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COUNTING DETERMINANTS OF FIBONACCI-HESSENBERG MATRICES USING LU FACTORIZATIONS

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Abstract

We investigate the LU factorizations of Fibonacci-Hessenberg matrices which enable us to find the determinant of a new class of Hessenberg matrices and yield a better computation of determinants of the old ones.

1. Introduction

The Fibonacci sequence is defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1}+F_{n-2}, n \geq 3$. Moreover, a matrix is said to be Hessenberg [2] if all entries above the superdiagonal are zero. For instance, the matrix [2]

$$E_n = \begin{bmatrix} 3 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 3 & 1 & \cdots & \cdots & \vdots \\ 0 & 1 & 3 & 1 & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 1 & 3 & 1 \\ 0 & \cdots & \cdots & \cdots & 1 & 3 \end{bmatrix}$$

is a Hessenberg matrix and its determinant is F_{2n+2}. Furthermore, a Hessenberg matrix is said to be a Fibonacci-Hessenberg matrix [2] if its determinant is in the form $tF_{n-1} + F_{n-2}$ or $F_{n-1} + tF_{n-2}$ for some real or complex number t. In [1] several types of Hessenberg matrices whose determinants are Fibonacci numbers were calculated by using the basic definition of the determinant as a signed sum over the symmetric group.

In this paper, we count the determinants of certain Hessenberg matrices by firstly investigating the feasibility of LU factorizations, i.e., a lower triangular matrix with unit main diagonal and an upper triangular matrix. Furthermore, the factorization is unique. As we know,the determinant of a triangular matrix is the product of its main diagonal entries. Then we can calculate easily the determinant of a given Hessenberg matrix by multiplying the diagonal entries of the corresponding upper triangular

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matrix. Trivially, the LU decomposition is better than the determinant definition approach used in [1].

2. LU Factorizations of Fibonacci-Hessenberg Matrices

Let $A_{n,t}$ be the $n \times n$ Hessenberg matrix in which the superdiagonal entries are 1, all main diagonal entries are 1 except the last one, which is $t+1$, and the entries of each column below the main diagonal alternate 0's and 1's, starting with 0, and t is an indeterminate. Let $B_{n,t}$ be the matrix obtained from $A_{n,t}$ by alternately replacing the 1's on the superdiagonal with $\imath$'s and $-\imath$'s, where $\imath = \sqrt{-1}$. Let $S_{n,t} := (A_{n,t} + B_{n,t})$; obviously $S_{n,t}$ is also a Hessenberg matrix. Let $C_{n,t}$ be the matrix in which the superdiagonal entries are -1's, all main diagonal are 2's except the last one, which is $t+1$, and all entries below the diagonal are 1's. Let D_n be the matrix with 1's on the main diagonal, $\imath$'s on the subdiagonal and superdiagonal, and 0's elsewhere. Let E_n be the matrix with 3's on the main diagonal, 1's on the subdiagonal and superdiagonal, and 0's elsewhere. Let G_n be the Hessenberg matrix in which the superdiagonal entries are 1's, the main diagonal entries are 2's, and the entries of each column below the main diagonal alternate -1's and 1's, starting with -1. Let H_n be the matrix obtained by changing the superdiagonal entries of G_n to -1's. Let $D_{n,t}$ be the matrix obtained from D_n by replacing the lowest superdiagonal $\imath$ with $\imath t$. Let $E_{n,t}$ be the matrix obtained from E_n by replacing the lowest superdiagonal 1 with t. Let $G_{n,t}$ be the matrix obtained from G_n by replacing the lowest diagonal 2 with t+1. Let $H_{n,t}$ be the matrix obtained from H_n by replacing the lowest diagonal 2 with t+1. Let $A_{n,t,s}$ be the matrix obtained from $A_{n,t}$ by replacing $t+1$ with $t+s$ in the lowest main diagonal entry and multiplying the lowest superdiagonal entry by s. Let $B_{n,t,s}$ be the matrix obtained from $B_{n,t}$ by replacing $t+1$ with $t+s$ in the lowest main diagonal entry and multiplying the lowest superdiagonal entry by s. Let $C_{n,t,s}$ be the matrix obtained from $C_{n,t}$ by replacing $t+1$ with $t+s$ in the lowest main diagonal entry and multiplying the lowest superdiagonal entry by s. Let $D_{n,t,s}$ be the matrix obtained from $D_{n,t}$ by replacing the lowest diagonal 1 with s. Let $G_{n,t,s}$ be the matrix obtained from $G_{n,t}$ by replacing $t+1$ with $t+s$ in the lowest main diagonal entry and multiplying the lowest superdiagonal entry by s. Let $H_{n,t,s}$ be the matrix obtained from $H_{n,t}$ by replacing $t+1$ with $t+s$ in the lowest main diagonal entry and multiplying the lowest superdiagonal entry by s. Because Theorem 2.1 and its proof are typical of the unproved theorems that follow, we omit the proofs of the remaining theorems.

Theorem 1 *For $n \geq 5$, $S_{n,t}$ can be factored as $S_{n,t} = L_{n,t}U_{n,t}$, where $L_{n,t} = [L_{n,t}(i,j)]_{1\leq i,j\leq n}$, so*

$$L_{n,t} = \begin{cases} 0 & i<j; \\ 0 & i = 2,4,\cdots,2\lfloor\frac{n}{2}\rfloor, j=1; \\ -\frac{a_j}{a_{j+1}}(1-\imath) & i=j+1,j+3,\cdots,2\lfloor\frac{n}{2}\rfloor, j=3,5,\cdots,2\lfloor\frac{n}{2}\rfloor-1; \\ -\frac{a_j}{2a_{j+1}}(1+\imath) & i=j+1,j+3,\cdots,2\lfloor\frac{n-1}{2}\rfloor+1, j=2,4,\cdots,2\lfloor\frac{n-1}{2}\rfloor; \\ 1 & \textit{otherwise}, \end{cases}$$

and

$$U_{n,t} = [U_{n,t}(i,j)]_{1\leq i,j\leq n} = \begin{cases} 2 & i=j=1; \\ \frac{2a_{j+1}}{a_j} & i=j=2,4,\cdots,2\lfloor\frac{n-1}{2}\rfloor; \\ \frac{a_{j+1}}{a_j} & i=j=3,5,\cdots,2\lfloor\frac{n}{2}\rfloor-1; \\ 2t+\frac{2a_{j+1}}{a_j} & i=j=n, n \textit{ is even}; \\ 2t+\frac{a_{j+1}}{a_j} & i=j=n, n \textit{ is odd}; \\ 1+(-1)^j\imath & i+1=j; \\ 0 & \textit{otherwise}, \end{cases}$$

where

$$< a_q >_{q\geq 2} := \begin{cases} 1 & q=2,3; \\ 2a_{q-1}+a_{q-2} & q=4,6,\cdots; \\ a_{q-1}+a_{q-2} & q=5,7,\cdots. \end{cases}$$

Proof. We use mathematical induction on n, the size of $S_{n,t}$.

