{6, 6, 4, 3, 3, 1\}$ of 23 can also be written as $\pi = [1, 3^2, 4, 6^2]$, suppressing the superscripts equal to 1. As this example of length 6 shows, in any partition, the multiplicities add up to the length: $\ell(\pi) = \sum_{i=1}^{n} \pi_i$. In other words, the multiset of multiplicities $\{\pi_1, \pi_2, ..., \pi_n\}$ is itself a partition of the integer $\ell(\pi)$. We shall refer to this as the *derived partition* of $\pi$, and denote it $\delta(\pi)$. For example, if $\pi = [1^5, 2^3, 4^2, 5^3, 9]$, a partition of 43 of length 14,then $\delta(\pi) = [1, 2, 3^2, 5]$, a partition of 14.

At times it may be useful to consider the order of the summands in a partition. In this case, we would be interested in an ordered $m$-tuple $\pi = (p_1, p_2, ..., p_m)$ of positive integers instead of a multiset. These objects are called *ordered partitions* or *compositions* of $n$. The terms "part", "length", and "multiplicity" retain their meanings. We will denote by $\mathcal{C}_n$ the set of all compositions of $n$ and by $\mathcal{C}_{n,m}$ the set of compositions of $n$ of length $m$. Given a composition $\alpha$ of $n$, we can obtain a partition of $n$ from it by simply forgetting the order of the parts. Such an operation will, of course, preserve the length and all multiplicities, and the underlying (unordered) partition of $\alpha$ we'll refer to as the *base* of $\alpha$, written $\phi(\alpha)$.

Let us introduce the following natural notation.

**Definition 1** Let $\alpha = \{p_1, p_2, ..., p_m\}$ be a partition (ordered or not) of $n$. The symbol $\alpha!$ will stand for the product of the factorials of the parts of $\alpha$, i.e., $\alpha! = \prod_{i=1}^{m}(p_i!)$. Similarly, we will use the notation $\binom{n}{\alpha}$ to stand for the multinomial coefficient:

$$\binom{n}{\alpha} = \frac{n!}{\alpha!} = \binom{n}{p_1, p_2, ..., p_m}.$$

To illustrate the utility of this notation, let us answer the following simple question: how many different compositions of $n$ have the same underlying partition $\pi$ when you forget the order of the summands? This is a standard counting problem, whose proof the reader can easily supply. We record the result for future reference:

**Lemma 2** Given $\pi \in \mathcal{P}_n$, the number of compositions of $n$ having base $\pi$ is given by the multinomial coefficient $\binom{\ell(\pi)}{\delta(\pi)}$.

## 2. The Higher Chain Rule

We begin this section by recalling the Faá di Bruno formula for computing the $n^{\text{th}}$ derivative of a composite function, expressed using our above notation for multinomial coefficients and derived partitions.

**Theorem 3** (Faá di Bruno, 1855) Suppose $u = f(x)$ and $x = g(t)$ are differentiable $n$ times. Then the composite function $u = (f \circ g)(t)$ is also differentiable $n$ times and:

$$(f \circ g)^{(n)}(t) = \sum_{\pi \in \mathcal{P}_n} \frac{\binom{n}{\pi}}{\delta(\pi)!} \cdot f^{(\ell(\pi))} \circ g(t) \cdot \prod_{i=1}^{n} \left[g^{(i)}(t)\right]^{\pi_i}. \tag{1}$$

From (1), a useful formula for the Taylor coefficients of $(f \circ g)(t)$ follows easily. It is a key result:

**Corollary 4** Let $T_n(f;a)$ denote the $n^{\text{th}}$ Taylor coefficient of the function $f(x)$ expanded about $x = a$, so $T_n(f;a) = \frac{f^{(n)}(a)}{n!}$. If both $f$ and $g$ have $n^{\text{th}}$ derivatives, then:

$$T_n(f \circ g; a) = \sum_{\pi \in \mathcal{P}_n} \binom{\ell(\pi)}{\delta(\pi)} \cdot T_{\ell(\pi)}(f; g(a)) \cdot \prod_{i=1}^{n} [T_i(g;a)]^{\pi_i} \tag{2}$$

