ON THE t-CORE OF AN s-CORE PARTITION

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Abstract

Given two relatively prime positive integers s and t, J. Olsson proved that the t-core of an s-core partition ρ is again an s-core. In this note we extend this result to the case where s and t are arbitrary distinct positive integers.

1. Main Result

Let $\mathbb{N}$ be the set of nonnegative integers and let $n \in \mathbb{N}$. Consider sequences $(\alpha_1, \cdots, \alpha_t)$ of integers from $\mathbb{N}$ with $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_t$ and $\sum_{i=1}^{t} \alpha_i = n$. Two such sequences $(\alpha_1, \cdots, \alpha_t)$ and $(\alpha'_1, \cdots, \alpha'_{t'})$ are said to be equivalent if their nonzero terms are the same. A *partition* λ of n will then be defined as an equivalence class of such sequences.

A sequence $(\alpha_1, \cdots, \alpha_t)$ representing λ determines a corresponding β-set, namely $\beta(\lambda) = \{x_1, \cdots, x_t\}$, where $x_i = \alpha_i + (i-1)$. The equivalence relation on sequences induces an equivalence relation on β-sets. Two β-sets $\beta(\lambda) = \{x_1, \cdots, x_t\}$ and $\beta(\lambda)' = \{x'_1, \cdots, x'_{t'}\}$ are equivalent if $t' - t = d \geq 0$ and $\{x'_1, \cdots, x'_{t'}\} = \{0, 1, 2, \cdots, d-1\} \cup \{x_1 + d, \cdots, x_t + d\}$.

We may write this as $\beta(\lambda)' = \{0, \cdots, d-1\} \cup \{\beta(\lambda) + d\}$. Then $f_d : (y, x] \longrightarrow (y+d, x+d]$ is a bijection between $\beta(\lambda)$ and $\beta(\lambda)'$. A *hook* h of λ is a pair of nonnegative integers $h = (y, x)$ where $x \in \beta(\rho)$, $y \notin \beta(\rho)$ and $y < x$. We say h has *length* s (t) if $x - y = s$ ($x - y = t$). A hook h of length s (t) is also called an *s-hook* (*t-hook*). A partition ρ is an *s-core* (*t-core*) if it contains no s-hooks (t-hooks). In particular, f_d preserves hook lengths.

Theorem 1.1 Suppose s and t are distinct positive integers and ρ is an s-core. Then the t-core of ρ is also an s-core.

To prove Theorem 1.1, we must define the *s-abacus* $\mathcal{A}^s(\beta(\rho))$ of ρ. We do so as follows: create s runners numbered $0, 1, \ldots, s-1$ running from north to south. In the i-th runner we place all non-negative integers of residue i modulo s in increasing order, and then underline the numbers that occur in $\beta(\rho)$. These underlined numbers will be referred to as *beads* while the numbers that are not underlined will be referred to as *spaces*.

Suppose $\rho = (5,5,4,3)$, $\beta(\rho) = \{3,5,7,8\}$. Then $\mathcal{A}^3(\beta(\rho))$ is defined to be:

$$\begin{array}{ccc} 0 & 1 & 2 \\ \underline{3} & 4 & \underline{5} \\ 6 & \underline{7} & \underline{8}. \end{array}$$

The subset of beads on the i-th runner will be denoted $\beta(\rho)_i$. Removing an s-hook from ρ is equivalent to replacing an $\underline{x}$ (in some $\beta(\rho)_i$) with $\underline{x-s}$. Notice $x - s$ is the position directly above x on the i-th runner. This will be described as moving a bead *one position north.* Then $\mathcal{A}^s(\beta(\rho))$ is the s-abacus of a s-core if for every $\beta(\rho)_i$ there are no available moves one position north (Theorem 2.7.16, [2]). The replacement $\underline{x} = i + ms$ on the i-th runner of the s-abacus with $\underline{x'} = i' + m's$ where $x' < x$, $i' \neq i$ will be described as moving a bead $x - x'$ positions *west.* (The reader is referred to Section 2.7, [2] and Sections I.1–I.3, [6] for further details on partitions, β-sets, hooks, and the s-abacus.)

