



## A MULTITUDE OF EXPRESSIONS FOR THE STIRLING NUMBERS OF THE FIRST KIND

**Jacob Katriel**

*Schulich Faculty of Chemistry, Technion, Israel Institute of Tech., Haifa, Israel*  
jkatriel@tx.technion.ac.il

*Received: 11/5/09, Revised: 2/11/10 Accepted: 2/22/10, Published: 6/10/10*

### Abstract

It is shown that the Stirling numbers of the first kind can be expressed in the form  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] = \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{(n-1)!}{\alpha Q[j_1, j_2, \dots, j_{k-1}]}$ , where  $Q$  is a product of  $k-1$  linear factors in the indices  $j_1, j_2, \dots, j_{k-1}$  and  $\alpha$  is a normalization coefficient determined by the condition  $\left[ \begin{smallmatrix} k \\ k \end{smallmatrix} \right] = 1$ . Several types of  $Q$ 's are shown to yield Stirling numbers (“be Stirling”), and some more are conjectured to do so. The complete characterization of the set of  $Q$ 's that are Stirling is not yet available. This set can be divided into subsets, within each of which different  $Q$ 's are related by permutational symmetries. The case  $Q = j_1 \cdot j_2 \cdots j_{k-1}$  is due to Adamchik (1997).

### 1. Introduction

The Stirling numbers of the first kind  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ , with  $n = 1, 2, \dots$  and  $k = 1, 2, \dots, n$ , are most commonly specified by the initial value  $\left[ \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] = 1$  and the recurrence relation

$$\left[ \begin{smallmatrix} n+1 \\ k \end{smallmatrix} \right] = n \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] + \left[ \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right]. \quad (1)$$

They can also be obtained from either the horizontal generating function

$$G_h(x; n) = \sum_{k=0}^n \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] x^k = x(x+1) \cdots (x+n-1) \quad (2)$$

or the vertical generating function [2]

$$G_v(z; k) = \sum_{n \geq k} \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] \frac{z^n}{n!} = \frac{1}{k!} \left( \log \frac{1}{1-z} \right)^k = \frac{1}{k!} \left( \sum_{i=1}^{\infty} \frac{z^i}{i} \right)^k. \quad (3)$$

The expression

$$\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] = \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{(n-1)!}{j_1 \cdot j_2 \cdots j_{k-1}} \quad (4)$$

was derived by Adamchik [3]. It can be shown to satisfy the recurrence relation and the initial value. It also follows from the horizontal generating function via

$$\begin{aligned}
 G_h(x; n) &= (n-1)!x \left(1 + \frac{x}{1}\right) \left(1 + \frac{x}{2}\right) \cdots \left(1 + \frac{x}{n-1}\right) \\
 &= (n-1)! \sum_{k=1}^n \sum_{1 \leq j_1 < j_2 < \cdots < j_{k-1} \leq n-1} \frac{x^k}{j_1 j_2 \cdots j_{k-1}}.
 \end{aligned}$$

Since the Stirling number  $\begin{bmatrix} n \\ k \end{bmatrix}$  is the number of permutations of  $n$  indices with  $k$  cycles it follows that it can be expressed in the form

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{\substack{i_1, i_2, \dots, i_k \geq 1 \\ (i_1 + i_2 + \dots + i_k = n)}} \frac{n!}{k! \cdot i_1 \cdot i_2 \cdots i_k}. \tag{5}$$

This expression also follows from the vertical generating function by noting that it can be written in the form

$$G_v(z; k) = \frac{1}{k!} \prod_{j=1}^k \left( \sum_{i_j=1}^{\infty} \frac{z^{i_j}}{i_j} \right)$$

and expanding in powers of  $z$ .

Equation (5) can be written as

$$\begin{aligned}
 \begin{bmatrix} n \\ k \end{bmatrix} &= \sum_{\substack{i_1, i_2, \dots, i_k \geq 1 \\ (i_1 + i_2 + \dots + i_k = n)}} \frac{(n-1)! \cdot (i_1 + i_2 + \dots + i_k)}{k! \cdot i_1 \cdot i_2 \cdots i_k} \\
 &= \sum_{\substack{i_1, i_2, \dots, i_{k-1} \geq 1 \\ (i_1 + i_2 + \dots + i_{k-1} \leq n-1)}} \frac{(n-1)!}{(k-1)! \cdot i_1 \cdot i_2 \cdots i_{k-1}}, \tag{6}
 \end{aligned}$$

where  $n$  in the numerator was replaced by the sum of indices that appear multiplicatively in the denominator, allowing a reduction of the summand into a sum of terms with slightly simplified denominators. The transition from the expression on the first line above to the second involves retaining one of  $k$  equivalent terms, and multiplying by  $k$ .

Equation (6) can be proved by induction, showing that it satisfies the initial value and the recurrence relation, Equation (1). The latter follows by noting that

Equation (6) yields

$$\begin{aligned} \begin{bmatrix} n+1 \\ k \end{bmatrix} &= \sum_{\substack{i_1, i_2, \dots, i_{k-1} \geq 1 \\ (i_1 + i_2 + \dots + i_{k-1} \leq n)}} \frac{n!}{(k-1)! \cdot i_1 \cdot i_2 \cdots i_{k-1}} \\ &= n \cdot \sum_{\substack{i_1, i_2, \dots, i_{k-1} \geq 1 \\ (i_1 + i_2 + \dots + i_{k-1} \leq n-1)}} \frac{(n-1)!}{(k-1)! \cdot i_1 \cdot i_2 \cdots i_{k-1}} \\ &\quad + \sum_{\substack{i_1, i_2, \dots, i_{k-1} \geq 1 \\ (i_1 + i_2 + \dots + i_{k-1} = n)}} \frac{n!}{(k-1)! \cdot i_1 \cdot i_2 \cdots i_{k-1}}. \end{aligned}$$

The first term on the right-hand-side is equal to  $n \cdot \begin{bmatrix} n \\ k \end{bmatrix}$  and the second is  $\begin{bmatrix} n \\ k-1 \end{bmatrix}$ , both by the induction hypothesis, the latter via Equation (5).

