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THE FOURIER TRANSFORM OF FUNCTIONS OF THE GREATEST COMMON DIVISOR

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Abstract

We study discrete Fourier transformations of functions of the greatest common divisor: $\sum_{k=1}^{n} f((k,n)) \cdot \exp(-2\pi ikm/n)$. Euler's totient function: $\varphi(n) = \sum_{k=1}^{n} (k,n) \cdot \exp(-2\pi ik/n)$ is an example. The greatest common divisor $(k,n) = \sum_{m=1}^{n} \exp(2\pi ikm/n) \cdot \sum_{d|n} \frac{c_d(m)}{d}$ is another result involving Ramanujan's sum $c_d(m)$. The last equation, interestingly, can be evaluated for k in the complex domain.

1. Introduction

This article is a study of discrete Fourier transformations of functions of the greatest common divisor (gcd). A special "Fourier transform," the gcd-sum function $P(n) := \sum_{k=1}^{n} (k,n)$, was investigated by S.S. Pillai in 1933 [1] (and therefore in the literature called Pillai's arithmetical function) followed by generalizations and analogues thereof: $\sum_{k=1}^{n} f((k,n))$ [2-8], where $f(n)$ is any arithmetic function.

Let $m \in \mathbb{Z},\ n \in \mathbb{N}$. For an arithmetic function $f : \mathbb{N} \to \mathbb{C}$, let

$$F_f(m,n) := \sum_{k=1}^{n} f((k,n)) \cdot \exp(-2\pi ikm/n)$$

denote the discrete Fourier transform of $f((k,n))$, where (k,n) is the gcd of k and n. Further, let

$$c_n(m) := \sum_{\substack{k=1 \\ (k,n)=1}}^{n} \exp(2\pi ikm/n) \tag{1}$$

denote Ramanujan's sum. Note that $c_n(m) = c_n(-m)$ for any $m \in \mathbb{N}$ (by complex conjugation since $c_n(m) \in \mathbb{R}$ for every $m \in \mathbb{Z}$) and $c_n(0) = \varphi(n)$ is Euler's totient function. For two arithmetic functions $f_1, f_2 : \mathbb{N} \to \mathbb{C}$ let $(f_1 * f_2)(n) := \sum_{d|n} f_1(d) \cdot f_2(n/d)$ denote the Dirichlet convolution, $\delta(n)$ the identity element for the Dirichlet convolution (i.e., $\delta(1) = 1$ and $\delta(n) = 0$ for every $n > 1$), $\mu(n)$ the Möbius function and $id(n) := n$ for every $n \in \mathbb{N}$.

Using this notation, the following easily proven theorem gives some already known and, until now unknown, relations for arithmetic functions and trigonometric relations.

Theorem. Let $f : \mathbb{N} \to \mathbb{C}$ be any arbitrary arithmetic function. Then
i) the discrete Fourier transform of $f((k, n))$ is given for every $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ by

$$F_f(m, n) = (f * c_\bullet(m))(n); \tag{2}$$

ii) the inverse Fourier transform thereof for every $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ by

$$f((k, n)) = \frac{1}{n} \sum_{m=1}^{n} (f * c_\bullet(m))(n) \cdot \exp(2\pi ikm/n) \tag{3}$$

Before proving the above theorem, we start with three motivating examples.

Example 1. Let $f(n) = id(n) := n$ in (3). Then

$$(k, n) = \sum_{m=1}^{n} \exp(2\pi ikm/n) \cdot \sum_{d|n} \frac{c_d(m)}{d} \tag{4}$$

gives a function for the gcd. Note that the right-hand side can be evaluated for k in the complex domain (for instance $(1/2, 3) = -5/3 \pm 2 \cdot i/\sqrt{3}$), although its interpretation for non-integer values is unclear. The function (4) is holomorphic everywhere on the whole complex plane and therefore for every $n \in \mathbb{N}$ is an entire function. Moreover, for fixed n it is n-periodic in the variable $k \in \mathbb{C}$ and not distributive, i.e., $(k \cdot j, n \cdot j) = j \cdot (k, n)$ does not hold in general, since $1 = (1, 6) \neq 2 \cdot (1/2, 3)$.

Interestingly, Keith Slavin [9] published an equation for the gcd that can be evaluated for complex k as well: $(k, n) = \log_2 \prod_{m=0}^{n-1} (1 + \exp(-2\pi ikm/n))$ for odd $n \geqslant 1$ (evaluating the last for $(1/2, 3)$ gives $\approx 1.79248 - 2.2661801 \cdot i$, not the same value, since Slavin's equation is not an entire function and is not defined for even n).

