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THE FINITE HEINE TRANSFORMATION AND CONJUGATE DURFEE SQUARES

Michael Rowell
Department of Mathematics, Pacific University, Forest Grove, OR
rowell@pacificu.edu

Ae Ja Yee
Department of Mathematics, Pennsylvania State University, University Park, PA
yee@math.psu.edu

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Abstract

We introduce the idea of a conjugate Durfee square and use it to answer a combinatorial question regarding a finite form of the Heine transformation posed by G. E. Andrews in a recent paper.

1. Introduction

In a recent publication [3], Andrews gave the following finite version of the Heine transformation:

Theorem 1. (Andrews) *For any n, we have*

$$\sum_{k=0}^{n} \frac{(q^{-n})_k(\alpha)_k(\beta)_k}{(q)_k(\gamma)_k(q^{1-n}/\tau)_k} q^k = \frac{(\beta)_n(\alpha\tau)_n}{(\gamma)_n(\tau)_n} \sum_{k=0}^{n} \frac{(q^{-n})_k(\gamma/\beta)_k(\tau)_k}{(q)_k(\alpha\tau)_k(q^{1-n}/\beta)_k} q^k. \tag{1}$$

(The q-shifted factorial $(a)_n$ is defined in Equation (3) in Section 2.) In [3] Andrews asked for a combinatorial proof of Theorem 1 along the lines of his proof of Heine's ${}_2\phi_1$ transformation formula when n tends to infinity [1]. This paper provides such a proof.

2. Conjugate Durfee Squares and Preliminary Results

We define a *partition* of a positive integer n as a sequence of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 + \cdots + \lambda_k = n$ with $\lambda_i \geq \lambda_{i+1}$. We refer to each λ_i as a part of our partition and denote by $|\lambda|$ the sum of its parts. We denote the number of non-zero parts of λ as $\ell(\lambda)$. For example, there are 7 partitions of 5, namely

$$(5), \quad (4,1), \quad (3,2), \quad (3,1,1), \quad (2,2,1), \quad (2,1,1,1), \quad (1,1,1,1,1).$$

To each partition we can associate a Ferrers diagram. Each part of the partition is given as a row of boxes, each row aligned and put in descending order. Figure 1 represents the Ferrers diagram of $(4,2,1)$.

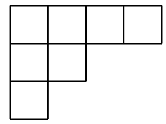

Figure 1: The Ferrers diagram of $(4,2,1)$.

For a partition λ into at most m parts less than or equal to n, we define the (m,n)-conjugate Durfee square as the largest square that can fit with the Ferrers diagram of λ inside of a $m \times n$ rectangle without the two overlapping. Figure 2 illustrates the (m,n)-conjugate Durfee square. It is simple to see that for a given partition, the (m,n)-conjugate Durfee square is unique.

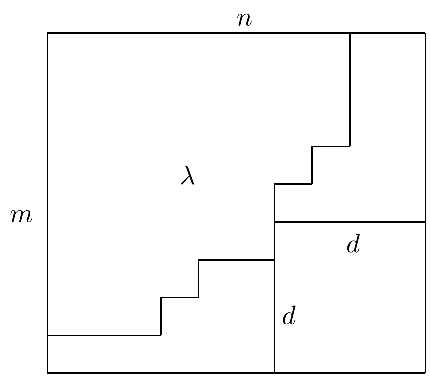

Figure 2: The (m,n)-conjugate Durfee square with side d.

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \dfrac{(q)_n}{(q)_k (q)_{n-k}}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$(a)_0 : = (a;q)_0 = 1, \tag{2}$$

$$(a)_n : = (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n \ge 1. \tag{3}$$

A partition theoretic interpretation of the q-binomial coefficient is as follows:

$$\begin{bmatrix} M+N \\ M \end{bmatrix} = \sum_{\lambda} q^{|\lambda|},$$

where the sum is over all partitions λ whose Ferrers diagram can fit inside an $M \times N$ rectangle.

For more information on partitions, Ferrers diagrams or the q-binomial coefficient, see [2].

We prove the following lemma combinatorially, which is well known in the literature.