The expression for the Stirling numbers of the first kind given by Equation (6) can be written in terms of the summation indices

$$\begin{aligned} j_1 &= i_1 \\ j_2 &= i_1 + i_2 \\ &\vdots \\ j_{k-1} &= i_1 + i_2 + \dots + i_{k-1} \end{aligned}$$

in the form

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{(n-1)!}{(k-1)! j_1 \cdot (j_2 - j_1)(j_3 - j_2) \cdots (j_{k-1} - j_{k-2})}. \tag{7}$$

Either one or the other of the two sets of summation indices  $j_1 < j_2 < \dots < j_{k-1} \leq n-1$  and  $1 \leq i_\ell \leq n-1$  ;  $\ell = 1, 2, \dots, k-1$  will be convenient in different contexts. We refer to the first set as the  $j$ -indices and to the second as the  $i$ -indices. The expression obtained from  $Q[j_1, j_2, \dots, j_{k-1}]$  by transforming to the  $i$ -indices will be denoted  $P[i_1, i_2, \dots, i_{k-1}]$ .

The two expressions presented above for the Stirling numbers of the first kind, Equation (4) and Equation (7), turn out to be members of a much richer set of expressions. They are, in fact, “extreme” members of that set, in a sense that will be clarified below.

In the present article we deal with a multitude of expressions of the form

$$\begin{bmatrix} n \\ k \end{bmatrix}_Q = \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{(n-1)!}{\alpha Q[j_1, j_2, \dots, j_{k-1}; n]}. \tag{8}$$

$Q$  is a product of  $k - 1$  factors, denoted  $q_m[j_1, j_2, \dots, j_{k-1}; n]$ , each of which is linear in the variables  $j_1, j_2, \dots, j_{k-1}, n$ , with integral coefficients. The normalization coefficient  $\alpha$  is determined by demanding that  $\left[ \begin{matrix} k \\ k \end{matrix} \right]_Q = 1$ . Note that in this case the sum reduces to a single term with  $j_1 = 1, j_2 = 2, \dots, j_{k-1} = k - 1$ . For a certain (fairly large) set of forms of  $Q$  the sum specified in Equation (8) will be found to be equal to the Stirling numbers of the first kind. Such  $Q$ 's will be said to possess the Stirling property (or "be Stirling"). When  $Q$  does not depend on  $n$  (such as in Equations (4) and (7)) we shall refer to it as homogeneous. Otherwise it will be referred to as inhomogeneous. Expressions for  $Q$  that involve  $k$  factors will also be encountered, in which case the numerator has to be modified to  $n!$ .  $Q$  will be referred to as minimal when it consists of precisely  $k - 1$  factors. When  $Q$  is a minimal homogeneous expression we refer to Equation (8) as the standard form.

The sum in Equation (8) depends on the multiset of  $(k - 1)$ -tuples  $\{(j_1, j_2, \dots, j_{k-1}) \ ; \ 1 \leq j_1 < j_2 < \dots < j_{k-1} \leq n - 1\}$ . Transformations of the summation indices under which this multiset is invariant will be referred to as multiset automorphisms. Obviously, while the form of  $Q$  is affected by such (non-trivial) transformations, the sum is invariant. Such transformations account for some of the multitude of expressions for the Stirling numbers of the first kind, mentioned above. Note, in particular, that the normalization coefficient is not affected by such transformations. The two "extreme" identities presented above, Equations (4) and (7), possess distinct normalization coefficients (1 and  $(k - 1)!$ , respectively), so they cannot be associated with one another via such a multiset automorphism. In fact, we will obtain identities with various other values of the normalization coefficient, and the following conjectures (or, if slightly reformulated, open problems) are proposed on the basis of the results presented below:

**Conjecture.** The normalization coefficients are integers.

This conjecture implies that the normalization coefficients are divisors of  $(k - 1)!$ .

**Conjecture.** The set of values of the normalization coefficients for  $Q$ 's with a minimal number of factors is the set of divisors of  $(k - 1)!$ .

The set of  $Q$ 's that possess the Stirling property has not been fully characterized. Nor has it (so far) been possible to specify a criterion that  $Q$  should satisfy to be Stirling. The following appears to hold, providing a necessary condition that  $Q$  has to satisfy to be Stirling,

**Conjecture.** If  $Q$  is Stirling, then, given  $n$ , the set of values of the factors  $q_m$  comprising  $Q$  determines the indices  $j_1, j_2, \dots, j_{k-1}$ .

An additional conjecture and an open problem are stated later on.

We have (arbitrarily) restricted our systematic attention to minimal homogeneous  $Q$ 's consisting of factors that are either single indices or differences of two indices. Equations (4) and (7) belong to this class. However, application of multiset-automorphisms yields  $Q$ 's of a more general form.

The structure of this paper is as follows: In section 2 we introduce two types of multiset automorphisms that generate families of symmetry related expressions for the Stirling numbers of the first kind. In section 3 we derive several expressions for the Stirling numbers of the first kind. In section 4 we present a comprehensive treatment of  $Q$ 's that consist of one single index and  $k - 2$  differences of pairs of indices, for  $k \leq 6$ . Most of the expressions presented follow from the general results presented in section 3, and a few depend on a single conjecture, stated below. For larger  $k$  more cases are expected to be encountered whose proof requires further types of general results.