(1) The case $n=2$:

$$L_{2,t}U_{2,t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1+\imath \\ 0 & 2t+2 \end{bmatrix} = \begin{bmatrix} 2 & 1+\imath \\ 0 & 2t+2 \end{bmatrix} = S_{2,t}.$$

(2) The case $k \Longrightarrow k+1$ with $k>1$: Assuming $S_{k,t} = L_{k,t}U_{k,t}$ holds, we want to prove $S_{k+1,t} = L_{k+1,t}U_{k+1,t}$. It is sufficient to show that

$$\begin{aligned} S_{k+1,t}(k,k) &= \sum_{m=1}^{k+1} L_{k+1,t}(k,m)U_{k+1,t}(m,k), \\ S_{k+1,t}(k+1,l) &= \sum_{m=1}^{k+1} L_{k+1,t}(k+1,m)U_{k+1,t}(m,l), \quad 1\leq l\leq k, \\ S_{k+1,t}(l,k+1) &= \sum_{m=1}^{k+1} L_{k+1,t}(l,m)U_{k+1,t}(m,k+1), \quad 1\leq l\leq k+1. \end{aligned}$$

First, if k is even, then $k+1$ is odd. So

$$\begin{aligned}\sum_{m=1}^{k+1} L_{k+1,t}(k,m)U_{k+1,t}(m,k) &= -\frac{a_{k-1}}{a_k}(1-\imath)\times(1+\imath)+1\times\frac{2a_{k+1}}{a_k}\\ &= \frac{2a_{k+1}-2a_{k-1}}{a_k} = \frac{2(a_k+a_{k-1})-2a_{k-1}}{a_k}\\ &= 2 = S_{k+1,t}(k,k),\end{aligned}$$

and

$$\sum_{m=1}^{k+1} L_{k+1,t}(k{+}1,m)U_{k+1,t}(m,1) = L_{k+1,t}(k{+}1,1)U_{k+1,t}(1,1) = 1{\times}2 = 2 = S_{k+1,t}(k{+}1,1).$$

Now, letting l be odd and $2 \le l \le k$,

$$\begin{aligned}&\sum_{m=1}^{k+1} L_{k+1,t}(k+1,m)U_{k+1,t}(m,l)\\ &\qquad = L_{k+1,t}(k+1,l-1)U_{k+1,t}(l-1,l)+L_{k+1,t}(k+1,l)U_{k+1,t}(l,l)\\ &\qquad = -\frac{a_{l-1}}{2a_l}(1+\imath)\times(1-\imath)+1\times\frac{a_{l+1}}{a_l}\\ &\qquad = \frac{a_{l+1}-a_{l-1}}{a_l} = \frac{(2a_l+a_{l-1})-a_{l-1}}{a_l}\\ &\qquad = 2 = S_{k+1,t}(k+1,l).\end{aligned}$$

Letting l be even and $2 \le l \le k$ we obtain

$$\begin{aligned}&\sum_{m=1}^{k+1} L_{k+1,t}(k+1,m)U_{k+1,t}(m,l)\\ &\qquad = L_{k+1,t}(k+1,l-1)U_{k+1,t}(l-1,l)+L_{k+1,t}(k+1,l)U_{k+1,t}(l,l)\\ &\qquad = 1\times(1+\imath)+(-\frac{a_l}{2a_{l+1}}(1+\imath))\times\frac{2a_{l+1}}{a_l} = 0 = S_{k+1,t}(k+1,l)\end{aligned}$$

and

$$\sum_{m=1}^{k+1} L_{k+1,t}(l,m)U_{k+1,t}(m,k+1) = 0 = S_{k+1,t}(l,k+1), \quad 1 \le l \le k-1,$$

$$\begin{aligned}\sum_{m=1}^{k+1} L_{k+1,t}(k,m)U_{k+1,t}(m,k+1) &= L_{k+1,t}(k,k)U_{k+1,t}(k,k+1)\\ &= 1\times(1-\imath) = (1-\imath) = S_{k+1,t}(k,k+1);\end{aligned}$$

$$\sum_{m=1}^{k+1} L_{k+1,t}(k+1,m)U_{k+1,t}(m,k+1)$$

$$= L_{k+1,t}(k{+}1,k)U_{k+1,t}(k,k{+}1){+}L_{k+1,t}(k{+}1,k{+}1)U_{k+1,t}(k{+}1,k{+}1)$$

$$= -\frac{a_k}{2a_{k+1}}(1+\imath)\times(1-\imath)+1\times(2t+\frac{a_{k+2}}{a_{k+1}}) = 2t+\frac{a_{k+2}-a_k}{a_{k+1}}$$

$$= 2t+\frac{(2a_{k+1}+a_k)-a_k}{a_{k+1}} = 2t+2 = S_{k+1,t}(k+1,k+1).$$

Secondly, if k is odd, then $k+1$ is even. So

$$\sum_{m=1}^{k+1} L_{k+1,t}(k,m)U_{k+1,t}(m,k) = -\frac{a_{k-1}}{2a_k}(1+\imath)\times(1-\imath)+1\times\frac{a_{k+1}}{a_k}$$

$$= \frac{a_{k+1}-a_{k-1}}{a_k} = \frac{(2a_k+a_{k-1})-a_{k-1}}{a_k}$$

$$= 2 = S_{k+1,t}(k,k),$$

and

$$\sum_{m=1}^{k+1} L_{k+1,t}(k{+}1,m)U_{k+1,t}(m,1) = L_{k+1,t}(k{+}1,1)U_{k+1,t}(1,1) = 0{\times}2 = 0 = S_{k+1,t}(k{+}1,1).$$

Now, letting l be odd and $2 \le l \le k$,

$$\sum_{m=1}^{k+1} L_{k+1,t}(k{+}1,m)U_{k+1,t}(m,l) = L_{k+1,t}(k{+}1,l{-}1)U_{k+1,t}(l{-}1,l){+}L_{k+1,t}(k{+}1,l)U_{k+1,t}(l,l)$$

$$= 1{\times}(1{-}\imath){+}({-}\frac{a_l}{a_{l+1}}(1{-}\imath)){\times}\frac{a_{l+1}}{a_l} = 0 = S_{k+1,t}(k{+}1,l),$$

