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CHRISTOFFEL WORDS AND MARKOFF TRIPLES

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Abstract
We construct a bijection between Markoff triples and Christoffel words.

1. Introduction

A Markoff triple is a triple of natural integers a, b, c which satisfies the Diophantine Equation

$$a^2 + b^2 + c^2 = 3abc.$$

This equation has been introduced in the work of Markoff [13], where he finished his earlier work [12] on minima of quadratic forms and approximation of real numbers by continued fractions. He studies for this certain bi-infinite sequences and shows that they must be periodic; each of these sequences is a repetition of the same pattern, which turns out to be one of the words introduced some years before by Christoffel [6]; Markoff was apparently not aware of this work of Christoffel. These words were called Christoffel words much later in [1] and may be constructed geometrically, see [14], [7], [15], [1], [5], and have also many different interpretations, in particular in the free group on two generators, see [11]. See[3] for the theory of Christoffel words.

In this note, we describe a mapping which associates to each Christoffel word a Markoff triple and show that this mapping is a bijection. The construction and the result are a variant of a theorem of Harvey Cohn [7], see also [8, Chapter 7] and [16]. Cohn uses a certain representation of the free group F_2 into $\mathrm{SL}_2(\mathbb{Z})$ in order to construct all Markoff triples, by restricting the representation to primitive elements of F_2 (he does not prove unicity). We use another representation, by positives matrices (naturally deduced from the continued fractions construction of Markoff) and prove unicity of this representation of the Markoff triples.

As does Cohn, we use the *Fricke identities* in order to prove that the Diophantine equation is satisfied. We use also the tree construction of the Christoffel words, see [5] and [2], and moreover the arithmetic recursive construction of the solutions of the previous Diophantine equation, see [13], [9], [8], [19], [16].

During the refereeing process, I became aware of the article of Bombieri [4]; he

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gives a new presentation of the whole Markoff theory, and our main result is equivalent to his Theorem 26, once are known several facts on Christoffel words, especially the equivalence between Christoffel words and positive primitive elements of the free group F_2, see, e.g., [7], [15], [11] or [3, Chapter 5]. The proof given here takes advantage of the theory of Sturmian words, which has close connections to the theory of Markoff, see [17], [18], [10], [3].

2. Results

We consider *lattice paths*, which are consecutive elementary steps in the x, y-plane; each *elementary step* is a segment $[(a, b), (a+1, b)]$ or $[(a, b), (a, b+1)]$, with $a, b \in \mathbf{Z}$.

Let p, q be relatively prime integers. Consider the segment from $(0, 0)$ to (p, q) and the lattice path from $(0, 0)$ to (p, q) located below this segment and such that the polygon delimited by the segment and the paths has no interior integer point.

The *Christoffel word of slope q/p* is the word in the free monoid $\{x, y\}^*$ coding the above path, where x (resp. y) codes an horizontal (resp. vertical) elementary step. See the figure, where is represented the path with $(p, q) = (7, 3)$ corresponding to the Christoffel word of slope $3/7$.

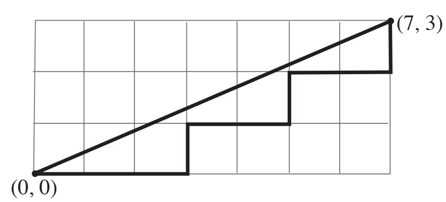

Figure 1: The Christoffel word $xxxyxxyxxy$ of slope $3/7$.

Note that the definition includes the particular cases $(p, q) = (1, 0)$ and $(0, 1)$, corresponding to the Christoffel words x and y. All other Christoffel words will be called *proper*.

Each proper Christoffel word w has a unique *standard factorization* $w = w_1 w_2$; it is obtained by cutting the path corresponding to w at the integer point closest to the segment. In the figure, the standard factorization is given by $w_1 = xxxyxxy$, $w_2 = xxy$. The words w_1 and w_2 are then Christoffel words.

Define the homomorphism μ from the free monoid $\{x, y\}^*$ into $SL_2(\mathbf{Z})$, defined by $\mu x = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\mu y = \left(\begin{smallmatrix} 5 & 2 \\ 2 & 1 \end{smallmatrix}\right)$.

A *Markoff triple* is a multiset $\{a, b, c\}$ of positive integers which satisfies the equation $a^2 + b^2 + c^2 = 3abc$. The triple is *proper* if a, b, c are distinct. It is classical that the only improper Markoff triples are $\{1, 1, 1\}$ and $\{1, 1, 2\}$, see [8, Chapter 2].

Theorem 1. *A set $\{a, b, c\}$ is a proper Markoff triple if and only if it is equal to*

$\{1/3Tr(\mu w_1), 1/3Tr(\mu w_2), 1/3Tr(\mu w)\}$ *for some unique proper Christoffel word* w *with standard factorization* $w = w_1 w_2$.

It is known that for each Christoffel word w, one has $1/3Tr(\mu w) = \mu(w)_{1,2}$, see [17, Lemma 3.2] or [3, Lemma 8.7]. We use this fact in the proof below.