It appears that a lot remains to be done to fully understand the multitude of expressions for the Stirling numbers of the first kind that the present paper suggests.

## 2. Multiset Automorphisms of the Set of Summation Indices

In the present section we examine transformations of the set of summation indices  $j_1, j_2, \dots, j_{k-1}$ , ( $1 \leq j_1 < j_2 < \dots < j_{k-1} \leq n - 1$ ) of the form  $j_i \rightarrow j_i^*$  ;  $i = 1, 2, \dots, k - 1$ , that maintain the property  $1 \leq j_1^* < j_2^* < \dots < j_{k-1}^* \leq n - 1$ . The multiset of  $(k - 1)$ -tuples  $(j_1, j_2, \dots, j_{k-1})$  is invariant under such transformations. We shall refer to transformations that satisfy this property as multiset automorphisms.

### 2.1. The $T$ -Transformations

Consider the transformation of summation variables specified by

$$t_1 : \begin{cases} j_1 \rightarrow \tilde{j}_1 = j_2 - j_1 \\ j_i \rightarrow \tilde{j}_i = j_i \quad ; \quad i = 2, 3, \dots, k - 1. \end{cases}$$

For any given value of  $j_2$ , the range of values of  $j_1$ , i.e.,  $1 \leq j_1 \leq j_2 - 1$ , is identical with the range of values of  $j_2 - j_1$ . Hence, the multiset of ordered pairs  $(j_1, j_2)$  is identical with the multiset of ordered pairs  $(j_2 - j_1, j_2)$ . This implies that for an arbitrary  $F(j_1, j_2)$  the identity  $\sum_{j_1 < j_2}^{n-1} F(j_1, j_2) = \sum_{j_1 < j_2}^{n-1} F(j_2 - j_1, j_2)$  holds.

This transformation can be generalized. Let

$$t_i : \begin{cases} j_i \rightarrow \tilde{j}_i = j_{i-1} + j_{i+1} - j_i \\ j_m \rightarrow \tilde{j}_m = j_m \text{ for } m \neq i \end{cases}$$

where  $i = 2, 3, \dots, k-2$ . Again, for a given set of values of  $j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_{k-1}$  the range of  $j_i$  is the same as that of  $j_{i-1} + j_{i+1} - j_i$ .

The transformations  $t_1, t_2, \dots, t_{k-2}$  satisfy the relations  $t_i^2 = e$ , ( $e$  is the identity)  $t_i \cdot t_{i+1} \cdot t_i = t_{i+1} \cdot t_i \cdot t_{i+1}$ ,  $i = 1, 2, \dots, k-3$ , and  $t_i \cdot t_j = t_j \cdot t_i$ ,  $|i - j| \geq 2$ . These are the relations satisfied by the generators of the symmetric group  $S_{k-1}$ .

**2.1.1. Some Elementary  $T$ -Transformations**

The following are easily derived from the defining relations

$$\begin{aligned} t_i(j_i - j_{i-1}) &= j_{i+1} - j_i \\ t_i(j_{i+1} - j_i) &= j_i - j_{i-1} \\ t_1 t_2 \cdots t_\ell j_\ell &= j_{\ell+1} - j_1 \\ t_{i+1} t_{i+2} \cdots t_{i+\ell} (j_{i+\ell} - j_i) &= j_{i+\ell+1} - j_{i+1}, \end{aligned}$$

and are often applicable.

**2.1.2.  $T$ -Transformations on the  $i$ -Indices**

The  $T$ -transformations obtain a very transparent form on the  $i$ -indices. Thus,

$$t_\ell \begin{Bmatrix} i_\ell \\ i_{\ell+1} \end{Bmatrix} = \begin{Bmatrix} i_{\ell+1} \\ i_\ell \end{Bmatrix} \quad ; \quad \ell = 1, 2, \dots, k-2, \tag{9}$$

i.e.,  $t_\ell$  acts as the transposition  $(\ell, \ell + 1)$ .

**2.1.3. Special Types of  $T$ -Transformations**

The  $U$  (up)-transformation is defined by

$$U : \begin{cases} j_i \rightarrow j_{k-1} - j_{k-1-i} \ ; \ i = 1, 2, \dots, k-2 \\ j_{k-1} \rightarrow j_{k-1} \end{cases} .$$

To express the  $U$ -transformation in terms of the generators of the  $T$ -transformations it is convenient to use the correspondence of the generators of the symmetric

group with transpositions of consecutive indices. For  $k - 1$  even consider the permutation  $u \equiv (1, k - 1)(2, k - 2) \cdots (i, k - i) \cdots (\frac{k-1}{2}, \frac{k+1}{2})$ , and for  $k - 1$  odd consider  $u \equiv (1, k - 1)(2, k - 2) \cdots (\frac{k}{2} - 1, \frac{k}{2} + 1)(\frac{k}{2})$ . For  $i < j$  the transposition  $(i, j)$  is associated with the product of generators  $t_i t_{i+1} \cdots t_{j-2} t_{j-1} t_{j-2} \cdots t_i$ . Replacing each transposition in  $u$  by the appropriate product of generators we obtain an expression that can be verified to be equal to  $U$ . As an illustration consider  $k - 1 = 4$ . From  $u = (1, 4)(2, 3)$  we obtain  $U = (t_1 t_2 t_3 t_4 t_3 t_2 t_1)(t_2 t_3 t_2)$ . From the definition of  $U$  it follows that it is an involution. This is even clearer noting the association with a product of mutually commuting transpositions.

The  $D$  (down)-transformation is

$$D : \begin{cases} j_i \rightarrow j_{i+1} - j_1 & ; i = 1, 2, \dots, k - 2 \\ j_{k-1} \rightarrow j_{k-1} \end{cases} .$$

It can be written in terms of the generators defined above in the form  $D = t_1 t_2 t_3 \cdots t_{k-2}$ , and it satisfies  $D^{k-1} = e$ .