*Proof.* By the definition of $T_n(f \circ g; a)$ and Theorem 3 one obtains:

$$\begin{aligned}
T_n(f \circ g; a) &= \frac{(f \circ g)^{(n)}(a)}{n!} = \frac{1}{n!}\left(\sum_{\pi \in \mathcal{P}_n} \frac{\binom{n}{\pi}}{\delta(\pi)!} \cdot f^{(\ell(\pi))} \circ g(a) \cdot \prod_{i=1}^{n} \left[g^{(i)}(a)\right]^{\pi_i}\right) \\
&= \sum_{\pi \in \mathcal{P}_n} \frac{1}{\pi!\delta(\pi)!} \cdot f^{(\ell(\pi))} \circ g(a) \cdot \prod_{i=1}^{n} \left[g^{(i)}(a)\right]^{\pi_i} \\
&= \sum_{\pi \in \mathcal{P}_n} \frac{f^{(\ell(\pi))} \circ g(a)}{\delta(\pi)!(1!)^{\pi_1}(2!)^{\pi_2} \cdot \ldots \cdot (n!)^{\pi_n}} \cdot \prod_{i=1}^{n} \left[g^{(i)}(a)\right]^{\pi_i} \\
&= \sum_{\pi \in \mathcal{P}_n} \frac{f^{(\ell(\pi))} \circ g(a)}{\delta(\pi)!} \cdot \prod_{i=1}^{n} \left[\frac{g^{(i)}(a)}{i!}\right]^{\pi_i}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\pi \in \mathcal{P}_n} \frac{\ell(\pi)!}{\delta(\pi)!} \frac{f^{(\ell(\pi))} \circ g(a)}{\ell(\pi)!} \cdot \prod_{i=1}^{n} \left[\frac{g^{(i)}(a)}{i!}\right]^{\pi_i} \\
&= \sum_{\pi \in \mathcal{P}_n} \binom{\ell(\pi)}{\delta(\pi)} \cdot T_{\ell(\pi)}(f; g(a)) \cdot \prod_{i=1}^{n} [T_i(g;a)]^{\pi_i},
\end{aligned}$$

as claimed. □

**Remark 5** Because of Lemma 2, we know there are $\binom{\ell(\pi)}{\delta(\pi)}$ compositions of $n$ having the same base $\phi(\pi)$. Since the map $\phi$ preserves length and multiplicities, it follows that we can view the above sum as being over $\mathcal{C}_n$ if desired, in which case the factor $\binom{\ell(\pi)}{\delta(\pi)}$ is not needed in the sum. Therefore, an equivalent way to state this result is:

$$T_n(f \circ g; a) = \sum_{\pi \in \mathcal{C}_n} T_{\ell(\pi)}(f; g(a)) \cdot \prod_{i=1}^{n} [T_i(g; a)]^{\pi_i}. \tag{3}$$

Furthermore, it is often convenient in (2) and (3) to collect together the partitions of a fixed length. Here is the resulting version of formula (2), for example:

$$T_n(f \circ g; a) = \sum_{m=1}^{n} T_m(f; g(a)) \cdot \sum_{\pi \in \mathcal{P}_{n,m}} \binom{m}{\delta(\pi)} \prod_{i=1}^{n} [T_i(g; a)]^{\pi_i}. \tag{4}$$

In this article, though we find the language of 'Taylor coefficients' to be a useful way to phrase our results, we emphasize that the manipulations we carry out are formal in nature. In particular, we do not consider any questions of convergence of the power series discussed. That is, we are using the functions merely in a formal way to aid in manipulating the sequences of coefficients, and therefore we are really operating within the realm of generating functions and exponential generating functions.