Proof. 1. Each couple (w_1, w_2) forming the standard factorization of a proper Christoffel word is obtained by applying iteratively the rules $(u, v) \to (u, uv)$ or $(u, v) \to (uv, v)$ starting from the couple (x, y), see [5, Proposition 2] or [2, Lemma 7.3]. As a consequence, an easy computation shows inductively that $w_1 w_2 w_1^{-1} w_2^{-1}$ is conjugate to $xyx^{-1}y^{-1}$ in the free group F_2 generated by x and y.

Now, let w be some proper Christoffel word with standard factorization $w = w_1 w_2$. Let $a = 1/3Tr(\mu w_1), b = 1/3Tr(\mu w_2), c = 1/3Tr(\mu w)$. We use the Fricke identity

$$Tr(A)^2 + Tr(B)^2 + Tr(AB)^2 = Tr(ABA^{-1}B^{-1}) + 2 + Tr(A)Tr(B)Tr(AB)$$

for any $A, B \in SL_2(\mathbf{Z})$. We take $A = \mu(w_1)$, $B = \mu(w_2)$ and therefore $AB = \mu(w)$. Thus $9a^2 + 9b^2 + 9c^2 = 27abc$ (and we are done), provided $Tr(ABA^{-1}B^{-1}) = -2$. Since $w_1 w_2 w_1^{-1} w_2^{-1}$ is conjugate to $xyx^{-1}y^{-1}$, it suffices to show the identity $Tr(\mu x \mu y \mu x^{-1} \mu y^{-1}) = -2$. Now the matrix $\mu x \mu y \mu x^{-1} \mu y^{-1}$ is equal to $\left(\begin{smallmatrix} 11 & -24 \\ -6 & -13 \end{smallmatrix}\right)$, which shows what we want.

2. It remains to show that a, b, c are distinct. In the special case where $w = xy$, this is seen by inspection. Furthermore, $\mu(w)_{12} = \mu(w_1)_{11}\mu(w_2)_{12} + \mu(w_1)_{12}\mu(w_2)_{22}$, which implies that $\mu(w)_{12} > \mu(w_1)_{12}, \mu(w_2)_{12}$, since the matrices have positive coefficients. Thus, by the remark before the proof, $c > a, b$. Now, since we assume $w \neq xy$ and by the tree construction recalled at the beginning of the proof, w_1 is a prefix of w_2 or w_2 is a suffix of w_1. Then, similarly, $a < b$ or $a > b$ respectively.

Observe that we obtain as a byproduct of the previous calculation, using the remark before the proof, that, if u, v are Christoffel words such that u is a proper prefix (resp. suffix) of v, then $Tr(\mu u) < Tr(\mu v)$.

3. Let $\{a, b, c\}$ be a proper Markoff triple and assume that $a < b < c$. We shall prove existence of the Christoffel word by using induction on the sum $a + b + c$. Observe that $\{a, b, 3ab - c\}$ is a Markoff triple and $0 < 3ab - c < b$, as shows a simple computation (or see [13], [9], [8], [19]). Hence $a + b + 3ab - c$ is smaller than $a + b + c$. Thus, if $a + b + c$ is minimal, the triple $\{a, b, 3ab - c\}$ must be improper. In this case, since $a \neq b$, we have $\{a, b, 3ab - c\} = \{1, 1, 2\}$ and $a = 1$ and $b = 2$ or conversely. Thus $3ab - c = 1$ and this implies $c = 3.2.1 - 1 = 5$ and $\{a, b, c\}$ corresponds to the Christoffel word xy.

Suppose now that the triple $\{a, b, 3ab - c\}$ is proper. By induction, the multiset $\{a, b, 3ab - c\}$ is equal to $\{1/3Tr(\mu w_1), 1/3Tr(\mu w_2), 1/3Tr(\mu w)\}$ for some Christoffel word w with standard factorization $w = w_1 w_2$. By the second part of the proof, $b = 1/3Tr(\mu w)$, since both numbers are the maximum of their multiset. Then either (i) $a = 1/3Tr(\mu w_1)$ and $3ab - c = 1/3Tr(\mu w_2)$, or (ii) $a = 1/3Tr(\mu w_2)$ and $3ab - c = 1/3Tr(\mu w_1)$.

In case (i), we have $c = 1/3Tr(\mu(w_1^2 w_2))$; indeed, for A, B in $SL_2(\mathbf{Z})$,

$$Tr(A^2B) + Tr(B) = Tr(A)Tr(AB), \tag{1}$$

hence $Tr(\mu(w_1^2 w_2)) = Tr(\mu w_1)Tr(\mu w) - Tr(\mu w_2) = 3a.3b - 3(3ab - c) = 3c$.

In case (ii), we have $c = Tr(\mu(w_1 w_2^2))$; indeed,

$$Tr(AB^2) + Tr(A) = Tr(AB)Tr(B), \tag{2}$$

hence $Tr(\mu(w_1 w_2^2)) = Tr(\mu w)Tr(\mu w_2) - Tr(\mu w_1) = 3b.3a - 3(3ab - c) = 3c$.