**2.1.4. An Invariance Property Under the  $T$ -Transformations**

Consider minimal homogeneous  $Q$ 's. For  $n = k$  the indices involved in  $Q$  obtain the values  $j_i = i$ ;  $i = 1, 2, \dots, k - 1$ . Evaluating the  $k - 1$  factors of  $Q$  for this case we obtain a multiset of integers, denoted  $(s_1, s_2, \dots, s_{k-1})$ , that we refer to as the shadow of  $Q$ . Defining  $\sigma = \prod_{i=1}^{k-1} s_i$ , the normalization coefficient is given by  $\frac{(k-1)!}{\sigma}$ .

The transformation  $t_i$  converts  $j_i$  into  $j_i^* = j_{i+1} - j_i + j_{i-1}$ . Obviously, for  $n = k$  the value of  $j_i^*$  is equal to that of  $j_i$ , i.e., to  $i$ . Hence, we have:

**Lemma 1.** *The shadow of  $Q$  is invariant under  $T$ -transformations.*

However, different  $Q$ 's with a common shadow are not necessarily related by a  $T$ -transformation. At least one instance involving two different  $Q$ 's with a common shadow, one of which,  $j_2(j_3 - j_1)(j_4 - j_1)(j_3 - j_2)$ , is Stirling and the other,  $j_1(j_3 - j_1)(j_4 - j_1)(j_4 - j_2)$ , is not, was encountered. This observation suggests the following.

**Open Problem.** Are minimal homogeneous  $Q$ 's with a common shadow, which are both Stirling, always related by a  $T$ -transformation?

**2.2. The  $I$ -Transformation**

Consider the “inversion” ( $I_{k-1}$ -) transformation

$$I : \begin{cases} j_1 \rightarrow j_1^* = n - j_{k-1} \\ j_2 \rightarrow j_2^* = n - j_{k-2} \\ \vdots \\ j_m \rightarrow j_m^* = n - j_{k-m} \\ \vdots \\ j_{k-1} \rightarrow j_{k-1}^* = n - j_1 \end{cases}$$

Clearly,  $1 \leq j_1^* < j_2^* < \dots < j_{k-1}^* \leq n - 1$ .

**2.3. Some Applications of the  $T$ - and  $I$ - Transformations**

As a first illustration we consider the two expressions for the Stirling numbers  $\left[ \begin{smallmatrix} n \\ 3 \end{smallmatrix} \right]$ , i.e.,

$$\left[ \begin{smallmatrix} n \\ 3 \end{smallmatrix} \right] = \sum_{j_1 < j_2}^{n-1} \frac{(n-1)!}{j_1 \cdot j_2},$$

which is a special case of Equation (4), and

$$\left[ \begin{smallmatrix} n \\ 3 \end{smallmatrix} \right] = \sum_{j_1 < j_2}^{n-1} \frac{(n-1)!}{2j_1 \cdot (j_2 - j_1)},$$

which is a special case of Equation (7). We note that the normalization coefficient is equal to 1 in the first identity and to 2 in the second, so they cannot be related to one another by a multiset automorphism. To illustrate this point we write the summands explicitly for  $n = 4$ :

$$\begin{aligned} \sum_{j_1 < j_2}^3 \frac{3!}{j_1 \cdot j_2} &= 3 + 2 + 1, \\ \sum_{j_1 < j_2}^3 \frac{3!}{2 \cdot j_1(j_2 - j_1)} &= \frac{1}{2}(6 + 3 + 3). \end{aligned}$$



The transformations of the  $Q$  factor of the first identity under  $t_1$  and  $I_2$  are presented in the diagram

$$\begin{array}{ccccc}
 (j_2 - j_1)j_2 & \xleftrightarrow{t_1} & j_1j_2 & \xleftrightarrow{I_2} & (n - j_1)(n - j_2) \\
 I_2 \downarrow & & & & \downarrow t_1 \\
 (j_2 - j_1)(n - j_1) & \xleftrightarrow{t_2} & j_1(n + j_1 - j_2) & \xleftrightarrow{t_1} & (n + j_1 - j_2)(n - j_2)
 \end{array} \tag{10}$$

Hence, six distinct expressions for the Stirling numbers  $\begin{bmatrix} n \\ 3 \end{bmatrix}$ , four of them inhomogeneous, have been generated. This hexagon indicates that  $t_1$  and  $I_2$  do not commute with one another. Rather, they satisfy the braiding relation  $t_1I_2t_1 = I_2t_1I_2$ , or  $(t_1I_2)^3 = e$ .

The  $Q$  factor in the second identity gives rise to the diagram

$$t_1 \subset j_1(j_2 - j_1) \xleftrightarrow{I_2} (n - j_2)(j_2 - j_1) \xleftrightarrow{t_1} (n - j_2)j_1 \supset I_2 \tag{11}$$

indicating that the leftmost term is invariant under  $t_1$  and the rightmost term is invariant under  $I_2$ .

Note that while  $j_1(n - j_2)$  is Stirling,  $j_2(n - j_1)$  is not.

A further illustration is provided by application of the  $D$ -transformation and of the  $U$ -transformation to Equation (4), yielding

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{(n - 1)!}{(j_2 - j_1)(j_3 - j_1) \cdots (j_{k-1} - j_1)j_{k-1}}, \tag{12}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{(n - 1)!}{(j_{k-1} - j_1)(j_{k-1} - j_2) \cdots (j_{k-1} - j_{k-2})j_{k-1}}, \tag{13}$$

respectively.