Proof. When $(s,t) = 1$ the result is true by [5]. Suppose $(s,t) \neq 1$. Either (1) s divides t or (2) $\gcd(s,t) \notin \{1, s\}$. If s divides t, then any s-core partition is itself a t-core, so we are done. Suppose s does not divide t and $\gcd(s,t) > 1$. Removing a t-hook from ρ is equivalent to taking a bead x on the ℓ-th runner (for some ℓ between 0 and $s-1$) of $\mathcal{A}^s(\beta(\rho))$ and placing it in empty position to the west in the $(\ell - t)$-th runner. (For the remainder of this note, if $t > \ell$ we will interpret this difference as $\ell - t \pmod s$.) Removing another t-hook starting from a bead on the $(\ell-t)$-th runner, we arrive at the $(\ell-2t)$-th runner, and so on, until eventually for some $j > 0$ we obtain $\ell - jt \equiv \ell \pmod s$. This suggests the following definition. A *t-orbit* of the s-abacus is a finite sequence (read from right-to-left) of distinct runners reached by repeated moves of t-positions west. Then, if $k' = gcd(s,t)$, each t-orbit of $\mathcal{A}^s(\beta(\rho))$ will have exactly k' distinct runners. Starting from $\ell = 1$, for each value $1, 2, 3 \cdots$ we denote by $\mathcal{O}_t(s-\ell)$ a t-orbit of runners which begins at the $(s-\ell)$-th runner. Since for $0 \leq z, z' \leq s-1$ and $z \neq z'$ either $\mathcal{O}_t(z) \cap \mathcal{O}_t(z') = \emptyset$ or $\mathcal{O}_t(z) = \mathcal{O}_t(z')$, there will be exactly $k = \frac{s}{k'}$ distinct t-orbits of $\mathcal{A}^s(\beta(\rho))$.

Now $\mathcal{O}_t(s-\ell)$ can itself be seen as a k'-abacus of runners plucked from $\mathcal{A}^{(s)}(\beta(\rho))$ and re-arranged in such a way that moving a bead one position west is equivalent to removing a t-hook from ρ. (Note: from the westmost runner removing a t-hook requires placing the bead one position north on the eastmost runner.) Viewed as a k'-abacus, moving a bead one position north in $\mathcal{O}_t(s-\ell)$ will still be equivalent to removing a s-hook from ρ, since it is comprised of runners of $\mathcal{A}^s(\beta(\rho))$. This construction is a variation of the (s,t)-abacus of J. Olsson and D. Stanton (see Section 5, [4]) which they define when s,t are relatively prime.

For all $1 \leq \ell \leq k$ the runners in $\mathcal{O}_t(s-\ell) \subset \mathcal{A}^s(\beta(\rho))$ will be labeled

$$\mathcal{O}_t(s-1) = (\beta(\rho)_{s-1-(k'-1)t}, \cdots, \beta(\rho)_{s-1})$$
$$\vdots$$
$$\mathcal{O}_t(s-k) = (\beta(\rho)_{s-k-(k'-1)t}, \cdots, \beta(\rho)_{s-k}).$$

For a fixed ℓ we obtain the *t-core of* $\mathcal{O}_t(s-\ell)$ by moving all available beads one position west on $\mathcal{O}_t(s-\ell)$ until we obtain a k'-abacus with no available westward moves. We denote this k'-abacus by $\widehat{\mathcal{O}}_t(s-\ell)$. It can be obtained systematically from $\mathcal{O}_t(s-\ell)$.