Letting l be even and $2 \le l \le k$ we obtain

$$\sum_{m=1}^{k+1} L_{k+1,t}(k{+}1,m)U_{k+1,t}(m,l) = L_{k+1,t}(k{+}1,l{-}1)U_{k+1,t}(l{-}1,l){+}L_{k+1,t}(k{+}1,l)U_{k+1,t}(l,l)$$

$$= -\frac{a_{l-1}}{a_l}(1{-}\imath)){\times}(1{+}\imath){+}1{\times}\frac{2a_{l+1}}{a_l} = \frac{2a_{l+1}-2a_{l-1}}{a_l}$$

$$= \frac{2(a_l+a_{l-1})-2a_{l-1}}{a_l} = 2 = S_{k+1,t}(k+1,l)$$

and

$$\sum_{m=1}^{k+1} L_{k+1,t}(l,m)U_{k+1,t}(m,k+1) = 0 = S_{k+1,t}(l,k+1), \quad 1 \le l \le k-1,$$

$$\begin{aligned}\sum_{m=1}^{k+1} L_{k+1,t}(k,m)U_{k+1,t}(m,k+1) &= L_{k+1,t}(k,k)U_{k+1,t}(k,k+1) \\ &= 1 \times (1+\imath) = (1+\imath) = S_{k+1,t}(k,k+1);\end{aligned}$$

$$\begin{aligned}&\sum_{m=1}^{k+1} L_{k+1,t}(k+1,m)U_{k+1,t}(m,k+1) \\ &= L_{k+1,t}(k+1,k)U_{k+1,t}(k,k+1)+L_{k+1,t}(k+1,k+1)U_{k+1,t}(k+1,k+1) \\ &= -\frac{a_k}{a_{k+1}}(1-\imath)\times(1+\imath)+1\times(2t+\frac{2a_{k+2}}{a_{k+1}}) = 2t+\frac{2a_{k+2}-2a_k}{a_{k+1}} \\ &= 2t+\frac{2(a_{k+1}+a_k)-2a_k}{a_{k+1}} = 2t+2 = S_{k+1,t}(k+1,k+1).\end{aligned}$$

So if $n = k+1$, the theorem holds. This completes the proof. □

Example 2. If $n = 5$, then $S_{5,t} = L_{5,t}U_{5,t}$, where

$$S_{5,t} = \begin{bmatrix} 2 & 1+\imath & 0 & 0 & 0 \\ 0 & 2 & 1-\imath & 0 & 0 \\ 2 & 0 & 2 & 1+\imath & 0 \\ 0 & 2 & 0 & 2 & 1-\imath \\ 2 & 0 & 2 & 0 & 2t+2 \end{bmatrix},$$

$$L_{5,t} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -\frac{1+\imath}{2} & 1 & 0 & 0 \\ 0 & 1 & -\frac{1-\imath}{3} & 1 & 0 \\ 1 & -\frac{1+\imath}{2} & 1 & -\frac{3}{8}(1+\imath) & 1 \end{bmatrix};$$

and

$$U_{5,t} = \begin{bmatrix} 2 & 1+\imath & 0 & 0 & 0 \\ 0 & 2 & 1-\imath & 0 & 0 \\ 0 & 0 & 3 & 1+\imath & 0 \\ 0 & 0 & 0 & \frac{8}{3} & 1-\imath \\ 0 & 0 & 0 & 0 & 2t+\frac{11}{4} \end{bmatrix}.$$

Corollary 3 *For $n \geq 2$,*

$$\det S_{n,t} = \begin{cases} 2^{\frac{n+1}{2}}(2ta_n + a_{n+1}) & n \text{ is odd;} \\ \\ 2^{\frac{n}{2}}(2ta_n + 2a_{n+1}) & n \text{ is even.} \end{cases}$$

Theorem 4 *For $n \geq 2$, $A_{n,t}$ can be factored as $A_{n,t} = L_{\{A;n,t\}}U_{\{A;n,t\}}$, where $L_{\{A;n,t\}} = [L_{\{A;n,t\}}(i,j)]_{1\leq i,j\leq n}$ so*

$$L_{\{A;n,t\}} = \begin{cases} 0 & i < j; \\ 1 & i = \begin{cases} j, j+2, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 1, 3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1; \\ j, j+2, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 2, 4, \cdots, 2\lfloor \frac{n}{2} \rfloor; \end{cases} \\ 0 & j = 1, i = 2, 4, \cdots, 2\lfloor \frac{n}{2} \rfloor; \\ -\frac{F_{j-1}}{F_j} & i = \begin{cases} j+1, j+3, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 3, 5, \cdots, 2\lfloor \frac{n}{2} \rfloor - 1; \\ j+1, j+3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 2, 4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor, \end{cases} \end{cases}$$

and

$$U_{\{A;n,t\}} = [U_{\{A;n,t\}}(i,j)]_{1\leq i,j\leq n} = \begin{cases} 1 & i = j = 1, j = i+1; \\ \frac{F_j}{F_{j-1}} & 2 \leq i = j \leq n-1; \\ \frac{tF_{n-1}+F_n}{F_{n-1}} & i = j = n; \\ 0 & \text{otherwise.} \end{cases}$$

Example 5 If $n = 5$, then $A_{5,t} = L_{\{A;5,t\}}U_{\{A;5,t\}}$, where

$$A_{5,t} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & t+1 \end{bmatrix};$$

$$L_{\{A;5,t\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 & 0 \\ 1 & -1 & 1 & -\frac{2}{3} & 1 \end{bmatrix};$$

and

$$U_{\{A;5,t\}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & \frac{3}{2} & 1 \\ 0 & 0 & 0 & 0 & \frac{3t+5}{3} \end{bmatrix}.$$

Corollary 6 *For $n \geq 2$,*

$$\det A_{n,t} = \prod_{i=1}^{n} U_{\{A;n,t\}}(i,i) = F_n + tF_{n-1}.$$

Theorem 7 *For $n \geq 2$, $B_{n,t}$ can be factored as $B_{n,t} = L_{\{B;n,t\}}U_{\{B;n,t\}}$, where $L_{\{B;n,t\}} = [L_{\{B;n,t\}}(i,j)]_{1\leq i,j\leq n}$ so*

$$L_{\{B;n,t\}} = \begin{cases} 0, i < j; & \\ 0 & j = 1, i = 2, 4, \cdots, 2\lfloor \frac{n}{2} \rfloor; \\ 1 & i = \begin{cases} j, j+2, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 1, 3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1; \\ j, j+2, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 2, 4, \cdots, 2\lfloor \frac{n}{2} \rfloor; \end{cases} \\ (-1)^{j-1} \frac{F_i}{F_{i-1}} \imath & i = \begin{cases} j+1, j+3, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 3, 5, \cdots, 2\lfloor \frac{n}{2} \rfloor - 1; \\ j+1, j+3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 2, 4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor, \end{cases} \end{cases}$$

and

$$U_{\{B;n,t\}} = [U_{\{B;n,t\}}(i,j)]_{1\leq i,j\leq n} = \begin{cases} 1 & i = j = 1; \\ (-1)^j \imath & i = j-1; \\ \frac{F_j}{F_{j-1}} & 2 \leq i = j \leq n-1; \\ \frac{tF_{n-1}+F_n}{F_{n-1}} & i = j = n; \\ 0 & \text{otherwise.} \end{cases}$$