Equation (7) is invariant under all the  $T$ -transformations, hence, in particular, both the  $U$ - and  $D$ -transformations.

### 2.3.1. Some Further $T$ -Transforms of Adamchik’s Expression

Equation (4) has normalization coefficient  $\alpha = 1$ . The  $(k - 1)!$   $T$ -transforms generate that many images of this expression. Equations (12) and (13) are two of these. Many others contain factors involving more than two indices, that we choose not to deal with. The following are particularly interesting, and can be readily demonstrated by induction.

Let  $\tau_\ell = t_1 t_2 \cdots t_\ell$ ,  $\ell = 1, 2, \dots, k - 2$ . Let  $\Theta_\ell = \tau_\ell \tau_{\ell-1} \cdots \tau_1$ . Then

$$\Theta_{m-1} j_1 j_2 \cdots j_{k-1} = (j_m - j_{m-1})(j_m - j_{m-2}) \cdots (j_m - j_1) j_m j_{m+1} \cdots j_{k-1}. \quad (14)$$

Let  $\mu_\ell = t_\ell t_{\ell-1} \cdots t_2 t_1 t_2 \cdots t_{\ell-1} t_\ell$ . Let  $\Xi_\ell = \mu_\ell \mu_{\ell-1} \cdots \mu_1$ . Then

$$\Xi_{m-1} j_1 j_2 \cdots j_{k-1} = (j_2 - j_1)(j_3 - j_1) \cdots (j_m - j_1) j_m j_{m+1} \cdots j_{k-1}. \quad (15)$$

### 3. Some Further Identities

In the present section some general techniques and results will be presented. However, they do not account for all the cases that appear to be of interest.

#### 3.1. Application of the Recurrence Relation: *S*-Derivation

Let  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]^*$  be an expression that depends on two positive integers  $n$  and  $k$ ,  $k \leq n$ , the asterisk serving to indicate that this is not necessarily a Stirling number.

**Definition.** *S*-derivation is defined by

$$S \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]^* = \left[ \begin{smallmatrix} n+1 \\ k \end{smallmatrix} \right]^* - n \cdot \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]^*.$$

From the recurrence relation for the Stirling numbers of the first kind, Equation (1), it follows that

**Lemma 2.** *If  $\left[ \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right]^* = 1$  and  $S \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]^* = \left[ \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right]^*$  then  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]^* = \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ .*

*S*-derivation can be written explicitly for  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_Q$ , when  $Q$  is homogeneous (independent of  $n$ ). Separating the sum over  $j_{k-1}$  in  $\left[ \begin{smallmatrix} n+1 \\ k \end{smallmatrix} \right]_Q$  into a sum up to  $n - 1$  and a term with  $j_{k-1} = n$ , one notes that the first term is equal to  $n \cdot \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_Q$ . Hence,

$$S \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_Q = \left[ \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right]_{\tilde{Q}} \quad \text{where} \quad \tilde{Q} = \frac{1}{n} \cdot Q|_{j_{k-1}=n} \quad (16)$$

$\left[ \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right]_{\tilde{Q}}$  is in general inhomogeneous although  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]_Q$  was assumed to be homogeneous. This fact requires that we deal with inhomogeneous identities even if all we desire is to establish the homogeneous ones. An exception of the inhomogeneity noted here involves the case in which  $Q$  depends on  $j_{k-1}$  only as a multiplicative factor. This case yields a simple but important special case specified by the following lemma, the proof of which is immediate.

**Lemma 3.** *If  $Q[j_1, j_2, \dots, j_{k-1}] = Q'[j_1, j_2, \dots, j_{k-2}] \cdot j_{k-1}$  then  $S$ -derivation transforms  $\begin{bmatrix} n \\ k \end{bmatrix}_Q$  into  $\begin{bmatrix} n \\ k-1 \end{bmatrix}_{Q'}$ .*

This lemma allows the elimination of any number of isolated indices from the top of  $Q$ . It means that if  $Q[j_1, j_2, \dots, j_m]$  is Stirling, so is  $Q[j_1, j_2, \dots, j_m] \cdot j_{m+1} \cdot j_{m+2} \cdots j_{k-1}$ .

When  $Q$  is homogeneous but not of the form specified in Lemma 3, Equation (16) can be written in the form  $S \begin{bmatrix} n \\ k \end{bmatrix}_Q = \sum_{j_1 < j_2 < \dots < j_{k-2}}^{n-1} \frac{n!}{\alpha Q|_{j_{k-1}=n}}$ . This is a non-minimal (inhomogeneous) expression,  $Q|_{j_{k-1}=n}$  being a product of  $k - 1$  factors. If  $Q|_{j_{k-1}=n}$  can be written in the form  $Q'' \cdot s_1 \cdot s_2$ , where  $s_1 + s_2 = n$ , then a procedure that we will refer to as “*partition of  $n$* ” can be applied, yielding

$$S \begin{bmatrix} n \\ k \end{bmatrix}_Q = \frac{\beta}{\alpha} \sum_{j_1 < j_2 < \dots < j_{k-2}}^{n-1} \frac{(n-1)!}{\beta Q'' \cdot s_1} + \frac{\gamma}{\alpha} \sum_{j_1 < j_2 < \dots < j_{k-2}}^{n-1} \frac{(n-1)!}{\gamma Q'' \cdot s_2}. \tag{17}$$

Since the normalization coefficients  $\alpha$ ,  $\beta$  and  $\gamma$  are readily evaluated, it is easy to show that  $\frac{\beta}{\alpha} + \frac{\gamma}{\alpha} = 1$ . If both sums on the right-hand-side of Equation (17) are equal to  $\begin{bmatrix} n \\ k-1 \end{bmatrix}$ , then so is the left-hand-side. This procedure can be generalized to any number of factors summing to  $n$ . A special case was applied to derive Equation (6) above.