Algorithm for finding $\widehat{\mathcal{O}}_t(s-\ell)$. Let $i \in \{1, \cdots, k'\}$ and $b_i = |\beta(\rho)_{(s-\ell)-(i-1)t}|$ be the number of beads in the runner i positions west on $\mathcal{O}_t(s-\ell)$. Then $B(1,2) = b_1 - b_2$ is the difference between the number of beads in the eastmost or $(s-\ell)$-th runner and the runner immediately to the west of it, the $(s-\ell-t)$-th runner. If $B(1,2)$ is negative or zero, move no beads. If $B(1,2) = 2f$, place f of the southmost beads from the $(s-\ell)$-th runner in the northmost empty spaces of the $(s-\ell-t)$-th runner, so that the number of beads in the two runners now become equal. If $B(1,2) = 2f+1$, place the $f+1$ of the southmost beads from the $(s-\ell)$-th runner in the northmost empty spaces of $(s-\ell-t)$-th runner, so that the runner to the west now has one more bead than the eastmost runner. For $i = 2, 3, \cdots$ etc. follow the same procedure for $B(i, i+1)$, except when $B = (k', 1) = 2f+1$. In this case, place only f beads from the westmost runner in the northmost available empty spaces on the eastmost or $(s-\ell)$-th runner. Repeating this procedure a finite number of times results in the modified subsequence $\widehat{\mathcal{O}}_t(s-\ell) = (\widehat{\beta}(\rho)_{s-\ell-(k'-1)t}, \cdots, \widehat{\beta}(\rho)_{s-\ell})$ which when viewed as a k'-abacus has no available westward moves.

Finding the t-core of ρ. For each ℓ, obtain $\widehat{\mathcal{O}}_t(s-\ell)$ from $\mathcal{O}_t(s-\ell)$ as above. Then $\mathcal{A}^s(\widehat{\beta}(\rho)) = (\widehat{\beta}(\rho)_0, \cdots, \widehat{\beta}(\rho)_{s-1})$ will be the s-abacus for the t-core of ρ. This follows by construction, since using the runners of the modified t-orbits $\widehat{\mathcal{O}}_t(s-\ell)$ implies there are no available moves t positions west. However $\mathcal{A}^s(\widehat{\beta}(\rho))$ still has no available moves one position north (by our algorithm) and hence remains an s-abacus of an s-core.

2. Examples

Example 2.1. Let $s = 6$ and $t = 3$. Consider the following 6-abacus $\mathcal{A}^6(\beta(\rho))$ of a 6-core ρ:

$\beta(\rho)_0$	$\beta(\rho)_1$	$\beta(\rho)_2$	$\beta(\rho)_3$	$\beta(\rho)_4$	$\beta(\rho)_5$
0	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
6	$\underline{7}$	$\underline{8}$	9	$\underline{10}$	11
12	13	$\underline{14}$	15	$\underline{16}$	17
18	19	20	21	$\underline{22}$	23.

Since gcd(6,3)=3, we have $k = 3$ and $k' = \frac{6}{3} = 2$. Hence we have three 3-orbits $\mathcal{O}_t(s-\ell)$ (each with $k' = 2$ runners) and their corresponding 3-cores $\widehat{\mathcal{O}}_t(s-\ell)$:

$$\mathcal{O}_3(1) = \begin{array}{ll} \beta(\rho)_2 & \beta(\rho)_5 \\ \underline{2} & \underline{5} \\ \underline{8} & 11 \\ \underline{14} & 17 \\ 20 & 23 \end{array} \qquad \widehat{\mathcal{O}}_3(1) = \begin{array}{ll} \widehat{\beta}(\rho)_2 & \widehat{\beta}(\rho)_5 \\ \underline{2} & \underline{5} \\ \underline{8} & \underline{11} \\ 14 & 17 \\ 20 & 23 \end{array}$$

$$\mathcal{O}_3(2) = \begin{array}{ll} \beta(\rho)_1 & \beta(\rho)_4 \\ \underline{1} & \underline{4} \\ \underline{7} & \underline{10} \\ 13 & \underline{16} \\ 19 & \underline{22} \end{array} \qquad \widehat{\mathcal{O}}_3(2) = \begin{array}{ll} \widehat{\beta}(\rho)_1 & \widehat{\beta}(\rho)_4 \\ \underline{1} & \underline{4} \\ \underline{7} & \underline{10} \\ \underline{13} & \underline{16} \\ 19 & 22 \end{array}$$

$$\mathcal{O}_3(3) = \begin{array}{ll} \beta(\rho)_0 & \beta(\rho)_3 \\ 0 & \underline{3} \\ 6 & 9 \\ 12 & 15 \\ 18 & 21 \end{array} \qquad \widehat{\mathcal{O}}_3(3) = \begin{array}{ll} \widehat{\beta}(\rho)_0 & \widehat{\beta}(\rho)_3 \\ \underline{0} & 3 \\ 6 & 9 \\ 12 & 15 \\ 18 & 21. \end{array}$$