Lemma 2. *We have*

$$\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-k \\ j \end{bmatrix} = \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n-j \\ k \end{bmatrix}$$

Proof. Note that the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ counts many interesting combinatorial objects including the partitions with Ferrers diagram fitting inside an $(n-k)\times k$ rectangle. Here, we use inversions of permutations, namely,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{w\in \mathrm{Per}(0^k,1^{n-k})} q^{\mathrm{inv}(w)},$$

where $Per(0^k, 1^{n-k})$ is the set of permutations of k 0's and $(n-k)$ 1's, and $inv(w)$ is the number of inversions in w. Adopting this interpretation, we see that

$$\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-k \\ j \end{bmatrix} = \sum_{w\in \mathrm{Per}(0^k,1^{n-k-j},2^j)} q^{\mathrm{inv}(w)},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ accounts for the inversions between k 0's and $(n-k)$ 1 or 2's, and $\begin{bmatrix} n-k \\ j \end{bmatrix}$ accounts for the inversions between $(n-k-j)$ 1's and j 2's. By counting the inversions between 2's and 0 or 1's first, and then the inversions between 0's and 1's, we obtain

$$\begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} n-j \\ k \end{bmatrix},$$

which completes the proof. □

It should be noted that one can combinatorially interchange the partition interpretation and the permutation interpretation of the q-binomial coefficient. Suppose we are given the partition $\lambda = (\lambda_1, \lambda_2, \dots)$ where $\lambda_1 \le n-k$ and $l(\lambda) \le k$. We can obtain the permutation $w \in Per(0^k, 1^{n-k})$ by first considering

$$(\underbrace{0\cdots 0}_{k}\underbrace{1\cdots 1}_{n-k}). \tag{4}$$

We move the rightmost 0 to the right past λ_1 1's, the rightmost unmoved 0 to the right past λ_2 1's, and so on. It should be clear that $|\lambda| = inv(w)$. We can consider an example with $n = 8$, $k = 3$ and $\lambda = (4, 2, 1)$. The corresponding permutation is (10101101). More can be found on this correspondence in [2].

We review a bijection that was first introduced by the second author in [5] to establish a combinatorial proof for Ramanujan's ${}_1\psi_1$ summation formula. Recall the q-binomial theorem [4]:

$$\sum_{n=0}^{\infty} \frac{(-a;q)_n}{(q;q)_n}(zq)^n = \frac{(-azq;q)_\infty}{(zq;q)_\infty}. \tag{5}$$

Yee's bijection. For a positive integer n, let π be a partition into nonnegative distinct parts less than n and σ a partition into exactly n parts. We define μ by

$$\mu_i = \sigma_{n-\pi_i} + \pi_i, \qquad \text{for all } 1 \le i \le \ell(\pi), \tag{6}$$

and let ν be the partition consisting of the remaining $n - \ell(\pi)$ parts of σ. Then, it can be easily seen that μ has distinct parts. It also follows from the construction that μ and ν are uniquely determined by π and σ. Thus, this map is reversible. The left-hand side of (5) generates the pairs of (π, σ) and the right-hand side generates the pairs of (μ, ν). The map is a bijection between the two sets of such pairs of partitions.

3. The Finite Heine Transformation

In this section, we will demonstrate a combinatorial proof of Theorem 1 along the lines of Andrews's proof of Heine's ${}_2\phi_1$ transformation formula. We start by proving a special case of Theorem 1. By replacing α, τ, γ by $-\alpha, \tau q, \gamma\beta$, respectively, and letting β approach 0 in (1), we obtain the following lemma.