Example 8 If $n = 5$, then $B_{5,t} = L_{\{B;5,t\}}U_{\{B;5,t\}}$, where

$$B_{5,t} = \begin{bmatrix} 1 & \imath & 0 & 0 & 0 \\ 0 & 1 & -\imath & 0 & 0 \\ 1 & 0 & 1 & \imath & 0 \\ 0 & 1 & 0 & 1 & -\imath \\ 1 & 0 & 1 & 0 & t+1 \end{bmatrix},$$

$$L_{\{B;5,t\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -\imath & 1 & 0 & 0 \\ 0 & 1 & \frac{\imath}{2} & 1 & 0 \\ 1 & -\imath & 1 & -\frac{2}{3}\imath & 1 \end{bmatrix}, \text{ and } U_{\{B;5,t\}} = \begin{bmatrix} 1 & \imath & 0 & 0 & 0 \\ 0 & 1 & -\imath & 0 & 0 \\ 0 & 0 & 2 & \imath & 0 \\ 0 & 0 & 0 & \frac{3}{2} & -\imath \\ 0 & 0 & 0 & 0 & \frac{3t+5}{3} \end{bmatrix}.$$

Corollary 9 *For $n \geq 2$,*

$$\det B_{n,t} = \prod_{i=1}^{n} U_{\{B;n,t\}}(i,i) = F_n + tF_{n-1}.$$

Theorem 10 *For $n \geq 1$, $C_{n,t}$ can be factored as $C_{n,t} = L_{\{C;n,t\}}U_{\{C;n,t\}}$, where*

$$L_{\{C;n,t\}} = [L_{\{C;n,t\}}(i,j)]_{1\leq i,j\leq n} = \begin{cases} 0 & i < j; \\ 1 & i = j; \\ \frac{F_{2j}}{F_{2j+1}} & i > j, \end{cases}$$

and

$$U_{\{C;n,t\}} = [U_{\{C;n,t\}}(i,j)]_{1\le i,j\le n} = \begin{cases} -1 & i=j+1, i=1,2,\cdots,n-1; \\ \frac{F_{2i+1}}{F_{2i-1}} & 1\le i=j\le n-1; \\ t+\frac{F_{2n}}{F_{2n-1}} & i=j=n; \\ 0 & \textit{otherwise}. \end{cases}$$

Example 11 If $n = 5$, then $C_{5,t} = L_{\{C;5,t\}}U_{\{C;5,t\}}$, where

$$C_{5,t} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 1 & t+1 \end{bmatrix},$$

$$L_{\{C;5,t\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{3}{5} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{3}{5} & \frac{8}{13} & 1 & 0 \\ \frac{1}{2} & \frac{3}{5} & \frac{8}{13} & \frac{21}{34} & 1 \end{bmatrix}, \textit{ and } U_{\{C;5,t\}} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{5}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{13}{5} & -1 & 0 \\ 0 & 0 & 0 & \frac{34}{13} & -1 \\ 0 & 0 & 0 & 0 & t+\frac{55}{34} \end{bmatrix}.$$

Corollary 12 For $n \ge 1$,

$$\det C_{n,t} = \prod_{i=1}^{n} U_{\{C;n,t\}}(i,i) = F_{2n} + tF_{2n-1}.$$

Theorem 13 *For $n \ge 1$, D_n can be factored as $D_n = L_{\{D;n\}}U_{\{D;n\}}$, where*

$$L_{\{D;n\}} = [L_{\{D;n\}}(i,j)]_{1\le i,j\le n} = \begin{cases} 1 & i=j; \\ \frac{F_j}{F_{j+1}}1 & i=j+1, i=2,3,\cdots,n; \\ 0 & \textit{otherwise}, \end{cases}$$

and

$$U_{\{D;n\}} = [U_{\{D;n\}}(i,j)]_{1\le i,j\le n} = \begin{cases} \frac{F_{j+1}}{F_j} & i=j; \\ 1 & i+1=j; \\ 0 & \textit{otherwise}. \end{cases}$$

Example 14 If $n = 5$, then $D_5 = L_{\{D;5\}}U_{\{D;5\}}$, where

$$D_5 = \begin{bmatrix} 1 & \imath & 0 & 0 & 0 \\ \imath & 1 & \imath & 0 & 0 \\ 1 & \imath & 1 & \imath & 0 \\ 0 & 0 & \imath & 1 & \imath \\ 0 & 0 & 0 & \imath & 1 \end{bmatrix},$$

$$L_{\{D;5\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \imath & 1 & 0 & 0 & 0 \\ 0 & \frac{\imath}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{3}\imath & 1 & 0 \\ 0 & 0 & 0 & \frac{3}{5}\imath & 1 \end{bmatrix}, \textit{ and } U_{\{D;5\}} = \begin{bmatrix} 1 & \imath & 0 & 0 & 0 \\ 0 & 2 & \imath & 0 & 0 \\ 0 & 0 & \frac{3}{2} & \imath & 0 \\ 0 & 0 & 0 & \frac{5}{3} & \imath \\ 0 & 0 & 0 & 0 & \frac{8}{5} \end{bmatrix}.$$

Corollary 15*For $n \geq 1$,*

$$\det D_n = \prod_{i=1}^{n} U_{\{D;n\}}(i,i) = F_{n+1}.$$

Theorem 16 *For $n \geq 1$, E_n can be factored as $E_n = L_{\{E;n\}}U_{\{E;n\}}$, where*

$$L_{\{E;n\}} = [L_{\{E;n\}}(i,j)]_{1 \leq i,j \leq n} = \begin{cases} 1 & i = j; \\ \frac{F_{2i-2}}{F_{2i}} & i = j+1; \\ 0 & \textit{otherwise}, \end{cases}$$

and

$$U_{\{E;n\}} = [U_{\{E;n\}}(i,j)]_{1 \leq i,j \leq n} = \begin{cases} \frac{F_{2i+2}}{F_{2i}} & i = j; \\ 1 & i+1 = j; \\ 0 & \textit{otherwise}. \end{cases}$$

Example 17 If $n = 5$, then $E_5 = L_{\{E;5\}}U_{\{E;5\}}$, where

$$E_5 = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix},$$

$$L_{\{E;5\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{8} & 1 & 0 & 0 \\ 0 & 0 & \frac{8}{21} & 1 & 0 \\ 0 & 0 & 0 & \frac{21}{55} & 1 \end{bmatrix}, \textit{ and } U_{\{E;5\}} = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & \frac{8}{3} & 1 & 0 & 0 \\ 0 & 0 & \frac{21}{8} & 1 & 0 \\ 0 & 0 & 0 & \frac{55}{21} & 1 \\ 0 & 0 & 0 & 0 & \frac{144}{55} \end{bmatrix}.$$