$S$ -derivation can also be written explicitly when  $Q$  is a product of a homogeneous expression in  $j_1, j_2, \dots, j_{k-2}$  and a factor of the form  $n - j_{k-1}$ . In this case we have:

**Lemma 4.** *Let  $Q[j_1, j_2, \dots, j_{k-1}; n] = Q'[j_1, j_2, \dots, j_{k-2}] \cdot (n - j_{k-1})$ . Then*

$$S \begin{bmatrix} n \\ k \end{bmatrix}_Q = \sum_{j_1 < j_2 < \dots < j_{k-2}}^{n-1} \frac{n!}{\alpha Q'[j_1, j_2, \dots, j_{k-2}] \cdot (n - j_{k-2})}. \tag{18}$$

*Proof.* Write  $\begin{bmatrix} n+1 \\ k \end{bmatrix}_Q$  and separate the sum over  $j_{k-1}$  into a term with  $j_{k-1} = j_{k-2} + 1$  and a sum over  $j_{k-1}$  from  $j_{k-2} + 2$  to  $n$ . After replacing  $j_{k-1}$  in the second sum by  $j_{k-1}^* = j_{k-1} - 1$ , that sum becomes  $n \cdot \begin{bmatrix} n \\ k \end{bmatrix}_Q$ . Hence, the right-hand-side of Equation (18) is the first term specified above.  $\square$

**Remark.** The numerator in Equation (18) is  $n!$ , not  $(n - 1)!$ . The lemma can be rewritten in the form

$$S : Q[j_1, j_2, \dots, j_{k-2}] \cdot (n - j_{k-1}) \rightarrow \frac{1}{n} \cdot Q[j_1, j_2, \dots, j_{k-2}](n - j_{k-2}).$$

Further reduction can be achieved when  $Q'[j_1, j_2, \dots, j_{k-2}] = Q''[j_1, j_2, \dots, j_{k-3}] \cdot j_{k-2}$ . In this case partition of  $n$  can be applied. Thus, for  $Q = Q''[j_1, j_2, \dots, j_{k-3}] \cdot j_{k-2} \cdot (n - j_{k-1})$  we obtain, after application of partition of  $n$ ,

$$S \begin{bmatrix} n \\ k \end{bmatrix}_Q = \frac{1}{k-1} \cdot \begin{bmatrix} n \\ k-1 \end{bmatrix}_{Q'' \cdot j_{k-2}} + \frac{k-2}{k-1} \cdot \begin{bmatrix} n \\ k-1 \end{bmatrix}_{Q'' \cdot (n-j_{k-2})}.$$

Hence,  $\begin{bmatrix} n \\ k \end{bmatrix}_Q$  is Stirling if both  $\begin{bmatrix} n \\ k-1 \end{bmatrix}_{Q'' \cdot j_{k-2}}$  and  $\begin{bmatrix} n \\ k-1 \end{bmatrix}_{Q'' \cdot (n-j_{k-2})}$  are.

Taking  $Q''[j_1, j_2, \dots, j_{k-3}] = j_1 j_2 \cdots j_{k-3}$  we note that  $\begin{bmatrix} n \\ k-1 \end{bmatrix}_{Q'' \cdot j_{k-2}}$  is Stirling by Equation (4). Hence,

$$S \begin{bmatrix} n \\ k \end{bmatrix}_{j_1 j_2 \cdots j_{k-2} (n-j_{k-1})} = \frac{1}{k-1} \cdot \begin{bmatrix} n \\ k-1 \end{bmatrix} + \frac{k-2}{k-1} \cdot \begin{bmatrix} n \\ k-1 \end{bmatrix}_{j_1 j_2 \cdots j_{k-3} (n-j_{k-2})},$$

allowing an inductive proof that  $\begin{bmatrix} n \\ k \end{bmatrix}_{j_1 j_2 \cdots j_{k-2} (n-j_{k-1})}$  is Stirling provided that  $\begin{bmatrix} n \\ 2 \end{bmatrix}_{n-j_1}$  is. The latter is readily established. Finally we obtain

**Lemma 5.** *We have*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{(n-1)!}{j_1 \cdot j_2 \cdots j_{k-2} (n - j_{k-1})}.$$

This lemma will be generalized in the following section.

**3.1.1. S-Derivation in the  $i$ -Indices**

For an expression of the form

$$\begin{bmatrix} n \\ k \end{bmatrix}_P = \sum_{\substack{i_1, i_2, \dots, i_{k-1} \geq 1 \\ (i_1 + i_2 + \dots + i_{k-1} \leq n-1)}} \frac{(n-1)!}{\alpha P[i_1, i_2, \dots, i_{k-1}]}$$

$S$ -derivation yields

$$S \begin{bmatrix} n \\ k \end{bmatrix}_P = \sum_{\substack{i_1, i_2, \dots, i_{k-1} \geq 1 \\ (i_1 + i_2 + \dots + i_{k-1} = n)}} \frac{n!}{\alpha P[i_1, i_2, \dots, i_{k-1}]} \tag{19}$$

Further reduction takes place when a partition of  $n$  in terms of some (or all) of the factors comprising  $P$  is feasible. The case  $P = i_1 i_2 \cdots i_{k-1}$  was treated above (Equation (6)), and additional cases will be dealt with below.