Thus in case (i), $\{a, b, c\}$ corresponds to the Christoffel word $w_1^2 w_2$ (with standard factorization $w_1.w_1w_2$) and in case (ii) to the Christoffel word $w_1 w_2^2$ (with standard factorization $w_1w_2.w_2$).

4. Concerning unicity, suppose that the proper Markoff triple $\{a, b, c\}$ with $a < b < c$ may be obtained from the two Christoffel words u and v with standard factorization $u = u_1u_2$ and $v = v_1v_2$. We may assume that they are both distinct from xy; indeed, a direct verification shows that the triple corresponding to xy, which is $(1, 2, 5)$, can be obtained only from the Christoffel word xy; indeed, 5 is the smallest c such that $\{a, b, c\}$ is a proper Markoff triple with greatest element c, since for any standard factorization (u, v) of a Christoffel word, with $(u, v) \neq (x, y)$, one has $Tr(\mu(uv)) < Tr(\mu(xy))$.

Then, assuming that $(u_1, u_2) \neq (x, y)$ and $(v_1, v_2) \neq (x, y)$, we see by the tree construction of Christoffel words, that u_1 is a prefix of u_2 or u_2 is a suffix of u_1, and similarly for v_1, v_2. Hence, we are by symmetry reduced to two cases: (i) u_1 is a prefix of u_2 and v_1 is a prefix of v_2, or (ii) u_1 is a prefix of u_2 and v_2 is a suffix of v_1.

In Case (i), u_2 (resp. v_2) has the standard factorization $u_2 = u_1u_2'$ (resp. $v_2 = v_1v_2'$). Moreover $a = 1/3Tr(\mu u_1), b = 1/3Tr(\mu u_2), c = 1/3Tr(\mu u)$ as follows from the second part. Then the Markoff triple corresponding to the Christoffel word u_1u_2' is equal to $\{a, 3ab - c, b\}$. Indeed, using Equation (1), $Tr(\mu u_2') = Tr(\mu u_1)Tr(\mu(u_1u_2')) - Tr(\mu(u_1^2u_2')) = 3a.3b - 3c$, because $u_1u_2' = u_2$ and $u_1^2u_2' = u$. Similarly, the Markoff triple corresponding to the Christoffel word v_1v_2' is also equal to $\{a, 3ab - c, b\}$. Thus, by induction, the Christoffel words u_1u_2' and v_1v_2' are equal, together with their standard factorization, hence $u = v$.

In Case (ii), by the same calculation, the Markoff triple corresponding to the Christoffel word u_1u_2' is equal to $\{a, 3ab - c, b\}$, where the standard factorization of u_2 is u_1u_2'. Symmetrically, v_1 has the standard factorization $v_1 = v_1'v_2$ and the Markoff triple corresponding to the Christoffel word $v_1'v_2$ is also equal to $\{a, 3ab - c, b\}$. Indeed, we note, as follows from the second part, that $a = 1/3Tr(\mu v_2)$, $b = 1/3Tr(\mu v_1)$, $c = 1/3Tr(\mu v)$. Then, using Equation (2), we have $Tr(\mu v_1') = Tr(\mu(v_1'v_2))Tr(\mu v_2) - Tr(\mu(v_1'v_2^2)) = 3b.3a - 3c$, since $v_1'v_2 = v_1$ and $v_1'v_2^2 = v$. Thus, by induction, the Christoffel words u_1u_2' and $v_1'v_2$ are equal, together with their standard factorization. Hence $u_1 = v_1'$ and $u_2' = v_2$. Thus $u = u_1u_2 = u_1^2u_2'$ and $v = v_1v_2 = v_1'v_2^2 = u_1{u_2'}^2$. Thus we are reduced to the following particular case, which shows that Case (ii) cannot happen.

5. Let $w = uv$ be the standard factorization of the proper Christoffel word w. Then the Markoff triple corresponding to the Christoffel words u^2v and uv^2 are distinct.

Indeed, disregarding the trivial case $w = xy$, we may assume that u is a prefix of v or v is a suffix of u. In the first case, each entry of the matrix μu is strictly smaller than the corresponding entry of the matrix μv. The same therefore holds for the matrices $\mu u\mu u\mu v$ and $\mu u\mu v\mu v$. Thus $Tr(\mu(u^2v))$ is smaller than $Tr(\mu(uv^2))$, which proves that the Markoff triples are distinct, since so are their greatest elements. The second case is similar. $\square$

It is known that each Markoff number (that is, a member of a Markoff triple) is of the form $1/3Tr(\mu w)$ for some Christoffel word w, see [17, Corollary 3.1] or [3, Theorem 8.4] (it follows also from the previous theorem). It is however not known if the mapping associating to each Christoffel word w the Markoff number $1/3Tr(\mu w)$ is injective. This is equivalent to the *Markoff numbers injectivity conjecture*; see [9, p. 614], [8], [19], [4].

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