## 3. Bernoulli and Euler Numbers

In [3], there is a section titled 'Operating With Power Series. Expansion of Composite Functions', where some basic properties of Bernoulli and Euler numbers are derived. Nowhere in that section does anything as explicit as formula (2) appear, and partitions are not mentioned at all, despite the fact that [3] was published 101 years after Faá di Bruno's work. Moreover, the author remarks on page 118 of [3] that the Bernoulli numbers "...may be regarded as 'known,' even though their values cannot be specified by a simple formula...", and goes on to describe some rather complicated recursive procedures for computing these numbers. Yet, as we see below, fairly simple explicit formulas for both the Bernoulli numbers and the Euler numbers can be derived quite easily from Corollary 4.

Recall that the Stirling numbers of the second kind, $S(n, m)$ are defined to be the number of ways of partitioning a set of size $n$ into exactly $m$ nonempty subsets. We would like to introduce the following generalizations of these numbers:

**Definition 6** Let $X$ be a set of cardinality $n$. Let $S(n, m, odd)$ (respectively, $S(n, m, even)$) be the number of set partitions of $X$ into $m$ nonempty parts where each part has odd (resp., even) cardinality.

**Definition 7** Let $X$ be a set of cardinality $n$, and let $\pi$ be a partition of $n$. Let $S_\pi$ denote the number of set partitions of $X$ into nonempty parts which have cardinality exactly equal to the parts of $\pi$.

**Remark 8** It is clear from the definitions that $\sum_{\pi\in\mathcal{P}_{n,m}} S_\pi = S(n,m)$.

**Lemma 9** Let $\pi \in \mathcal{P}_n$. Then:

$$S_\pi = \frac{\binom{n}{\pi}}{\delta(\pi)!}, \tag{5}$$

where $\delta(\pi)$ is the derived partition of $\pi$.

*Proof.* See [2], page 39. □

**Remark 10** Notice that $S_\pi$ is the coefficient in Faá di Bruno's formula corresponding to the partition $\pi$.

We conclude this article by presenting our identities for the Euler and Bernoulli numbers, which we believe are new.

**Theorem 11** If $B_n$ is the $n^{\text{th}}$ Bernoulli number and $E_n$ is the $n^{\text{th}}$ Euler number, then:

$$\text{(a)}\quad B_n = \sum_{\pi\in\mathcal{P}_n} \frac{(-1)^{\ell(\pi)}}{1+\ell(\pi)} \cdot \binom{\ell(\pi)}{\delta(\pi)} \cdot \binom{n}{\pi} = \sum_{\pi\in\mathcal{C}_n} \frac{(-1)^{\ell(\pi)}}{1+\ell(\pi)} \cdot \binom{n}{\pi}$$

$$\text{(b)}\quad B_n = \sum_{m=1}^{n} \frac{(-1)^m m!}{1+m} S(n,m)$$

$$\text{(c)}\quad E_n = \sum_{\substack{\pi\in\mathcal{P}_n \\ \text{all even parts}}} (-1)^{\ell(\pi)} \cdot \binom{\ell(\pi)}{\delta(\pi)} \cdot \binom{n}{\pi} = \sum_{\substack{\pi\in\mathcal{C}_n \\ \text{all even parts}}} (-1)^{\ell(\pi)} \binom{n}{\pi}$$

$$\text{(d)}\quad E_n = \sum_{m=1}^{n} (-1)^m\, m!\, S(n,m,even)$$

$$\text{(e)}\quad 1 = \sum_{r=1}^{j} \frac{(-1)^r}{(2r)!} E_{2r} \sum_{\substack{\pi\in\mathcal{P}_{2j,2r} \\ \text{all odd parts}}} \binom{2r}{\delta(\pi)} \cdot \binom{2j}{\pi} \cdot \prod_{s=0}^{j} (E_{2s})^{\pi_{2s+1}} \quad \forall\, j>0$$

