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ON THE NUMBER OF SUBSETS OF $[1, M]$ RELATIVELY PRIME TO N AND ASYMPTOTIC ESTIMATES

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Abstract

A set A of positive integers is *relatively prime to* n if $\gcd(A \cup \{n\}) = 1$. Given positive integers $l \le m \le n$, let $\Phi([l,m],n)$ denote the number of nonempty subsets of $\{l, l+1, \dots, m\}$ which are relatively prime to n and let $\Phi_k([l,m],n)$ denote the number of such subsets of cardinality k. In this paper we give formulas for these functions for the case $l = 1$. Intermediate consequences include identities for the number of subsets of $\{1, 2, \dots, n\}$ with elements in both $\{1, 2, \dots, m\}$ and $\{m, m+1, \dots, n\}$ which are relatively prime to n and the number of such subsets having cardinality k. Some of our proofs use the Möbius inversion formula extended to functions of several variables.

1. Introduction

Let k and $l \le m \le n$ denote positive integers, let $[l,m] = \{l, l+1, \dots, m\}$, and let $[x]$ be the floor of x. For nonnegative integers $0 \le M \le N$ we have the following basic identity for binomial coefficients:

$$\sum_{j=k}^{N} \binom{j}{k} = \binom{N+1}{k+1}. \tag{1}$$

Definition 1. *Let* $\Phi([l,m],n) = \#\{A \subseteq [l,m] : \ A \ne \emptyset, \ \text{and} \ \gcd(A \cup \{n\}) = 1\}$ *and* $\Phi_k([l,m],n) = \#\{A \subseteq [l,m] : \ \#A = k \ \text{and} \ \gcd(A \cup \{n\}) = 1\}$.

Nathanson [2] found

$$\begin{aligned} \Phi([1,n],n) &= \sum_{d \mid n} \mu(d) 2^{n/d}, \\ \Phi_k([1,n],n) &= \sum_{d \mid n} \mu(d) \binom{n/d}{k}. \end{aligned} \tag{2}$$

El Bachraoui [1] obtained

$$\Phi([m,n],n) = \sum_{d|n} \mu(d) 2^{n/d} - \sum_{i=1}^{m-1} \sum_{d|(i,n)} \mu(d) 2^{(n-i)/d},$$

$$\Phi_k([m,n],n]) = \sum_{d|n} \mu(d) \binom{n/d}{k} - \sum_{i=1}^{m-1} \sum_{d|(i,n)} \mu(d) \binom{(n-i)/d}{k-1}.$$

Nathanson-Orosz [3] simplified the latter two identities and found

$$\begin{aligned} \Phi([m,n],n) &= \sum_{d|n} \mu(d) 2^{(n/d)-[(m-1)/d]}, \\ \Phi_k([m,n],n]) &= \sum_{d|n} \mu(d) \binom{n/d - [(m-1)/d]}{k}. \end{aligned} \tag{3}$$

2. Phi Functions for $[1,m]$

Our main goal is to give identities for the functions $\Phi([1,m],n)$ and $\Phi_k([1,m],n)$. We need the following lemma.

Lemma 2. *Let* $\Psi([1,m],n) = \#\{A \subseteq [1,m] : \ m \in A \ \text{and} \ \gcd(A \cup \{n\}) = 1\}$ *and* $\Psi_k([1,m],n) = \#\{A \subseteq [1,m] : \ m \in A, \ \#A = k, \ \text{and} \ \gcd(A \cup \{n\}) = 1\}$. *Then*

$$(a) \qquad \Psi([1,m],n) = \sum_{d|(m,n)} \mu(d) 2^{m/d-1}.$$

$$(b) \qquad \Psi_k([1,m],n) = \sum_{d|(m,n)} \mu(d) \binom{m/d-1}{k-1}.$$

Proof. (a) Let $\mathcal{P}(m)$ denote the set of subsets of $[1,m]$ containing m and let $\mathcal{P}(m,d)$ be the set of subsets A of $[1,m]$ such that $m \in A$ and $\gcd(A \cup \{n\}) = d$. It is clear that the set $\mathcal{P}(m)$ of cardinality 2^{m-1} can be partitioned using the equivalence relation of having the same gcd. Moreover, the mapping $A \mapsto \frac{1}{d}A$ is a one-to-one correspondence between the subsets of $\mathcal{P}(m,d)$ and the set of subsets B of $[1,m/d]$ such that $m/d \in B$ and $\gcd(B \cup \{n/d\}) = 1$. Then $\#\mathcal{P}(m,d) = \Psi([1,m/d],n/d)$. Thus,

$$2^{m-1} = \sum_{d|(m,n)} \#\mathcal{P}(m,d) = \sum_{d|(m,n)} \Psi([1,m/d],n/d),$$

which by the Möbius inversion formula extended to multivariable functions [1, Theorem 2(c)] is equivalent to

$$\Psi([1,m],n) = \sum_{d|(m,n)} \mu(d) 2^{m/d-1}.$$

(*b*) Noting that the correspondence $A \mapsto \frac{1}{d}A$ defined above preserves the cardinality and using an argument similar to the one in part (a), we obtain the following identity

$$\binom{m-1}{k-1} = \sum_{d|(m,n)} \Psi_k([1, m/d], n/d)$$

which by the Möbius inversion formula [1, Theorem 2(c)] is equivalent to

$$\Psi_k([1, m], n) = \sum_{d|(m,n)} \mu(d)\binom{m/d-1}{k-1}.$$

□

Theorem 3. *We have*

$$(a)\ \Phi([1, m], n) = \sum_{d|n} \mu(d)2^{[m/d]}.$$

$$(b)\ \Phi_k([1, m], n) = \sum_{d|n} \mu(d)\binom{[m/d]}{k}.$$

Proof. (*a*) Repeatedly applying Lemma 2(a) together with Equation (2) yield the following identities:

$$\begin{aligned}
\Phi([1, m], n) &= \Phi([1, m+1], n) - \Psi([1, m+1], n) \\
&= \Phi([1, m+2], n) - \big(\Psi([1, m+2], n) + \Psi([1, m+1], n)\big) \\
&= \Phi([1, n], n) - \sum_{i=1}^{n-m} \Psi([1, m+i], n) \\
&= \sum_{d|n} \mu(d)2^{n/d} - \sum_{i=1}^{n-m} \sum_{d|(m+i,n)} \mu(d)2^{(m+i)/d-1} \\
&= \sum_{d|n} \mu(d)2^{n/d} - \sum_{d|n} \mu(d) \sum_{\substack{i=1 \\ d|m+i}}^{n-m} 2^{(m+i)/d-1} \\
&= \sum_{d|n} \mu(d)2^{n/d} - \sum_{d|n} \mu(d) \sum_{j=[m/d]+1}^{n/d} 2^{j-1} \\
&= \sum_{d|n} \mu(d)\left(2^{n/d} - 2^{[m/d]}(2^{n/d-[m/d]} - 1)\right) \\
&= \sum_{d|n} \mu(d)2^{[m/d]}.
\end{aligned}$$

(*b*) Similar to (*a*), repeatedly applying Lemma 2(b) together with Equation (2) we find

$$\Phi_k([1, m], n) = \sum_{d|n} \mu(d)\binom{n/d}{k} - \sum_{i=1}^{n-m} \sum_{d|(m+i,n)} \mu(d)\binom{(m+i)/d-1}{k-1}.$$

Then

$$\begin{aligned}
\Phi_k([1,m],n) &= \sum_{d|n}\mu(d)\binom{n/d}{k} - \sum_{d|n}\mu(d)\sum_{\substack{i=1\\ d|m+i}}^{n-m}\binom{(m+i)/d-1}{k-1}\\
&= \sum_{d|n}\mu(d)\binom{n/d}{k} - \sum_{d|n}\mu(d)\sum_{j=[m/d]+1}^{n/d}\binom{j-1}{k-1}\\
&= \sum_{d|n}\mu(d)\left(\binom{n/d}{k} - \sum_{j=[m/d]+1}^{n/d}\binom{j-1}{k-1}\right)\\
&= \sum_{d|n}\mu(d)\left(\binom{n/d}{k} - \sum_{j=1}^{n/d}\binom{j-1}{k-1} + \sum_{j=1}^{[m/d]}\binom{j-1}{k-1}\right)\\
&= \sum_{d|n}\mu(d)\left(\binom{n/d}{k} - \binom{n/d}{k} + \binom{[m/d]}{k}\right)\\
&= \sum_{d|n}\mu(d)\binom{[m/d]}{k},
\end{aligned}$$

where the next-to-last identity follows by (1). □

Corollary 4. *Let $U(m,n)$ be the number of nonempty subsets of $[1,n]$ with elements in both $[1,m]$ and $[m,n]$ which are relatively prime to n and let $U_k(m,n)$ be the number of such sets of cardinality k. Then*

$$U(m,n) = \sum_{d|n}\mu(d)(2^{n/d} - 2^{[(m-1)/d]} - 2^{n/d-[m/d]})$$

and

$$U_k(m,n) = \sum_{d|n}\mu(d)\left(\binom{n/d}{k} - \binom{[(m-1)/d]}{k} - \binom{n/d-[m/d]}{k}\right).$$

Proof. The are immediate from equations (2) and (3), Theorem 3, and the obvious facts that

$$U(m,n) = \Phi([1,n],n) - \Phi([1,m-1],n) - \Phi([m+1,n],n)$$

and

$$U_k(m,n) = \Phi_k([1,n],n) - \Phi_k([1,m-1],n) - \Phi_k([m+1,n],n).$$

□

3. Asymptotic Estimates

Theorem 5. *Let p be the smallest prime divisor of n in the interval $[1,m]$. Then we have the following inequalities:*

$$(a)\quad 0 \le 2^m - 2^{[m/p]} - \Phi([1,m],n) \le m2^{[m/p]}.$$

$$(b) \quad 0 \leq \binom{m}{k} - \binom{[m/p]}{k} - \Phi_k([1,m],n) \leq m\binom{[m/p]}{k}.$$

Proof. (a) The number $2^m - 2^{[m/p]}$ of sets consisting of multiples of p in $[1,m]$ is an upper bound for $\Phi([1,m],n)$. As to the lower bound we have

$$\Phi([1,m],n) - (2^m - 2^{[m/p]}) = \sum_{\substack{d|n \\ d>p}} \mu(d) 2^{[m/d]} \leq m 2^{[m/p]}.$$

(b) The number $\binom{m}{k} - \binom{[m/p]}{k}$ of sets consisting of multiples of p in $[1,m]$ and having cardinality k is an upper bound for $\Phi_k([1,m],n)$. As to the lower bound we find

$$\begin{aligned} \Phi_k([1,m],n) &= \binom{m}{k} - \binom{[m/p]}{k} + \sum_{\substack{d|n \\ d>p}} \mu(d) \binom{[m/d]}{k} \\ &\geq \binom{m}{k} - \binom{[m/p]}{k} - \sum_{\substack{d|n \\ d>p}} \binom{[m/d]}{k} \\ &\geq \binom{m}{k} - \binom{[m/p]}{k} - m\binom{[m/p]}{k}. \end{aligned}$$

□

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References

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