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ON A RELATION BETWEEN THE RIEMANN ZETA FUNCTION AND THE STIRLING NUMBERS

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Abstract

Let $\zeta(z)$ be the Riemann zeta function and $s(k,n)$ the Stirling numbers of the first kind. Shen proved the identity $\zeta(n+1) = \sum_{k=n}^{\infty} \frac{s(k,n)}{k\cdot k!} \qquad (1 \le n \in \mathbb{Z})$. We give a short proof by elementary methods.

1. The Result

Let $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$ be the Riemann zeta function, and let $s(k,n)$ denote the Stirling numbers of the first kind, which are defined by

$$s(0,0) = 1\ , \quad s(k,0) = s(0,n) = 0 \quad (k \ne 0, n \ne 0), \tag{1}$$

$$s(k+1,n+1) = s(k,n) + k \cdot s(k,n+1) \quad (k \in \mathbb{Z}, n \in \mathbb{Z}). \tag{2}$$

Shen [2] proved the following identity, which shows an interesting relation between $\zeta(n)$ and $s(k,n)$ by using Gauss's summation theorem of the hypergeometric series:

$$\zeta(n+1) = \sum_{k=n}^{\infty} \frac{s(k,n)}{k \cdot k!} \qquad (1 \le n \in \mathbb{Z}). \tag{3}$$

In this paper we give a short proof of (3) by elementary methods.

First we show the outline of the proof. We denote

$$(k)_{-n} = \frac{1}{k(k+1)(k+2)\cdots(k+n-1)} \qquad (1 \le n \in \mathbb{Z}, 1 \le k \in \mathbb{Z})$$

and put $\xi(n) = \sum_{k=1}^{\infty} (k)_{-n}$. Then we have

$$\xi(n+1) = \sum_{k=1}^{\infty} \frac{1}{n}\{(k)_{-n} - (k+1)_{-n}\} = \frac{1}{n \cdot n!}. \tag{4}$$

Proposition. For $1 \leq x \in \mathbb{R}$ and $0 \leq n \in \mathbb{Z}$ we have

$$x^{-(n+1)} = \sum_{k=n}^{\infty} s(k,n) \cdot (x)_{-(k+1)}. \tag{5}$$

By this proposition we have

$$\zeta(n+1) = \sum_{m=1}^{\infty} m^{-(n+1)} = \sum_{m=1}^{\infty}\sum_{k=n}^{\infty} s(k,n) \cdot (m)_{-(k+1)}.$$

Since it is a convergent series with positive terms, we can change the order of summation. Noting (4), we obtain

$$\zeta(n+1) = \sum_{k=n}^{\infty} s(k,n)\xi(k+1) = \sum_{k=n}^{\infty} \frac{s(k,n)}{k \cdot k!}.$$

Now we prove the proposition above. We need the following result [1, Section 54, p. 160].

Lemma. For fixed $1 \leq k \in \mathbb{Z}$, we have $\lim_{N\to\infty} \frac{s(N,k)}{N!}$.

We prove (5) by induction on n. The case $n = 0$, which is $x^{-1} = \sum_{k=0}^{\infty} s(k,0) \cdot (x)_{-(k+1)}$, follows from (1) and the definition of $(x)_{-k}$. Now let N be a sufficiently large integer. From (2) we have

$$\begin{aligned}
\sum_{k=n}^{N} s(k,n) \cdot (x)_{-(k+1)} &= \sum_{k=n}^{N} (s(k+1,n+1) - k \cdot s(k,n+1)) \cdot (x)_{-(k+1)} \\
&= \sum_{k=n}^{N} s(k+1,n+1) \cdot (x)_{-(k+1)} - \sum_{k=n}^{N} k \cdot s(k,n+1) \cdot (x)_{-(k+1)} \\
&= \sum_{k=n}^{N} s(k+1,n+1) \cdot (x)_{-(k+2)} \cdot (x+k+1) - \sum_{k=n}^{N} k \cdot s(k,n+1) \cdot (x)_{-(k+1)} \\
&= \sum_{k=n+1}^{N+1} s(k,n+1) \cdot (x)_{-(k+1)} \cdot (x+k) - \sum_{k=n+1}^{N} k \cdot s(k,n+1) \cdot (x)_{-(k+1)}
\end{aligned}$$

(Note $s(n, n+1) = 0$.)

$$= x \cdot \sum_{k=n+1}^{N+1} s(k,n+1) \cdot (x)_{-(k+1)} + s(N+1,n+1) \cdot (x)_{-(N+2)} \cdot (N+1).$$

Noting $x \geq 1$, we obtain

$$\begin{aligned}
s(N+1,n+1) \cdot (x)_{-(N+2)} \cdot (N+1) &\leq s(N+1,n+1) \cdot \frac{N+1}{1 \cdot 2 \cdots (N+2)} \\
&= \frac{s(N+1,n+1)}{(N+1)!} \cdot \frac{N+1}{N+2},
\end{aligned}$$

which tends to 0 as $N \to \infty$ because of the lemma above. Therefore as $N \to \infty$ we obtain by the induction assumption

$$x^{-(n+1)} = x \cdot \sum_{k=n+1}^{\infty} s(k, n+1) \cdot (x)_{-(k+1)}.$$

Hence we have complete the proof of (5).

Remark. Let $S(n,k)$ be the Stirling number of the second kind and denote $(x)_n = x(x-1)\cdots(x-n+1)$ for $1 \le n \in \mathbb{Z}$. Equation (5) can be viewed as the negative n case of the well-known identity

$$x^n = \sum_{k=0}^{n} S(n,k) \cdot (x)_k \quad (0 \le n \in \mathbb{Z}).$$

References

[1] C. Jordan, Calculus of Finite Differences, 3rd ed., Chelsea, 1965.

[2] L. C. Shen, Remarks on some integrals and series involving the Stirling numbers and $\zeta(n)$, *Trans. Amer. Math. Soc.* **347** (1995), 1391-1399.