Finally we obtain the 6-abacus $\mathcal{A}^6(\widehat{\beta}(\rho))$ of $\widehat{\rho}$ the 3-core of ρ:

$$\begin{array}{llllll} \widehat{\beta}(\rho)_0 & \widehat{\beta}(\rho)_1 & \widehat{\beta}(\rho)_2 & \widehat{\beta}(\rho)_3 & \widehat{\beta}(\rho)_4 & \widehat{\beta}(\rho)_5 \\ \underline{0} & \underline{1} & \underline{2} & 3 & \underline{4} & \underline{5} \\ 6 & \underline{7} & \underline{8} & 9 & \underline{10} & \underline{11} \\ 12 & \underline{13} & 14 & 15 & \underline{16} & 17. \end{array}$$

Example 2.2. Let $s = 12$ and $t = 8$. Consider the following 12-abacus $\mathcal{A}^{12}(\beta(\rho))$ of a 12-core ρ:

$$\begin{array}{llllllllllll} \beta(\rho)_0 & \beta(\rho)_1 & \beta(\rho)_2 & \beta(\rho)_3 & \beta(\rho)_4 & \beta(\rho)_5 & \beta(\rho)_6 & \beta(\rho)_7 & \beta(\rho)_8 & \beta(\rho)_9 & \beta(\rho)_{10} & \beta(\rho)_{11} \\ 0 & \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & 7 & \underline{8} & \underline{9} & \underline{10} & \underline{11} \\ 12 & \underline{13} & \underline{14} & \underline{15} & \underline{16} & 17 & \underline{18} & 19 & 20 & \underline{21} & \underline{22} & \underline{23} \\ 24 & \underline{25} & 26 & \underline{27} & \underline{28} & 29 & \underline{30} & 31 & 32 & \underline{33} & \underline{34} & 35 \\ 36 & \underline{37} & 38 & \underline{39} & 40 & 41 & 42 & 43 & 44 & \underline{45} & \underline{46} & 47 \\ 48 & 49 & 50 & \underline{51} & 52 & 53 & 54 & 55 & 56 & \underline{57} & 58 & 59 \\ 60 & 61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 & \underline{69} & 70 & 71. \end{array}$$

Since gcd(12,8)=4, we have $k = 4$ and $k' = \frac{12}{4} = 3$. Hence we have four 8-orbits $\mathcal{O}_8(s - \ell)$ (each with $k' = 3$ runners) and their corresponding 8-cores $\widehat{\mathcal{O}}_8(s - \ell)$:

$$\mathcal{O}_8(1) = \begin{array}{lll} \beta(\rho)_7 & \beta(\rho)_3 & \beta(\rho)_{11} \\ 7 & \underline{3} & \underline{11} \\ 19 & \underline{15} & \underline{23} \\ 31 & \underline{27} & 35 \\ 43 & \underline{39} & 47 \\ 55 & \underline{51} & 59 \\ 67 & 63 & 71 \end{array} \qquad \widehat{\mathcal{O}}_8(1) = \begin{array}{lll} \widehat{\beta}(\rho)_7 & \widehat{\beta}(\rho)_3 & \widehat{\beta}(\rho)_{11} \\ \underline{7} & \underline{3} & \underline{11} \\ \underline{19} & \underline{15} & \underline{23} \\ \underline{31} & 27 & 35 \\ 43 & 39 & 47 \\ 55 & 51 & 59 \\ 67 & 63 & 71 \end{array}$$