Corollary 18 *For $n \geq 1$,*

$$\det E_n = \prod_{i=1}^{n} U_{\{E;n\}}(i,i) = F_{2n+2}.$$

Theorem 19 *For $n \geq 1$, G_n can be factored as $G_n = L_{\{G;n\}} U_{\{G;n\}}$, where $L_{\{G;n\}} = [L_{\{G;n\}}(i,j)]_{1 \leq i,j \leq n}$ so*

$$L_{\{G;n\}} = \begin{cases} 0 & i < j; \\ 1 & i = j; \\ -\frac{F_{2j}}{F_{2j+1}} & i = \begin{cases} j+1, j+3, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 1, 3, \cdots, 2\lfloor \frac{n}{2} \rfloor - 1; \\ j+1, j+3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 2, 4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor; \end{cases} \\ \frac{F_{2j}}{F_{2j+1}} & i = \begin{cases} j+2, j+4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 1, 3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor - 1; \\ j+2, j+4, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 2, 4, \cdots, 2\lfloor \frac{n}{2} \rfloor - 2, \end{cases} \end{cases}$$

and

$$U_{\{G;n\}} = [U_{\{G;n\}}(i,j)]_{1 \leq i,j \leq n} = \begin{cases} \frac{F_{2i+1}}{F_{2i-1}} & i = j; \\ 1 & i+1 = j; \\ 0 & \textit{otherwise}. \end{cases}$$

Example 20 If $n = 5$, then $G_5 = L_{\{G;5\}} U_{\{G;5\}}$, where

$$G_5 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ -1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 1 & -1 & 2 \end{bmatrix},$$

$$L_{\{G;5\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{3}{5} & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{5} & -\frac{8}{13} & 1 & 0 \\ \frac{1}{2} & -\frac{3}{5} & \frac{8}{13} & -\frac{21}{34} & 1 \end{bmatrix}, \textit{ and } U_{\{G;5\}} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{5}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{13}{5} & 1 & 0 \\ 0 & 0 & 0 & \frac{34}{13} & 1 \\ 0 & 0 & 0 & 0 & \frac{89}{34} \end{bmatrix}.$$

Corollary 21 For $n \geq 1$,

$$\det G_n = \prod_{i=1}^{n} U_{\{G;n\}}(i,i) = F_{2n+1}.$$

Theorem 22 *For $n \geq 1$, H_n can be factored as $H_n = L_{\{H;n\}}U_{\{H;n\}}$, where $L_{\{H;n\}} = [L_{\{H;n\}}(i,j)]_{1\leq i,j\leq n}$ so*

$$L_{\{H;n\}} = \begin{cases} 0 & i < j; \\ 1 & i = j; \\ -\frac{F_j}{F_{j+2}} & i = \begin{cases} j+1, j+3, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 1, 3, \cdots, 2\lfloor \frac{n}{2} \rfloor - 1; \\ j+1, j+3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 2, 4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor; \end{cases} \\ \frac{F_j}{F_{j+2}} & i = \begin{cases} j+2, j+4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 1, 3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor - 1; \\ j+2, j+4, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 2, 4, \cdots, 2\lfloor \frac{n}{2} \rfloor - 2, \end{cases} \end{cases}$$

and

$$U_{\{H;n\}} = [U_{\{H;n\}}(i,j)]_{1\leq i,j\leq n} = \begin{cases} \frac{F_{i+2}}{F_{i+1}} & i = j; \\ -1 & i+1 = j; \\ 0 & \text{otherwise.} \end{cases}$$

Example 23 If $n = 5$, then $H_5 = L_{\{H;5\}}U_{\{H;5\}}$, where

$$H_5 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ -1 & 1 & -1 & 2 & -1 \\ 1 & -1 & 1 & -1 & 2 \end{bmatrix}'$$

$$L_{\{H;5\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{3} & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{3} & -\frac{2}{5} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{3} & \frac{2}{5} & -\frac{3}{8} & 1 \end{bmatrix}, \text{ and } U_{\{H;5\}} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{5}{3} & -1 & 0 \\ 0 & 0 & 0 & \frac{8}{5} & -1 \\ 0 & 0 & 0 & 0 & \frac{13}{8} \end{bmatrix}.$$

Corollary 24 *For $n \geq 1$,*

$$\det H_n = \prod_{i=1}^{n} U_{\{H;n\}}(i,i) = F_{n+2}.$$

Theorem 25 *For $n \geq 1$, $D_{n,t}$ can be factored as $D_{n,t} = L_{\{D;n,t\}}U_{\{D;n,t\}}$, where*

$$L_{\{D;n,t\}} = [L_{\{D;n,t\}}(i,j)]_{1\leq i,j\leq n} = \begin{cases} 1 & i = j; \\ \frac{F_j}{F_{j+1}}1 & i = j+1; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$U_{\{D;n,t\}} = [U_{\{D;n,t\}}(i,j)]_{1\le i,j\le n} = \begin{cases} \frac{F_{i+1}}{F_i} & 1 \le i = j \le n-1; \\ 1 + \frac{F_{n-1}}{F_n}t & i = j = n; \\ \imath & 2 \le i+1 = j \le n-1; \\ \imath t & i+1 = j = n; \\ 0 & otherwise. \end{cases}$$

Example 26 If $n = 5$, then $D_{5,t} = L_{\{D;5,t\}}U_{\{D;5,t\}}$, where

$$D_{5,t} = \begin{bmatrix} 1 & \imath & 0 & 0 & 0 \\ \imath & 1 & \imath & 0 & 0 \\ 0 & \imath & 1 & \imath & 0 \\ 0 & 0 & \imath & 1 & \imath t \\ 0 & 0 & 0 & \imath & 1 \end{bmatrix},$$

$$L_{\{D;5,t\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \imath & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{3}\imath & 1 & 0 \\ 0 & 0 & 0 & \frac{3}{5}\imath & 1 \end{bmatrix}, \text{ and } U_{\{D;5,t\}} = \begin{bmatrix} 1 & \imath & 0 & 0 & 0 \\ 0 & 2 & \imath & 0 & 0 \\ 0 & 0 & \frac{3}{2} & \imath & 0 \\ 0 & 0 & 0 & \frac{5}{3} & \imath t \\ 0 & 0 & 0 & 0 & 1 + \frac{3}{5}t \end{bmatrix}.$$

Corollary 27 *For* $n \ge 1$*,*

$$\det D_{n,t} = \prod_{i=1}^{n} U_{\{D;n,t\}}(i,i) = F_n + tF_{n-1}.$$

Theorem 28 *For* $n \ge 1$*,* $E_{n,t}$ *can be factored as* $E_{n,t} = L_{\{E;n,t\}}U_{\{E;n,t\}}$*, where*

$$L_{\{E;n,t\}} = [L_{\{E;n,t\}}(i,j)]_{1\le i,j\le n} = \begin{cases} 1 & i = j; \\ \frac{F_{2j}}{F_{2j+2}} & i = j+1; \\ 0 & otherwise, \end{cases}$$

and

$$U_{\{E;n,t\}} = [U_{\{E;n,t\}}(i,j)]_{1\le i,j\le n} = \begin{cases} \frac{F_{2j+2}}{F_{2j}} & 1 \le i = j \le n-1; \\ 3 - \frac{F_{2n-2}}{F_{2n}}t & i = j = n; \\ 1 & 2 \le i+1 = j \le n-1; \\ t & i+1 = j = n; \\ 0 & otherwise. \end{cases}$$