**3.2. Some Z Identities**

The following identities are representative of a class of identities that we will refer to as the Z (for zipper) identities:

For  $m = 0, 1, \dots, k - 1$  let

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\alpha, m} = \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{(n-1)!}{\binom{k-1}{m} j_1 \cdot j_2 \cdots j_m (n-j_{m+1})(n-j_{m+2}) \cdots (n-j_{k-1})} \tag{20}$$

and

$$\begin{aligned} & \begin{bmatrix} n \\ k \end{bmatrix}_{\beta, m} \\ &= \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{n!}{\binom{k}{m+1} j_1 \cdot j_2 \cdots j_m \cdot j_{m+1} (n-j_{m+1})(n-j_{m+2}) \cdots (n-j_{k-1})} \end{aligned} \tag{21}$$

**Theorem 6.** For all  $m$ ,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\alpha, m} = \begin{bmatrix} n \\ k \end{bmatrix} \tag{22}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\beta, m} = \begin{bmatrix} n \\ k \end{bmatrix}. \tag{23}$$

*Proof.* Write  $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{\alpha, m+1}$  and separate the sum over  $j_{m+2}$  into a term with  $j_{m+2} = j_{m+1} + 1$  and a sum over  $j_{m+2}$  from  $j_{m+1} + 2$  to  $n - k + m + 2$  (the maximal value allowed by the condition  $j_{m+2} < j_{m+3} < \dots < j_k \leq n$ ). In the first term introduce the summation indices  $j_i^* = j_i - 1$ ,  $i = m + 3, m + 4, \dots, k$ . In the second term do the same for the index  $j_{m+2}$  as well. One obtains

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{\alpha, m+1} = \begin{bmatrix} n \\ k \end{bmatrix}_{\beta, m} + n \cdot \begin{bmatrix} n \\ k+1 \end{bmatrix}_{\alpha, m+1}. \tag{24}$$

Apply partition of  $n$  to the expression on the right-hand-side of Equation (21), to obtain

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\beta, m} = \frac{m+1}{k} \cdot \begin{bmatrix} n \\ k \end{bmatrix}_{\alpha, m} + \frac{k-(m+1)}{k} \cdot \begin{bmatrix} n \\ k \end{bmatrix}_{\alpha, m+1}. \tag{25}$$

After verifying that the theorem holds for  $n = 1$  (or, less trivially, for  $n = 2$ ), we make the induction hypothesis that Equation (22) holds for  $n$  (and all consistent

values of  $k$  and  $m$ , i.e.,  $k = 1, 2, \dots, n$  and  $m = 0, 1, \dots, k - 1$ . It follows from Equation (25) that Equation (23) holds for  $n$ . Finally, it follows from Equation (24) that Equation (22) holds for  $n + 1$  and all  $m \geq 1$ . The case  $m = 0$  was established above. This completes the proof of the theorem.  $\square$

**Theorem 7.** *We have*

$$\begin{aligned} & \begin{bmatrix} n \\ k \end{bmatrix} \\ &= \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{(n-1)!}{\binom{k-2}{m} j_1 \cdot j_2 \cdots j_m (j_{k-1} - j_{m+1})(j_{k-1} - j_{m+2}) \cdots (j_{k-1} - j_{k-2}) j_{k-1}}. \end{aligned} \tag{26}$$

*Proof.* Apply  $S$  followed by  $I_{k-2}$  to obtain the expression in the first part of Theorem 6.  $\square$

**Theorem 8.** *We have*

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix} &= \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{(n-1)!}{\binom{k-1}{m} j_1 \cdot j_2 \cdots j_m (j_{k-1} - j_m)(j_{k-1} - j_{m+1}) \cdots (j_{k-1} - j_{k-2})}. \end{aligned} \tag{27}$$

*Proof.* Apply  $S$  and then partition of  $n$  into  $j_m + (n - j_m)$ . Each of the two sums obtained is equal to  $\begin{bmatrix} n \\ k-1 \end{bmatrix}$  by Theorem 6.  $\square$

**Remark.** For  $m = 0$  both Theorem 7 and Theorem 8 reduce to Equation (13) (setting  $j_0 = 0$ ). For  $m > 0$  they correspond to distinct shadows, that of Theorem 7 containing  $k - 1$ .

**3.3. Identities Involving  $(j_2 - j_1)(j_3 - j_2) \cdots (j_{k-1} - j_{k-2})$**

In the present section let  $Q_\pi = (j_2 - j_1)(j_3 - j_2) \cdots (j_{k-1} - j_{k-2})$ . Equation (7) states that  $j_1 \cdot Q_\pi$  is Stirling.

**Lemma 9.**  $j_{k-1} \cdot Q_\pi$  is Stirling.

*Proof.* Writing  $\begin{bmatrix} n \\ k \end{bmatrix}^* = \sum_{j_1 < j_2 < \dots < j_{k-1}} \frac{(n-1)!}{(k-2)!j_{k-1} \cdot Q_\pi}$ , we note that  $\begin{bmatrix} k \\ k \end{bmatrix}^* = 1$  and obtain

$$S \begin{bmatrix} n \\ k \end{bmatrix}^* = \sum_{j_1 < j_2 < \dots < j_{k-2}}^{n-1} \frac{(n-1)!}{(k-2)!(j_2 - j_1)(j_3 - j_2) \cdots (j_{k-2} - j_{k-3})(n - j_{k-2})}. \tag{28}$$

Application of the  $I_{k-2}$ -transformation yields

$$S \begin{bmatrix} n \\ k \end{bmatrix}^* = \sum_{j_1 < j_2 < \dots < j_{k-2}}^{n-1} \frac{(n-1)!}{(k-2)!(j_{k-2} - j_{k-3})(j_{k-3} - j_{k-4}) \cdots (j_2 - j_1)j_1} \tag{29}$$

which is equal to  $\begin{bmatrix} n \\ k-1 \end{bmatrix}$ , by Equation (7). (The same result could be derived by application of  $U$  to Equation (7), followed by application of  $S$ .)  $\square$

Since  $Q_\pi$  is invariant under  $I_{k-1}$  it follows immediately that

**Lemma 10.**  $(n - j_1)Q_\pi$  and  $(n - j_{k-1})Q_\pi$  are both Stirling.

**Theorem 11.**  $\begin{bmatrix} n \\ k \end{bmatrix}_{j_\ell \cdot Q_\pi} = \sum_{j_1 < j_2 < \dots < j_{k-1}} \frac{(n-1)!}{\alpha_\ell j_\ell \cdot Q_\pi}$ ,  $\ell = 1, 2, \dots, k - 1$ , where  $\alpha_\ell = \frac{(k-1)!}{\ell}$ , is a Stirling number of the first kind.

