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# A NOTE ON FIBONACCI-TYPE POLYNOMIALS

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**Abstract**

We opt to study the convergence of maximal real roots of certain Fibonacci-type polynomials given by $G_n = x^k G_{n-1} + G_{n-2}$. The special cases $k = 1$ and $k = 2$ were done by Moore, and Zeleke and Molina, respectively.

## 1. Main Results

In the sequel, $\mathbb{P}$ denotes the set of positive integers. The Fibonacci polynomials [2] are defined recursively by $F_0(x) = 1, F_1(x) = x$ and

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \qquad n \geq 2.$$

**Fact 1.** *Let $n \geq 1$. Then the roots of $F_n(x)$ are given by*

$$x_k = 2i \cos\left(\frac{\pi k}{n+1}\right), \qquad 1 \leq k \leq n.$$

*In particular a Fibonacci polynomial has no positive real roots.*

*Proof.* The Fibonacci polynomials are essentially Tchebycheff polynomials. This is well-known (see, for instance [2]). □

Let $k \in \mathbb{P}$ be fixed. Several authors ([3]-[7]) have investigated the so-called *Fibonacci-type* polynomials. In this note, we focus on a particular group of polynomials recursively defined by

$$G_n^{(k)}(x) = \begin{cases} -1, & n = 0 \\ x - 1, & n = 1 \\ x^k G_{n-1}^{(k)}(x) + G_{n-2}^{(k)}(x), & n \geq 2. \end{cases}$$

When there is no confusion, we suppress the index $k$ to write $G_n$ for $G_n^{(k)}(x)$. We list a few basic properties relevant to our work here.

**Fact 2.** *For each $k \in \mathbb{P}$, there is a rational generating function for $G_n$; namely,*

$$\sum_{n\geq 0} G_n^{(k)}(x)t^n = \frac{(x^k+x-1)t-1}{1-x^kt-t^2}.$$

*Proof.* The claim follows from the definition of $G_n$. □

**Fact 3.** *The following relation holds*

$$G_n^{(k)}(x) = \frac{G_{n-1}^{(k)}F_{n-1}(x^k)+(-1)^{n-1}}{F_{n-2}(x^k)}.$$

*Proof.* We write the equivalent formulation

$$G_n^{(k)}(x) = \det\begin{pmatrix} x-1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & x^k & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & x^k & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & x^k & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & x^k & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & x^k \end{pmatrix},$$

then we apply Dodgson′s determinantal formula [1]. □

**Fact 4.** *For a fixed $k$, let $\{g_n^{(k)}\}_{n\in\mathbb{P}}$ be the maximal real roots of $\{G_n^{(k)}(x)\}_n$. Then $\{g_{2n}^{(k)}\}_n$ and $\{g_{2n-1}^{(k)}\}_n$ are decreasing and increasing sequences, respectively.*

*Proof.* First, each $g_n$ exists since $G_n(0) = 1 < 0$ and $G_n(\infty) = \infty$. Assume $x > 0$. Invoking Fact 3 from above, twice, we find that

$$F_{2n-3}(x^k)G_{2n}^{(k)}(x) = F_{2n-1}(x^k)G_{2n-2}^{(k)}(x) + x^k,$$

$$F_{2n-2}(x^k)G_{2n+1}^{(k)}(x) = F_{2n}(x^k)G_{2n-1}^{(k)}(x) - x^k.$$

From these equations and $F_n(x) > 0$ (see Fact 1), it is clear that $G_{2n-2}(x) > 0$ implies $G_{2n}(x) > 0$; also if $G_{2n-2}(x) = 0$ then $G_{2n}(x) > 0$. Thus $g_{2n-2} > g_{2n}$. A similar argument shows $g_{2n+1} > g_{2n-1}$. The proof is complete. □

Define the quantities

$$\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}, \qquad \beta(x) = \frac{x-\sqrt{x^2+4}}{2},$$

$$p(x) = \frac{(x-1)+\beta(x^k)}{\alpha(x^k)-\beta(x^k)}, \qquad q(x) = \frac{(x-1)+\alpha(x^k)}{\alpha(x^k)-\beta(x^k)}.$$

**Fact 5.** *For $n \geq 0$ and $k \in \mathbb{P}$, we have the explicit formula*

$$G_n^{(k)}(x) = p(x)\alpha^n(x^k) - q(x)\beta^n(x^k).$$

*Proof.* This is the standard generalized Binet formula. □

For each $k \in \mathbb{P}$, we introduce another set of polynomials

$$H^{(k)}(x) = x^k - x^{k-1} + x - 2.$$

**Fact 6.** *For each $k \in \mathbb{P}$, the polynomial $H^{(k)}(x)$ has exactly one positive real root $\xi^{(k)} > 1$.*

*Proof.* Since $H^{(k)}(x) = (x-1)(x^{k-1}+1) - 1 < 0$, whenever $0 < x \leq 1$, there are no roots in the range $0 < x \leq 1$. On the other hand, $H^{(k)}(1) < 0, H^{(k)}(\infty) = \infty$ and the derivative

$$\frac{d}{dx}H^{(k)}(x) = x^{k-1}(k(x-1)+1) + 1 > 0 \qquad \text{whenever } x \in \mathbb{P},$$

suggest there is only one positive root (necessarily greater than 1). □

**Fact 7.** *If $k$ is odd (even), then $H^{(k)}(x)$ has no (exactly one) negative real root.*

*Proof.* For $k$ odd, $H^{(k)}(-x) = (-x-1)(x^{k-1}+1) - 1 < 0$. For $k$ even, $H^{(k)}(-x) = x^k + x^{k-1} - x - 2$ changes sign only once. We now apply Descartes′ Rule to infer the claim. □

Now, we state and prove the main result of the present note.