*Proof.* For part (a), let $g(t) = e^t - 1$ with $a = 0$ and let $f(x) = \frac{\ln(1+x)}{x}$. Then $T_0(g;0) = 0$ and $T_i(g;0) = \frac{1}{i!}$ if $i > 0$, while $T_m(f;g(0)) = T_m(f;0) = \frac{(-1)^m}{1+m}$. Then (4) implies that:

$$T_n(f\circ g;0) = \sum_{m=1}^{n} \frac{(-1)^m}{1+m} \cdot \sum_{\pi\in\mathcal{P}_{n,m}} \binom{m}{\delta(\pi)} \prod_{i=1}^{n} \left[\frac{1}{i!}\right]^{\pi_i}. \tag{6}$$

But the composite function $f\circ g(t) = \frac{t}{e^t-1}$ is by definition the exponential generating function of the Bernoulli numbers $B_n$ (see page 466 of [4]), so $T_n(f\circ g;0) = \frac{B_n}{n!}$. Combined with (6), this yields

$$\begin{aligned} B_n &= n! T_n(f \circ g; 0) = \sum_{m=1}^{n} \frac{(-1)^m}{1+m} \cdot \sum_{\pi \in \mathcal{P}_{n,m}} \binom{m}{\delta(\pi)} \frac{n!}{\pi!} \\ &= \sum_{m=1}^{n} \frac{(-1)^m}{1+m} \cdot \sum_{\pi \in \mathcal{P}_{n,m}} \binom{m}{\delta(\pi)} \cdot \binom{n}{\pi} = \sum_{\pi \in \mathcal{P}_n} \frac{(-1)^{\ell(\pi)}}{1+\ell(\pi)} \cdot \binom{\ell(\pi)}{\delta(\pi)} \cdot \binom{n}{\pi}, \end{aligned}$$

which is the first sum in part (a). The second sum follows from our usual trick of trading the factor $\binom{\ell(\pi)}{\delta(\pi)}$ in for summing over compositions instead of partitions.

Part (b) follows from part (a), because $\binom{m}{\delta(\pi)} \cdot \binom{n}{\pi} = m! \cdot S_\pi$ (by Lemma 9), so if we collect together all partitions of a fixed length, we obtain:

$$B_n = \sum_{m=1}^{n} \frac{(-1)^m}{1+m} \cdot \sum_{\pi \in \mathcal{P}_{n,m}} \binom{m}{\delta(\pi)} \cdot \binom{n}{\pi} = \sum_{m=1}^{n} \frac{(-1)^m m!}{1+m} \cdot \sum_{\pi \in \mathcal{P}_{n,m}} S_\pi = \sum_{m=1}^{n} \frac{(-1)^m m!}{1+m} S(n,m),$$

the last equality by Remark 8.

For part (c), let $g(t) = \cosh(t)$ with $a = 0$ and $f(x) = \frac{1}{x}$. Then $T_i(g; 0) = \frac{1}{i!}$ if $i$ is even, and 0 otherwise, while $T_m(f; 1) = (-1)^m$. Then (4) yields:

$$T_n(f \circ g; 0) = \sum_{m=1}^{n} (-1)^m \cdot \sum_{\pi \in \mathcal{P}_{n,m}} \binom{m}{\delta(\pi)} \prod_{i=1}^{n} [T_i(g; 0)]^{\pi_i},$$

but if $\pi_i > 0$ with $i$ odd, the product then vanishes. Thus, we may sum over partitions with only even parts, obtaining:

$$T_n(f \circ g; 0) = \sum_{m=1}^{n} (-1)^m \cdot \sum_{\substack{\pi \in \mathcal{P}_{n,m} \\ \text{all even parts}}} \binom{m}{\delta(\pi)} \prod_{i=1}^{n} \left[\frac{1}{i!}\right]^{\pi_i} = \sum_{m=1}^{n} (-1)^m \cdot \sum_{\substack{\pi \in \mathcal{P}_{n,m} \\ \text{all even parts}}} \binom{m}{\delta(\pi)} \frac{1}{\pi!}. \quad (7)$$