Example 29 If $n = 5$, then $E_{5,t} = L_{\{E;5,t\}}U_{\{E;5,t\}}$, where

$$E_{5,t} = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & t \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

$$L_{\{E;5,t\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{8} & 1 & 0 & 0 \\ 0 & 0 & \frac{8}{21} & 1 & 0 \\ 0 & 0 & 0 & \frac{21}{55} & 1 \end{bmatrix},$$

and

$$U_{\{E;5,t\}} = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & \frac{8}{3} & 1 & 0 & 0 \\ 0 & 0 & \frac{21}{8} & 1 & 0 \\ 0 & 0 & 0 & \frac{55}{21} & t \\ 0 & 0 & 0 & 0 & 3 - \frac{21}{55}t \end{bmatrix}.$$

Corollary 30 *For $n \geq 1$,*

$$\det E_{n,t} = \prod_{i=1}^{n} U_{\{E;n,t\}}(i,i) = F_{2n+2} - (t-1)F_{2n-2}.$$

Theorem 31 *For $n \geq 1$, $G_{n,t}$ can be factored as $G_{n,t} = L_{\{G;n,t\}}U_{\{G;n,t\}}$, where $L_{\{G;n,t\}} = [L_{\{G;n,t\}}(i,j)]_{1 \leq i,j \leq n}$ so*

$$L_{\{G;n,t\}} = \begin{cases} 0 & i < j; \\ 1 & i = j; \\ -\frac{F_{2j}}{F_{2j+1}} & i = \begin{cases} j+1, j+3, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 1, 3, \cdots, 2\lfloor \frac{n}{2} \rfloor - 1; \\ j+1, j+3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 2, 4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor; \end{cases} \\ \frac{F_{2j}}{F_{2j+1}} & i = \begin{cases} j+2, j+4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 1, 3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor - 1; \\ j+2, j+4, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 2, 4, \cdots, 2\lfloor \frac{n}{2} \rfloor - 2, \end{cases} \end{cases}$$

and

$$U_{\{G;n,t\}} = [U_{\{G;n,t\}}(i,j)]_{1 \leq i,j \leq n} = \begin{cases} \frac{F_{2j+1}}{F_{2j-1}} & 1 \leq i = j \leq n-1; \\ t + \frac{F_{2n}}{F_{2n-1}} & i = j = n; \\ 1 & 2 \leq i+1 = j \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

Example 32 If $n = 5$, then $G_{5,t} = L_{\{G;5,t\}}U_{\{G;5,t\}}$, where

$$G_{5,t} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ -1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 1 & -1 & t+1 \end{bmatrix},$$

$$L_{\{G;5,t\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{3}{5} & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{5} & -\frac{8}{13} & 1 & 0 \\ \frac{1}{2} & -\frac{3}{5} & \frac{8}{13} & -\frac{21}{34} & 1 \end{bmatrix},$$

$$U_{\{G;5,t\}} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{5}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{13}{5} & 1 & 0 \\ 0 & 0 & 0 & \frac{34}{13} & 1 \\ 0 & 0 & 0 & 0 & t+\frac{55}{34} \end{bmatrix}.$$

Corollary 33 *For* $n \geq 1$,

$$\det G_{n,t} = \prod_{i=1}^{n} U_{\{G;n,t\}}(i,i) = F_{2n} + tF_{2n-1}.$$

Theorem 34 *For* $n \geq 1$, $H_{n,t}$ *can be factored as* $H_{n,t} = L_{\{H;n,t\}}U_{\{H;n,t\}}$, *where* $L_{\{H;n,t\}} = [L_{\{H;n,t\}}(i,j)]_{1\leq i,j\leq n}$ *so*

$$L_{\{H;n,t\}} = \begin{cases} 0 & i < j; \\ 1 & i = j; \\ -\frac{F_j}{F_{j+2}} & i = \begin{cases} j+1, j+3, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 1, 3, \cdots, 2\lfloor \frac{n}{2} \rfloor - 1; \\ j+1, j+3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 2, 4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor; \end{cases} \\ \frac{F_j}{F_{j+2}} & i = \begin{cases} j+2, j+4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 1, 3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor - 1; \\ j+2, j+4, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 2, 4, \cdots, 2\lfloor \frac{n}{2} \rfloor - 2, \end{cases} \end{cases}$$

and

$$U_{\{H;n,t\}} = [U_{\{H;n,t\}}(i,j)]_{1\leq i,j\leq n} = \begin{cases} \frac{F_{j+2}}{F_{j+1}} & 1 \leq i = j \leq n-1; \\ t + \frac{F_n}{F_{n+1}} & i = j = n; \\ -1 & i+1 = j; \\ 0 & \textit{otherwise}. \end{cases}$$

Example 35 If $n = 5$, then $H_{5,t} = L_{\{H;5,t\}}U_{\{H;5,t\}}$, where

$$H_{5,t} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ -1 & 1 & -1 & 2 & -1 \\ 1 & -1 & 1 & -1 & t+1 \end{bmatrix},$$

$$L_{\{H;5,t\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{3} & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{3} & -\frac{2}{5} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{3} & \frac{2}{5} & -\frac{3}{8} & 1 \end{bmatrix}, \text{ and } U_{\{H;5,t\}} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{5}{3} & -1 & 0 \\ 0 & 0 & 0 & \frac{8}{5} & -1 \\ 0 & 0 & 0 & 0 & t+\frac{5}{8} \end{bmatrix}.$$

Corollary 36 *For $n \geq 1$,*

$$\det H_{n,t} = \prod_{i=1}^{n} U_{\{H;n,t\}}(i,i) = F_n + tF_{n+1}.$$

Theorem 37 *For $n \geq 2$, $A_{n,t,s}$ can be factored as $A_{n,t,s} = L_{\{A;n,t,s\}}U_{\{A;n,t,s\}}$, where $L_{\{A;n,t,s\}} = [L_{\{A;n,t,s\}}(i,j)]_{1\leq i,j\leq n}$ so*

$$L_{\{A;n,t,s\}} = \begin{cases} 1 & i = j; \\ -\frac{F_{j-1}}{F_j} & i = \begin{cases} j+1, j+3, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 3, 5, \cdots, 2\lfloor \frac{n}{2} \rfloor - 1; \\ j+1, j+3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 2, 4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor; \end{cases} \\ 1 & i = \begin{cases} j+2, j+4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 1, 3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor - 1; \\ j+2, j+4, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 2, 4, \cdots, 2\lfloor \frac{n}{2} \rfloor - 2; \end{cases} \\ 0 & \text{otherwise}, \end{cases}$$

and

$$U_{\{A;n,t,s\}} = [U_{\{A;n,t,s\}}(i,j)]_{1\leq i,j\leq n} = \begin{cases} 1 & i = j = 1, 2 \leq i+1 = j \leq n-1; \\ s & i+1 = j = n; \\ \frac{F_j}{F_{j-1}} & 2 \leq i = j \leq n-1; \\ t + \frac{F_n}{F_{n-1}}s & i = j = n; \\ 0 & \text{otherwise}. \end{cases}$$