Example 2. Let $f(n) = 1$ and $m = 1$ in (2); then because of $\sum_{k=1}^{n} \exp(2\pi ik/n) = \delta(n)$, the well known relation $(c_\bullet(1) * 1)(n) = \delta(n) \Leftrightarrow$

$$c_n(1) = \mu(n) \tag{5}$$

for the Möbius function follows.

Example 3. Let $f(n) = id(n) := n$ and $m = 1$ in (2). Then with (5) a nice relation for Euler's totient function follows: $\sum_{k=1}^{n} (k,n) \cdot \exp(-2\pi ik/n) = (id*\mu)(n) =: \varphi(n)$; and splitting up into real and imaginary parts gives the trigonometric relations:

$$\sum_{k=1}^{n} (k,n) \cdot \cos(2\pi k/n) = \varphi(n) \quad \text{and} \quad \sum_{k=1}^{n} (k,n) \cdot \sin(2\pi k/n) = 0.$$

Proof of the theorem. We prove this in three small steps.

`Step A`: Let $f(n) = \delta(n)$. Then (2) gives the definition (1) of the (complex conjugated) Ramanujan sum.

`Step B`: Now let $f(n) = \delta_j(n) := \begin{cases} 1 & \text{for } \ j = n \\ 0 & \text{else} \end{cases}$ and $j > 1$. For $n \equiv 0 \,(\text{mod } j)$ we have $\sum_{k=1}^{q\cdot j} \delta_j((k, q \cdot j)) \cdot \exp(\frac{\pm 2\pi ikm}{q\cdot j}) = \sum_{k=1}^{q} \delta_j((k \cdot j, q \cdot j)) \cdot \exp(\frac{\pm 2\pi ikm\cdot j}{q\cdot j})$, since j does not divide $(k \cdot j \pm l, q\cdot j)$ for $1 \leqslant l < j$. Further, because of the distributive law – $(k\cdot j, q\cdot j) = j\cdot(k,q)$ – of the gcd, the last sum equals $\sum_{k=1}^{q} \delta_j(j \cdot (k,q)) \cdot \exp(\pm 2\pi ikm/q) = \sum_{k=1}^{q} \delta((k,q)) \cdot \exp(\pm 2\pi ikm/q)$. Comparing this result with Step A gives $\sum_{k=1}^{n} \delta_j((k,n)) \cdot \exp(\pm 2\pi ikm/n) = c_{n/j}(m)$. Otherwise, for $n \not\equiv 0 \,(\text{mod } j)$ it is obvious that $\sum_{k=1}^{n} \delta_j((k,n)) \cdot \exp(\pm 2\pi ikm/n) = 0$. Because the Dirichlet convolution of $\delta_j(n)$ with any arithmetic function $g(n)$ gives

$$(g * \delta_j)(n) = \begin{cases} g(n/j) & \text{for } n \equiv 0(j) \\ 0 & \text{else} \end{cases}$$

we have

$$\sum_{k=1}^{n} \delta_j((k,n)) \cdot \exp(-2\pi ikm/n) = (\delta_j * c_{\bullet}(m))\,(n). \tag{6}$$

`Step C`: Now let f be any arithmetic function. Multiplying (6) with $f(j)$ and summing up $1 \leqslant j \leqslant n$ gives finally (2) and immediately (3) by the inverse Fourier transform thereof. □

We can also give a short proof of the theorem.

Short proof of the theorem (2). By grouping the terms according to the values $(k,n) = d$, where $d\,|n$, $k = dj$, $(j, n/d) = 1$, $1 \leqslant j \leqslant n/d$, we have

$$F_f(m,n) = \sum_{d|n} f(d) \sum_{\substack{1\leqslant j\leqslant n/d \\ (j,n/d)=1}} \exp(-2\pi ijm/(n/d)) = \sum_{d|n} f(d) \cdot c_{n/d}(m) = (f * c_{\bullet}(m))(n).$$

□

Corollary. If f is a multiplicative function, then $F_f(m,n)$ is also multiplicative in variable n.

Proof. The Ramanujan sum $c_n(m)$ is multiplicative in n and the Dirichlet convolution preserves the multiplicativity of functions. □

Here are some more examples.

Example 4. Let $f(n) = id(n) := n$ and $m = 0$ in (2). Then the well-known Pillai sum [1]:

$$\sum_{k=1}^{n}(k,n) = (\varphi * id)(n)$$

follows.