But the composite function $f \circ g(t) = sech(t)$ is by definition the exponential generating function of the Euler numbers (consult the tables in [4]), so $T_n(f \circ g; 0) = \frac{E_n}{n!}$. Combined with (7) this yields:

$$\begin{aligned} E_n &= n! T_n(f \circ g; 0) = n! \sum_{m=1}^{n} (-1)^m \cdot \sum_{\substack{\pi \in \mathcal{P}_{n,m} \\ \text{all even parts}}} \binom{m}{\delta(\pi)} \frac{1}{\pi!} = \sum_{m=1}^{n} (-1)^m \cdot \sum_{\substack{\pi \in \mathcal{P}_{n,m} \\ \text{all even parts}}} \binom{m}{\delta(\pi)} \frac{n!}{\pi!} \\ &= \sum_{m=1}^{n} (-1)^m \cdot \sum_{\substack{\pi \in \mathcal{P}_{n,m} \\ \text{all even parts}}} \binom{m}{\delta(\pi)} \cdot \binom{n}{\pi}, \quad (8) \end{aligned}$$

as claimed.

Part (d) follows from (8) because $\binom{m}{\delta(\pi)} \cdot \binom{n}{\pi} = m! \cdot S_\pi$, so for a fixed length $m$, summing over all partitions with even parts gives $m! \cdot S(n, m, even)$.

For part (e), take $g(t) = gd(t)$, the *gudermannian function* with definition $gd(t) = 2\tan^{-1}(e^t) - \frac{\pi}{2}$, and take $f(x) = \sec(x)$. Apply Corollary 4 with $n = 2j$, $m = 2r$ and $i = 2s+1$ together with the fact (see page 213 of [4]) that $\sec(gd(t)) = \cosh(t)$. Details are left for the reader. □

Of course, another way to rewrite (8) is to sum over compositions instead of partitions to eliminate the factor $\binom{m}{\delta(\pi)}$. If we do that and do not group terms of the same length $m$, we obtain the rather simple looking formula:

$$E_n = \sum_{\substack{\pi \in \mathcal{C}_n \\ \text{all even parts}}} (-1)^{\ell(\pi)} \binom{n}{\pi}, \tag{9}$$

which is the second sum in part (c). From (9), it is immediate that $E_n$ is an integer, and is equal to 0 if $n$ is odd. Similarly, from part (a) or part (b), it is immediate that $B_n$ is a rational number.

While the formulas in parts (a)-(d) are explicit, the formula in part (e) can be used (though it is not very efficient) to compute the Euler numbers recursively. For example, when $j = 3$, the only partitions which appear in the right side of the formula of part (e) are $[1,5]$ and $[3^2]$ (when $r = 1$), $[1^3, 3]$ (when $r = 2$), and $[1^6]$ (when $r = 3$), so that the formula becomes:

$$\begin{aligned} 1 \;=\; & \frac{-1}{2!} E_2 \left[ \binom{2}{1\;\;1} \cdot \binom{6}{1\;\;5} E_0 E_4 + \binom{2}{2} \cdot \binom{6}{3\;\;3} E_2^2 \right] + \\ & \frac{1}{4!} E_4 \binom{4}{1\;\;3} \cdot \binom{6}{3\;\;1\;\;1\;\;1} E_0^3 E_2 + \frac{-1}{6!} E_6 \binom{6}{6} \cdot \binom{6}{1\;\;1\;\;1\;\;1\;\;1\;\;1} E_0^6, \end{aligned}$$

or

$$1 = -6E_2E_0E_4 - 10E_2^3 + 20E_4E_0^3E_2 - E_6E_0^6.$$

Now $E_6$ appears only once in this equation, so substituting in the values $E_0 = 1$, $E_2 = -1$ and $E_4 = 5$, we may solve for $E_6$, obtaining the correct value $-61$ (one may also check this using (9).)

## References


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