Example 38 If $n = 5$, then $A_{5,t,s} = L_{\{A;5,t,s\}}U_{\{A;5,t,s\}}$, where

$$A_{5,t,s} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & s \\ 1 & 0 & 1 & 0 & t+s \end{bmatrix},$$

$$L_{\{A;5,t,s\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 & 0 \\ 1 & -1 & 1 & -\frac{2}{3} & 1 \end{bmatrix}, \text{ and } U_{\{A;5,t,s\}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & \frac{3}{2} & s \\ 0 & 0 & 0 & 0 & t+\frac{5}{3}s \end{bmatrix}.$$

Corollary 39 *For $n \geq 2$,*

$$\det A_{n,t,s} = \prod_{i=1}^{n} U_{\{A;n,t,s\}}(i,i) = sF_n + tF_{n-1}.$$

Theorem 40 *For $n \geq 2$, $B_{n,t,s}$ can be factored as $B_{n,t,s} = L_{\{B;n,t,s\}}U_{\{B;n,t,s\}}$, where $L_{\{B;n,t,s\}} = [L_{\{B;n,t,s\}}(i,j)]_{1 \leq i,j \leq n}$ so*

$$L_{\{B;n,t,s\}} = \begin{cases} 0 & i < j; \\ 1 & i = j; \\ 0 & i = j+1, j+3, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 1; \\ (-1)^{j-1}\frac{F_{j-1}}{F_j}1 & i = \begin{cases} j+1, j+3, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 3, 5, \cdots, 2\lfloor \frac{n}{2} \rfloor - 1; \\ j+1, j+3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 2, 4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor; \end{cases} \\ 1 & i = \begin{cases} j+2, j+4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 1, 3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor - 1; \\ j+2, j+4, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 2, 4, \cdots, 2\lfloor \frac{n}{2} \rfloor - 2, \end{cases} \end{cases}$$

and

$$U_{\{B;n,t,s\}} = [U_{\{B;n,t,s\}}(i,j)]_{1 \leq i,j \leq n} = \begin{cases} (-1)^j 1 & 2 \leq i+1 = j \leq n-1; \\ (-1)^n s 1 & i = j = n; \\ 1 & i = j = 1; \\ \frac{F_j}{F_{j-1}} & 2 \leq i = j \leq n-1; \\ t + \frac{F_n}{F_{n-1}}s & i = j = n; \\ 0 & \text{otherwise.} \end{cases}$$

Example 41 If $n = 5$, then $B_{5,t,s} = L_{\{B;5,t,s\}}U_{\{B;5,t,s\}}$, where

$$B_{5,t,s} = \begin{bmatrix} 1 & \imath & 0 & 0 & 0 \\ 0 & 1 & -\imath & 0 & 0 \\ 1 & 0 & 1 & \imath & 0 \\ 0 & 1 & 0 & 1 & -s\imath \\ 1 & 0 & 1 & 0 & t+s \end{bmatrix},$$

$$L_{\{B;5,t,s\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -\imath & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 1 & 0 \\ 1 & -\imath & 1 & -\frac{2}{3}\imath & 1 \end{bmatrix},$$

and

$$U_{\{B;5,t,s\}} = \begin{bmatrix} 1 & \imath & 0 & 0 & 0 \\ 0 & 1 & -\imath & 0 & 0 \\ 0 & 0 & 2 & \imath & 0 \\ 0 & 0 & 0 & \frac{3}{2} & -s\imath \\ 0 & 0 & 0 & 0 & t+\frac{5}{3}s \end{bmatrix}.$$

Corollary 42 *For $n \geq 2$,*

$$\det B_{n,t,s} = \prod_{i=1}^{n} U_{\{B;n,t,s\}}(i,i) = sF_n + tF_{n-1}.$$

Theorem 43 *For $n \geq 1$, $C_{n,t,s}$ can be factored as $C_{n,t,s} = L_{\{C;n,t,s\}}U_{\{C;n,t,s\}}$, where*

$$L_{\{C;n,t,s\}} = [L_{\{C:n,t,s\}}(i,j)]_{1\leq i,j\leq n} = \begin{cases} 0 & i < j; \\ 1 & i = j; \\ \frac{F_{2j}}{F_{2j+1}} & i > j, \end{cases}$$

and

$$U_{\{C;n,t,s\}} = [U_{\{C;n,t,s\}}(i,j)]_{1\leq i,j\leq n} = \begin{cases} \frac{F_{2j+1}}{F_{2j-1}} & 1 \leq i = j \leq n-1; \\ t + \frac{F_{2n}}{F_{2n-1}}s & i = j = n; \\ -1 & 2 \leq i+1 = j \leq n-1; \\ -s & i+1 = j = n; \\ 0 & \textit{otherwise}. \end{cases}$$

Example 44 If $n = 5$, then $C_{5,t,s} = L_{\{C;5,t,s\}}U_{\{C;5,t,s\}}$, where

$$C_{5,t,s} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 1 & 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 2 & -s \\ 1 & 1 & 1 & 1 & t+s \end{bmatrix},$$

$$L_{\{C;5,t,s\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{3}{5} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{3}{5} & \frac{8}{13} & 1 & 0 \\ \frac{1}{2} & \frac{3}{5} & \frac{8}{13} & \frac{21}{34} & 1 \end{bmatrix},$$

and

$$U_{\{C;5,t,s\}} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{5}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{13}{5} & -1 & 0 \\ 0 & 0 & 0 & \frac{34}{13} & -s \\ 0 & 0 & 0 & 0 & t+\frac{55}{34}s \end{bmatrix}.$$

Corollary 45 *For $n \geq 1$,*

$$\det C_{n,t,s} = \prod_{i=1}^{n} U_{\{C;n,t,s\}}(i,i) = sF_{2n} + tF_{2n-1}.$$

Theorem 46 *For $n \geq 1$, $D_{n,t,s}$ can be factored as $D_{n,t,s} = L_{\{D;n,t,s\}}U_{\{D;n,t,s\}}$, where*

$$L_{\{D;n,t,s\}} = [L_{\{D;n,t,s\}}(i,j)]_{1\leq i,j\leq n} = \begin{cases} 1 & i = j; \\ \frac{F_j}{F_{j+1}}1 & i = j+1, j = 1, 2, \cdots, n-1; \\ 0 & otherwise, \end{cases}$$

and

$$U_{\{D;n,t,s\}} = [U_{\{D;n,t,s\}}(i,j)]_{1\leq i,j\leq n} = \begin{cases} \frac{F_{j+1}}{F_j} & 1 \leq i = j \leq n-1; \\ t + \frac{F_{n+1}}{F_n}s & i = j = n; \\ 1 & 2 \leq i+1 = j \leq n-1; \\ s1 & i+1 = j = n; \\ 0 & otherwise. \end{cases}$$