$$\mathcal{O}_8(2) = \begin{array}{lll} \beta(\rho)_6 & \beta(\rho)_2 & \beta(\rho)_{10} \\ \underline{6} & \underline{2} & \underline{10} \\ \underline{18} & \underline{14} & \underline{22} \\ \underline{30} & 26 & \underline{34} \\ 42 & 38 & \underline{46} \\ 54 & 50 & 58 \\ 66 & 62 & 70 \end{array} \qquad \widehat{\mathcal{O}}_8(2) = \begin{array}{lll} \widehat{\beta}(\rho)_6 & \widehat{\beta}(\rho)_2 & \widehat{\beta}(\rho)_{10} \\ \underline{6} & \underline{2} & \underline{10} \\ \underline{18} & \underline{14} & \underline{22} \\ \underline{30} & \underline{26} & \underline{34} \\ 42 & 38 & 46 \\ 54 & 50 & 58 \\ 66 & 62 & 70 \end{array}$$

$$\mathcal{O}_8(3) = \begin{array}{lll} \beta(\rho)_5 & \beta(\rho)_1 & \beta(\rho)_9 \\ \underline{5} & \underline{1} & \underline{9} \\ 17 & \underline{13} & \underline{21} \\ 29 & \underline{25} & \underline{33} \\ 41 & \underline{37} & \underline{45} \\ 53 & 49 & \underline{57} \\ 65 & 61 & \underline{69} \end{array} \qquad \widehat{\mathcal{O}}_8(3) = \begin{array}{lll} \widehat{\beta}(\rho)_5 & \widehat{\beta}(\rho)_1 & \widehat{\beta}(\rho)_9 \\ \underline{5} & \underline{1} & \underline{9} \\ \underline{17} & \underline{13} & \underline{21} \\ \underline{29} & \underline{25} & \underline{33} \\ \underline{41} & \underline{37} & 45 \\ 53 & 49 & 57 \\ 65 & 61 & 69 \end{array}$$

$$\mathcal{O}_8(4) = \begin{array}{lll} \beta(\rho)_4 & \beta(\rho)_0 & \beta(\rho)_8 \\ \underline{4} & 0 & \underline{8} \\ \underline{16} & 12 & 20 \\ \underline{28} & 24 & 32 \\ 40 & 36 & 44 \\ 52 & 48 & 56 \\ 64 & 60 & 68 \end{array} \qquad \widehat{\mathcal{O}}_8(4) = \begin{array}{lll} \widehat{\beta}(\rho)_4 & \widehat{\beta}(\rho)_0 & \widehat{\beta}(\rho)_8 \\ \underline{4} & \underline{0} & \underline{8} \\ \underline{16} & 12 & 20 \\ 28 & 24 & 32 \\ 40 & 36 & 44 \\ 52 & 48 & 56 \\ 64 & 60 & 68. \end{array}$$

Finally we obtain the 12-abacus $\mathcal{A}^{12}(\widehat{\beta}(\rho))$ of $\widehat{\rho}$ the 8-core of ρ:

$$\begin{array}{llllllllllll} \widehat{\beta}(\rho)_1 & \widehat{\beta}(\rho)_2 & \widehat{\beta}(\rho)_3 & \widehat{\beta}(\rho)_4 & \widehat{\beta}(\rho)_4 & \widehat{\beta}(\rho)_5 & \widehat{\beta}(\rho)_6 & \widehat{\beta}(\rho)_7 & \widehat{\beta}(\rho)_8 & \widehat{\beta}(\rho)_9 & \widehat{\beta}(\rho)_{10} & \widehat{\beta}(\rho)_{11} \\ \underline{0} & \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & \underline{7} & \underline{8} & \underline{9} & \underline{10} & \underline{11} \\ 12 & \underline{13} & \underline{14} & \underline{15} & \underline{16} & \underline{17} & \underline{18} & \underline{19} & 20 & \underline{21} & \underline{22} & \underline{23} \\ 24 & \underline{25} & \underline{26} & 27 & 28 & \underline{29} & \underline{30} & \underline{31} & 32 & \underline{33} & \underline{34} & 35 \\ 36 & \underline{37} & 38 & 39 & 40 & \underline{41} & 42 & 43 & 44 & 45 & 46 & 47 \\ 48 & 49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 & 59 \\ 60 & 61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 & 69 & 70 & 71. \end{array}$$

For other results on partitions that are simultaneously s-cores and t-cores see [1], [3], [7].

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