Lemma 3. *We have*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-\alpha)_k}{(\tau q^{n-k+1})_k}(\tau q)^k = \frac{(-\alpha\tau q)_n}{(\tau q)_n}. \tag{7}$$

Proof. Let μ be a partition into distinct parts less than or equal to n and ν be a partition into parts less than or equal to n. Then the right-hand side of (7) generates such pairs of partitions, namely

$$\frac{(-\alpha\tau q)_n}{(\tau q)_n} = \sum_{\mu,\nu} \tau^{\ell(\mu)+\ell(\nu)}\alpha^{\ell(\mu)}q^{|\mu|+|\nu|}.$$

Let $m = \ell(\mu)+\ell(\nu)$. We apply the reverse map of Yee's bijection to μ and ν and denote the resulting partitions by π and σ, where π is a partition into $\ell(\mu)$ nonnegative distinct parts and σ is a partition into exactly m parts less than or equal to n. We

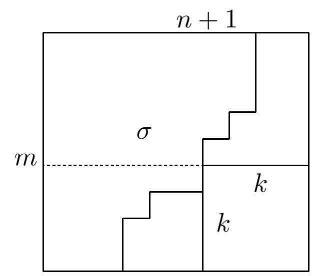

Figure 3: The $(m, n+1)$-conjugate Durfee square of σ with side k.

find the $(m, n+1)$-conjugate Durfee square of σ and denote its side as k. Figure 3 illustrates the conjugate Durfee square of σ. Note that the k parts of σ below the dashed line are less than or equal to $n-k+1$; the other parts above the dashed line are larger than or equal to $n-k+1$, and less than or equal to n. Thus, the generating function of σ is

$$\sum_{\sigma} \tau^{\ell(\sigma)} q^{|\sigma|} = \begin{bmatrix} n \\ k \end{bmatrix} \frac{(\tau q)^k}{(\tau q^{n-k+1})_k}.$$

Furthermore, our process ensures that π has no part exceeding $k-1$. Suppose that $\pi_1 \geq k$. Then, by Yee's bijection (6), we see that

$$\mu_1 = \sigma_{m-\pi_i} + \pi_1 \geq \sigma_{m-k} + k \geq n+1-k+k = n+1,$$

which is a contradiction to the fact that μ has parts less than or equal to n. Thus, the generating function of π is $(-\alpha)_k$. Therefore, summing over all possible values of π and σ, we obtain

$$\sum_{\pi,\sigma} \tau^{\ell(\sigma)} \alpha^{\ell(\pi)} q^{|\pi|+|\sigma|} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-\alpha)_k}{(\tau q^{n-k+1})_k} (\tau q)^k,$$

which completes the proof. □

We now prove Theorem 1 combinatorially. We first make some change of variables. Allowing $\alpha, \tau, \gamma \to -\alpha, \tau q, -\gamma\beta$ followed by $\beta \to \beta q$ in Theorem 1 yields the equivalent identity,

$$\begin{aligned} &\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-\alpha)_k}{(\tau q^{n-k+1})_k} (\tau q)^k \frac{(-\gamma\beta q^{k+1})_{n-k}}{(\beta q^{k+1})_{n-k}} \\ &\quad = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-\gamma)_k}{(\beta q^{n-k+1})_k} (\beta q)^k \frac{(-\alpha\tau q^{k+1})_{n-k}}{(\tau q^{k+1})_{n-k}}. \end{aligned} \tag{8}$$

Theorem 4. *Equation* (8) *is valid.*

Proof. We start by interpreting the left-hand side of (8). We will show that the left- and right-hand side of (8) generate 7-tuples of partitions. We first note that the term

$$\frac{(-\gamma\beta q^{k+1})_{n-k}}{(\beta q^{k+1})_{n-k}}$$

on the left-hand side of (8) can be interpreted as a strict partition μ with all parts exceeding k and no part exceeding n, and a partition ν with all parts exceeding k and no part exceeding n. As we did in the proof of Lemma 3, we apply the reverse map of Yee's bijection to μ and ν to obtain a pair of partitions π and σ, where π is a partition with nonnegative distinct parts and σ is a partition with all parts exceeding k and no part exceeding n. Let j be the side of the $(\ell(\sigma), n+1)$-conjugate Durfee square of σ. Then, all the parts of π are less than j. Thus, using Lemma 3, we can see that

$$\begin{aligned}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} &\frac{(-\alpha)_k}{(\tau q^{n-k+1})_k} (\tau q)^k \frac{(-\gamma\beta q^{k+1})_{n-k}}{(\beta q^{k+1})_{n-k}} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-\alpha)_k}{(\tau q^{n-k+1})_k} (\tau q)^k \begin{bmatrix} n-k \\ j \end{bmatrix} \frac{(-\gamma)_j}{(\beta q^{n-j+1})_j} (\beta q^{k+1})^j.\end{aligned}$$