Example 5. Let $m = 0$ in (2). Then $\sum_{k=1}^{n} f((k,n)) = (\varphi * f)(n)$ gives the generalization thereof [2-8], already known to E. Cesàro in 1885 [6].

Example 6. Let $m = 1$, let $f(n) = \omega(n)$ be the number of distinct prime factors of n, and let $\mathrm{X}_{Primes}(n)$ be the characteristic function of the primes. Then because of

$$\omega(n) = \sum_{\substack{p|n\\ p\in\mathbb{P}}} 1 = \sum_{d|n} \mathrm{X}_{Primes}(d) \Leftrightarrow \mathrm{X}_{Primes}(n) = (\omega * \mu)(n),$$

part (2) of the theorem with (5) gives

$$\sum_{k=1}^{n} \omega((k,n)) \cdot \exp(-2\pi i k/n) = \mathrm{X}_{Primes}(n).$$

Analogously let $m = 1$ and $f(n) = \Omega(n)$ the total number of prime factors of n (counting multiple factors multiple times) and $\mathrm{X}_{PrimePower}(n)$ the characteristic function of prime powers, then because of

$$\Omega(n) = \sum_{\substack{p^k|n\\ p\in\mathbb{P};\, k\in\mathbb{N}}} 1 = \sum_{d|n} \mathrm{X}_{PrimePower}(d) \Leftrightarrow \mathrm{X}_{PrimePower}(n) = (\Omega * \mu)(n),$$

part (2) of the theorem with (5) gives

$$\sum_{k=1}^{n} \Omega((k,n)) \cdot \exp(-2\pi i k/n) = \mathrm{X}_{PrimePower}(n).$$

Example 7. Let $m = 1$ and let $f(n) = \sigma_z(n) := \sum_{d|n} d^z$ be the divisor function for any $z \in \mathbb{C}$. Then (2) with (5) gives $\sum_{k=1}^{n} \sigma_z((k,n)) \cdot \exp(-2\pi i k/n) = (\sigma_z * \mu)(n) = n^z$; and for $z \in \mathbb{R}$, splitting up into real and imaginary parts gives the trigonometric relations

$$\sum_{k=1}^{n} \sigma_z((k,n)) \cdot \cos(2\pi k/n) = n^z \quad \text{and} \quad \sum_{k=1}^{n} \sigma_z((k,n)) \cdot \sin(2\pi k/n) = 0.$$

Example 8. Let $r(n) = \sum_{n_1^2+n_2^2=n} 1$ be the number of ways that n can be expressed as the sum of two squares, and denote the Dirichlet character:

$$\chi_{(\overline{1,0,-1,0})}(n) := \frac{i^{n-1}+(-i)^{n-1}}{2} = \begin{cases} 1 & \text{for } n \equiv 1(4) \\ -1 & \text{for } n \equiv 3(4) \\ 0 & \text{for } n \equiv 2(4). \end{cases}$$

Then (2) with (5) gives $\sum_{k=1}^{n} r((k,n)) \cdot \exp(-2\pi ik/n) = (r * \mu)(n) = 4 \cdot \chi_{(\overline{1,0,-1,0})}(n)$.

Example 9. Let $f(n) = \log(n)$ and $m = 0$. Then (2) gives $\log \prod_{k=1}^{n}(k,n) = (\log * \varphi)(n)$, whereof $\prod_{k=1}^{n} \sqrt[n]{(k,n)} = \exp \frac{(\log*\varphi)(n)}{n} = \prod_{p_j^{\alpha_j}|n} p_j^{\frac{1-p_j^{-\alpha_j}}{p_j-1}}$ for $n = \prod_{p_j \in \mathbb{P}} p_k^{\alpha_j}$ a multiplicative arithmetic function follows (see [10]). A generalization thereof for $f(n) = \log(h(n))$ and $m = 0$, where $h : \mathbb{N} \to \mathbb{C} \backslash \{z \cdot \mathbb{R}^+\}$ is any arithmetical function for a fixed $0 \neq z \in \mathbb{C}$, gives the general gcd-product function:

$$\prod_{k=1}^{n} h((k,n)) = \begin{cases} 0 & \text{if any factor is 0} \\ \exp((\log(h) * \varphi)(n)) & \text{else} \end{cases}$$

as investigated in [10].