Example 47 If $n = 5$, then $D_{5,t,s} = L_{\{D;5,t,s\}}U_{\{D;5,t,s\}}$, where

$$D_{5,t,s} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & s1 \\ 0 & 0 & 0 & 1 & t+s \end{bmatrix},$$

$$L_{\{D;5,t,s\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{3}1 & 1 & 0 \\ 0 & 0 & 0 & \frac{3}{5}1 & 1 \end{bmatrix},$$

and

$$U_{\{D;5,t,s\}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 & \frac{5}{3} & s1 \\ 0 & 0 & 0 & 0 & t+\frac{8}{5}s \end{bmatrix}.$$

Corollary 48 *For $n \geq 1$,*

$$\det D_{n,t,s} = \prod_{i=1}^{n} U_{\{D;n,t,s\}}(i,i) = sF_n + tF_{n-1}.$$

Theorem 49 *For $n \geq 1$, $G_{n,t,s}$ can be factored as $G_{n,t,s} = L_{\{G;n,t,s\}}U_{\{G;n,t,s\}}$, where $L_{\{G;n,t,s\}} = [L_{\{G;n,t,s\}}(i,j)]_{1\leq i,j\leq n}$ so*

$$L_{\{G;n,t,s\}} = \begin{cases} 0 & i < j; \\ 1 & i = j; \\ -\frac{F_{2j}}{F_{2j+1}} & i = \begin{cases} j+1, j+3, \cdots, 2\lfloor\frac{n}{2}\rfloor, j = 1, 3, \cdots, 2\lfloor\frac{n}{2}\rfloor - 1; \\ j+1, j+3, \cdots, 2\lfloor\frac{n-1}{2}\rfloor + 1, j = 2, 4, \cdots, 2\lfloor\frac{n-1}{2}\rfloor; \end{cases} \\ \frac{F_{2j}}{F_{2j+1}} & i = \begin{cases} j+2, j+4, \cdots, 2\lfloor\frac{n-1}{2}\rfloor + 1, j = 1, 3, \cdots, 2\lfloor\frac{n-1}{2}\rfloor - 1; \\ j+2, j+4, \cdots, 2\lfloor\frac{n}{2}\rfloor, j = 2, 4, \cdots, 2\lfloor\frac{n}{2}\rfloor - 2, \end{cases} \end{cases}$$

and

$$U_{\{G;n,t,s\}} = [U_{\{G;n,t,s\}}(i,j)]_{1\leq i,j\leq n} = \begin{cases} \frac{F_{2j+1}}{F_{2j-1}} & 1 \leq i = j \leq n-1; \\ t + \frac{F_{2n}}{F_{2n-1}}s & i = j = n; \\ 1 & 2 \leq i+1 = j \leq n-1; \\ s & i+1 = j = n; \\ 0 & \text{otherwise.} \end{cases}$$

Example 50 If $n = 5$, then $G_{5,t,s} = L_{\{G;5,t,s\}} U_{\{G;5,t,s\}}$, where

$$G_{5,t,s} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ -1 & 1 & -1 & 2 & s \\ 1 & -1 & 1 & -1 & t+s \end{bmatrix},$$

$$L_{\{G;5,t,s\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{3}{5} & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{5} & -\frac{8}{13} & 1 & 0 \\ \frac{1}{2} & -\frac{3}{5} & \frac{8}{13} & -\frac{21}{34} & 1 \end{bmatrix},$$

and

$$U_{\{G;5,t,s\}} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & \frac{5}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{13}{5} & 1 & 0 \\ 0 & 0 & 0 & \frac{34}{13} & s \\ 0 & 0 & 0 & 0 & t + \frac{55}{34}s \end{bmatrix}.$$

Corollary 51 *For $n \geq 1$,*

$$\det G_{n,t,s} = \prod_{i=1}^{n} U_{\{G;n,t,s\}}(i,i) = sF_{2n} + tF_{2n-1}.$$

Theorem 52 *For $n \geq 1$, $H_{n,t,s}$ can be factored as $H_{n,t,s} = L_{\{H;n,t,s\}} U_{\{H;n,t,s\}}$, where $L_{\{H;n,t,s\}} = [L_{\{H;n,t,s\}}(i,j)]_{1 \leq i,j \leq n}$ so*

$$L_{\{H;n,t,s\}} = \begin{cases} 0 & i < j; \\ 1 & i = j; \\ -\frac{F_j}{F_{j+2}} & i = \begin{cases} j+1, j+3, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 1, 3, \cdots, 2\lfloor \frac{n}{2} \rfloor - 1; \\ j+1, j+3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 2, 4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor; \end{cases} \\ \frac{F_j}{F_{j+2}} & i = \begin{cases} j+2, j+4, \cdots, 2\lfloor \frac{n-1}{2} \rfloor + 1, j = 1, 3, \cdots, 2\lfloor \frac{n-1}{2} \rfloor - 1; \\ j+2, j+4, \cdots, 2\lfloor \frac{n}{2} \rfloor, j = 2, 4, \cdots, 2\lfloor \frac{n}{2} \rfloor - 2, \end{cases} \end{cases}$$

and

$$U_{\{H;n,t,s\}} = [U_{\{H;n,t,s\}}(i,j)]_{1 \leq i,j \leq n} = \begin{cases} \frac{F_{j+2}}{F_{j+1}} & 1 \leq i = j \leq n-1; \\ t + \frac{F_n}{F_{n+1}}s & i = j = n; \\ -1 & 2 \leq i+1 = j \leq n-1; \\ -s & i+1 = j = n; \\ 0 & \text{otherwise}. \end{cases}$$

Example 53 If $n = 5$, then $H_{5,t,s} = L_{\{H;5,t,s\}}U_{\{H;5,t,s\}}$, where

$$H_{5,t,s} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ -1 & 1 & -1 & 2 & -s \\ 1 & -1 & 1 & -1 & t+s \end{bmatrix},$$

$$L_{\{H;5,t,s\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{3} & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{3} & -\frac{2}{5} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{3} & \frac{2}{5} & -\frac{3}{8} & 1 \end{bmatrix},$$

and

$$U_{\{H;5,t,s\}} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{5}{3} & -1 & 0 \\ 0 & 0 & 0 & \frac{8}{5} & -s \\ 0 & 0 & 0 & 0 & t+\frac{5}{8}s \end{bmatrix}.$$

Corollary 54 For $n \geq 1$,

$$\det H_{n,t,s} = \prod_{i=1}^{n} U_{\{H;n,t,s\}}(i,i) = tF_n + sF_{n-1}.$$

3. Conclusions

In the near future, we will discuss the sums of $A_{n,t}$, $B_{n,t}$, $C_{n,t}$, $\cdots$, and so on and try to find another class of Fibonacci-Hessenberg Matrices by using the technique of LU factorization.

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