The interpretation is the same for the right-hand side of (8), namely

$$\begin{aligned}\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} &\frac{(-\gamma)_j}{(\beta q^{n-j+1})_j} (\beta q)^j \frac{(-\alpha\tau q^{j+1})_{n-j}}{(\tau q^{j+1})_{n-j}} \\ &= \sum_{j=0}^{n} \sum_{k=0}^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(-\gamma)_j}{(\beta q^{n-j+1})_j} (\beta q)^j \begin{bmatrix} n-j \\ k \end{bmatrix} \frac{(-\alpha)_k}{(\tau q^{n-k+1})_k} (\tau q^{j+1})^k.\end{aligned}$$

We can now see that the left-hand side of (8) generates 7-tuples of partitions

$$(\lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5, \lambda^6, \lambda^7),$$

where $\lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5, \lambda^6$ and λ^7 are generated by $\begin{bmatrix} n \\ k \end{bmatrix}$, $(\tau q)^k (\beta q^{k+1})^j$, $(-\alpha)_k$, $1/(\tau q^{n-k+1})_k$, $\begin{bmatrix} n-k \\ j \end{bmatrix}$, $(-\gamma)_j$, $1/(\beta q^{n-j+1})_j$, respectively; while the right-hand side generates 7-tuples of partitions $(\mu^1, \mu^2, \mu^3, \mu^4, \mu^5, \mu^6, \mu^7)$, where $\mu^1, \mu^2, \mu^3, \mu^4, \mu^5, \mu^6$ and μ^7 are generated by $\begin{bmatrix} n \\ j \end{bmatrix}$, $(\beta q)^j (\tau q^{j+1})^k$, $(-\gamma)_j$, $1/(\beta q^{n-j+1})_j$, $\begin{bmatrix} n-j \\ k \end{bmatrix}$, $(-\alpha)_k$, $1/(\tau q^{n-k+1})_k$, respectively.

To show (8), given $\lambda^3, \lambda^4, \lambda^6$ and λ^7, we take $\mu^3 = \lambda^6$, $\mu^4 = \lambda^7$, $\mu^6 = \lambda^3$ and $\mu^7 = \lambda^4$. To construct a bijection between (λ^1, λ^5) and (μ^1, μ^5) we apply Lemma 2, noting that we can combinatorially interchange between partitions and permutations as seen in Section 2. Lastly, we must construct the bijection between λ^2 and μ^2. We note that λ^2 is a partition with k 1's each marked with a τ and j $k+1$'s each marked with a β. We subtract k from each of the j parts of size $k+1$ and add j to each of the k parts of size 1. Thus, we have a partition with j 1's each marked with a β and k $j+1$'s each marked with a τ. It is easy to see this is μ^2. □

4. Conclusion

We see as $n \to \infty$ that our conjugate Durfee square gets pushed further to the right, eliminating all of the parts which lie above it and reducing our proof down to a proof similar to Andrews'. In terms of Ferrers diagrams, the integral part of Andrews' proof of the Heine transformation is removing a rectangle, flipping it on its diagonal and reinserting it. We can see this in our proof when we show the bijection from λ^2 to μ^2.

It should be noted that Theorem 1 does not directly follow from Sears ${}_3\phi_2$ transformation [4, Appendix (III.11)] nor its iterate, but can be deduced from the terminating ${}_3\phi_2$ transformation in [4, Appendix (III.13)] in the following way: The left-hand side of the transformation [4, Appendix (III.13)] is clearly symmetric in b, c and in d, e, i.e., (b, c, d, e) can be replaced by (c, b, e, d). Therefore, the right-hand side of [4, Appendix (III.13)] must satisfy the same symmetry, and we obtain an identity by equating it with its $(b, c, d, e) \to (c, b, e, d)$ case. This turns out to be (up to a substitution of parameters) exactly Theorem 1.

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