**Theorem.** *Preserve the notations of Facts 4 and 6. Then, depending on the parity of $n$, the roots $\{g_n^{(k)}\}_n$ converge from above or below so that $g_n^{(k)} \to \xi^{(k)}$ as $n \to \infty$. Note also $\xi^{(k)} \to 1$ as $k \to \infty$.*

*Proof.* For notational brevity, we suppress $k$ and write $g_n$ and $\xi$. From $G_n(g_n) = 0$ and Fact 5 above, we resolve

$$\frac{p(g_n)}{q(g_n)} = \frac{\beta^n(g_n^k)}{\alpha^n(g_n^k)},$$

or

$$\frac{2(g_n-1) + g_n^k - \sqrt{g_n^{2k}+4}}{2(g_n-1) + g_n^k + \sqrt{g_n^{2k}+4}} = (-1)^n \left(1 - \frac{2g_n}{g_n^k + \sqrt{g_n^{2k}+4}}\right)^n. \qquad (1)$$

Using Gershgorin′s Circle theorem, it is easy to see that $1 \leq g_n \leq 2$. When combined with Fact 4, the monotonic sequences $\{g_{2n}\}_n$ and $\{g_{2n-1}\}_n$ converge to finite limits, say $\xi_+$ and $\xi_-$ respectively.

The right-hand side of (1) vanishes in the limit $n \to \infty$, thus

$$2(\xi - 1) + \xi^k - \sqrt{\xi^{2k} + 4} = 0.$$

Further simplification leads to $H^{(k)}(\xi) = \xi^k - \xi^{k-1} + \xi - 2 = 0$. From Fact 6, such a solution is unique. So, $\xi_+ = \xi_- = \xi$ completes the proof. □

## 2. Further Comments

In this section, we discuss the *bivariate Fibonacci* polynomials, of the *first kind* (BFP1), defined as

$$g_n(x,y) = xg_{n-1}(x,y) + yg_{n-2}(x,y), \qquad g_0(x,y) = x, \qquad g_1(x,y) = y.$$

If $x = y = 1$ then the resulting sequence is the Fibonacci numbers.
The following is a generating function for the BFP1:

$$\sum_{n \geq 0} g_n(x,y)t^n = \frac{x + (y - x^2)t}{1 - xt - yt^2}.$$

It is also possible to give an explicit expression:

$$g_n(x,y) = \sum_{k \geq 1} \frac{2n - 3k + 1}{n - k} \binom{n-k}{k-1} x^{n-2k+1} y^k.$$

This shows clearly that each BFP1 has nonnegative coefficients.

The other variant that appears often in the literature is what we call *bivariate Fibonacci* polynomials of the *second kind* (BFP2). These are recursively defined as

$$f_n(x,y) = xf_{n-1}(x,y) + yf_{n-2}(x,y), \qquad f_0(x,y) = y, \qquad f_1(x,y) = x.$$

Obviously $f_n(1,1)$ yields the Fibonacci numbers. We also find the ordinary generating function

$$\sum_{n \geq 0} f_n(x,y)t^n = \frac{y + (x - xy)t}{1 - xt - yt^2}.$$

One interesting contrast between the two families is the following. While the roots of $f_n(x,1)$ are all imaginary, the roots of $g_n(1,y)$ are all real numbers.

Using the corresponding generating functions for BFP2 $f_n(x,y)$ and the classical Fibonacci polynomials $F_n(x) = f_n(x,1)$ proves the below affine relation:

$$f_n(x,y^2) = xy^{n-1}F_{n-1}(x/y) + y^{n+2}F_{n-2}(x/y).$$

In particular, the *Jacobsthal-Lucas* numbers $J_n = f_n(2,1)$ can be expreesed in terms of values of the Fibonacci polynomials, at $1/\sqrt{2}$, namely that

$$J_n = 2^{\frac{n-1}{2}}F_{n-1}\left(\frac{1}{\sqrt{2}}\right) + 2^{\frac{n}{2}+1}F_{n-2}\left(\frac{1}{\sqrt{2}}\right).$$

This translates to the identity

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{5n-9k}{n-k}\binom{n-k}{k}2^k = 2^{n+1} + (-1)^{n+1}, \qquad \text{for } n \in \mathbb{P}.$$

Since we have

$$\sum_{n\geq 0} F_n(x)t^n = \frac{1}{1-xt-t^2} \qquad \text{and} \qquad F_n(x) = \sum_{k\geq 0}\binom{n-k}{k}x^{n-2k},$$

we obtain

$$f_n(x,y^2) = \sum_{k\geq 0}\binom{n-k-1}{k}x^{n-2k}y^k + \sum_{k\geq 0}\binom{n-k-2}{k}x^{n-2k-2}y^{k+2}.$$

In particular, when $x = 1$ there holds

$$f_n(1,y) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \frac{(n-k-1)!}{k!(n-2k+2)}Q(n,k)y^k$$

where $Q(n,k) = n^3 - 3(2k-1)n^2 + (13k(k-1)+2)n - k(k-1)(9k-4)$.
If we alter the definition of BFP2 and specialize as $h_0 = 2, h_1 = 1, h_n(x) = h_{n-1}(x) + xh_{n-2}(x)$ then the resulting sequence of polynomials become intimately linked to the Lucas polynomials $L_n(x)$ and the so-called Jacobsthal-Lucas polynomials $J_n(x)$ as follows:

$$L_n(x) = x^n h_n(1/x^2), \qquad \text{and} \qquad J_n(x) = h_n(2x).$$

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