Example 10. Let $f(n) = \log(n)$ and $m = 1$. Then (2) with (5) gives the Mangoldt function

$$\sum_{k=1}^{n} \log((k,n)) \cdot \exp(-2\pi ik/n) = (\log * \mu)(n) =: \Lambda(n).$$

Example 11. Let $f(n) = \delta(n)$ and $k \in \mathbb{N}$. Then (3) gives

$$\delta((k,n)) = \frac{1}{n} \cdot \sum_{m=1}^{n} c_n(m) \cdot \exp(2\pi ikm/n).$$

This means that k and n are not coprime $\Leftrightarrow \sum_{m=1}^{n} c_n(m) \cdot \exp(2\pi ikm/n) = 0 \Leftrightarrow$ the n-vectors $(\exp(2\pi ikm/n))_{m=1..n}$ and $(c_n(m))_{m=1..n}$ are orthogonal. This, as a definition, could allow a generalization of the coprime concept for $k \in \mathbb{C}$.

Example 12. Let $f(n) = 1$ in (2) and $\eta_n(m) := \sum_{k=1}^{n} \exp(-2\pi ikm/n)$ then $\eta_n(m) = (1 * c_\bullet(m))(n)$ if and only if

$$c_n(m) = (\mu * \eta_\bullet(m))(n), \tag{7}$$

a well known relation for Ramanujan's sum follows. Since $\eta_n(m) = n$ for $n\,|m$ otherwise zero, $c_n(m) \in \mathbb{R}$ for every $m \in \mathbb{Z}$ as already mentioned in the introduction. Using (7), the gcd-function (4) can be transformed to

$$(k,n) = \sum_{m=1}^{n} \frac{(\varphi * \eta_\bullet(m))\,(n)}{n} \cdot \exp(2\pi ikm/n).$$

Example 13. Let $m = 1$ and $f(n) = n^z$ for any $z \in \mathbb{C}$. Then (2) with (5) gives

$$\sum_{k=1}^{n} (k,n)^z \cdot \exp(\,-\,2\pi ik/n) = \sum_{d|n} d^z \cdot \mu(n/d) \;=: J_z(n),$$

the Jordan function $J_z(n)$, a generalization of Euler's totient function [11].

Note that the relations in the Examples $6, 8, 9, 10$, and 13 (the last for $z \in \mathbb{R}$), when split up into real and imaginary parts, give (analogously to Examples 3 and 7) trigonometric relations as well.

2 Summary

The table below summarizes the examples concerning theorem (2). The number in the last row denotes the example number in the manuscript. White spots, e.g., $F_{\log(h)}(1,n)$, in the landscape of this table might be of further scientific interest.

f	$F_f(0,n) = (f * \varphi)(n)$	$F_f(1,n) = (f * \mu)(n)$	$F_f(m,n) = (f * c_\bullet(m))(n)$	Ex.
1	$id(n)$	$\delta(n)$	$\eta_n(m)$	2, 12
$id(n)$	Pillai sum $P(n)$	$\varphi(n)$	$(\eta_\bullet(m) * \varphi)(n)$	3,4
$f(n)$	"Cesàro sum"		Theorem (2)	5
$\log(h(n))$	$\log \prod_{k=1}^{n} h((k,n))$			9
$\log(n)$	$(id * \Lambda)(n)$	$\Lambda(n)$	$(\eta_\bullet(m) * \Lambda)(n)$	10
$r(n)$	$4 \cdot (id * \chi_{\overline{(1,0,-1,0)}})(n)$	$4 \cdot \chi_{\overline{(1,0,-1,0)}}(n)$	$4 \cdot (\eta_\bullet(m) * \chi_{\overline{(1,0,-1,0)}})(n)$	8
$\omega(n)$	$(id * \mathrm{X}_{Primes})(n)$	$\mathrm{X}_{Primes}(n)$	$(\eta_\bullet(m) * \mathrm{X}_{Primes})(n)$	6
$\Omega(n)$	$(id * \mathrm{X}_{PrimePower})(n)$	$\mathrm{X}_{PrimePower}(n)$	$(\eta_\bullet(m) * \mathrm{X}_{PrimePower})(n)$	6
$\sigma_z(n)$ $z \in \mathbb{C}$	$n \cdot \sum_{d\|n} d^{z-1}$ $F_{\sigma_0}(0,n) = \sigma_1(n)$	n^z	$\sum_{d\|n} \eta_{n/d}(m) \cdot d^z$	7
n^z $z \in \mathbb{C}$	$(id * J_z)(n)$	$J_z(n)$	$(\eta_\bullet(m) * J_z)(n)$	13

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