*Proof.* The proof is by induction, using Lemma 9 with  $k - 1 = \ell$  to start the induction. To proceed, it is convenient to use the  $i$ -indices.  $Q_\pi$  becomes  $P_\pi = i_2 i_3 \cdots i_{k-1}$ . An application of Equation (19) to the sum in the theorem, followed by partition of  $n$  into  $(i_1 + i_2 + \cdots + i_\ell) + i_{\ell+1} + i_{\ell+2} + \cdots + i_{k-1}$ , yields

$$S \begin{bmatrix} n \\ k \end{bmatrix}_{j_\ell \cdot Q_\pi} = \frac{\ell}{k-1} \sum_{\substack{i_1, i_2, \dots, i_{k-1} \\ (i_1 + i_2 + \dots + i_{k-1} = n)}} \frac{(n-1)!}{\beta_\ell i_2 i_3 \cdots i_{k-1}} \tag{30}$$

$$+ \frac{k-1-\ell}{k-1} \sum_{\substack{i_1, i_2, \dots, i_{k-1} \\ (i_1 + i_2 + \dots + i_{k-1} = n)}} \frac{(n-1)!}{\gamma_\ell (i_1 + i_2 + \cdots + i_\ell) i_2 i_3 \cdots i_\ell i_{\ell+2} \cdots i_{k-1}}$$

where  $\beta_\ell = (k-2)!$  and  $\gamma_\ell = \frac{(k-2)!}{\ell}$ . The second sum on the right-hand-side incorporates the  $k-1-\ell$  equivalent sums obtained by cancellation of any one of the factors  $i_{\ell+1}, i_{\ell+2}, \dots, i_{k-1}$ . The summand in the first sum on the right-hand-side of Equation (30) does not depend on  $i_1$ , so this sum can be written as a sum over  $i_2, i_3, \dots, i_{k-1}$  with the condition  $i_2 + i_3 + \cdots + i_{k-1} \leq n - 1$ , showing that it is equal to  $\begin{bmatrix} n \\ k-1 \end{bmatrix}$  by Equation (5). The summand in the second sum does not depend on  $i_{\ell+1}$ , which can similarly be eliminated from the set of summation indices. This sum is also equal to  $\begin{bmatrix} n \\ k-1 \end{bmatrix}$ , by the induction hypothesis.  $\square$

**3.4. Expressions Involving  $(j_2 - j_1)(j_3 - j_1) \cdots (j_{k-1} - j_1)$**

In the present section let  $Q_\delta = (j_2 - j_1)(j_3 - j_1) \cdots (j_{k-1} - j_1)$ .

The case  $j_{k-1} \cdot Q_\delta$  is taken care of by Equation (12).

**Lemma 12.**  $j_1 \cdot Q_\delta$  is Stirling.

*Proof.* Apply  $S$  followed by a partition of  $n$  into  $(n - j_1) + j_1$  to obtain

$$S \begin{bmatrix} n \\ k \end{bmatrix}_{j_1 \cdot Q_\delta} = \frac{k-2}{k-1} \sum_{j_1 < j_2 < \cdots < j_{k-2}}^{n-1} \frac{(n-1)!}{(k-2)j_1(j_2-j_1)(j_3-j_1) \cdots (j_{k-2}-j_1)} \quad (31)$$

$$+ \frac{1}{k-1} \sum_{j_1 < j_2 < \cdots < j_{k-2}}^{n-1} \frac{(n-1)!}{(j_2-j_1)(j_3-j_1) \cdots (j_{k-2}-j_1)(n-j_1)}.$$

Apply  $I_{k-2}$  to the second sum on the right-hand-side to obtain

$$\sum_{j_1 < j_2 < \cdots < j_{k-2}}^{n-1} \frac{(n-1)!}{(j_{k-2}-j_1)(j_{k-2}-j_2) \cdots (j_{k-2}-j_{k-3})j_{k-2}},$$

which is equal to  $\begin{bmatrix} n \\ k-1 \end{bmatrix}$  by Equation (13). The first sum on the right-hand-side of Equation (31) equals  $\begin{bmatrix} n \\ k-1 \end{bmatrix}$  by the induction hypothesis. Hence, the lemma follows.  $\square$

**3.5. Expressions Involving  $(j_{k-1} - j_1)(j_{k-1} - j_2) \cdots (j_{k-1} - j_{k-2})$**

The expression  $(j_{k-1} - j_1)(j_{k-1} - j_2) \cdots (j_{k-1} - j_{k-2})j_{k-1}$  is Stirling by Equation (13).

We shall now prove

**Lemma 13.** *We have*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j_1 < j_2 < \cdots < j_{k-1}}^{n-1} \frac{(n-1)!}{(k-1)j_1(j_{k-1}-j_1)(j_{k-1}-j_2) \cdots (j_{k-1}-j_{k-2})} \quad (32)$$

*Proof.* Application of  $S$ -derivation to Equation (32), followed by the application of  $I_{k-2}$ , yields

$$S \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j_1 < j_2 < \cdots < j_{k-2}}^{n-1} \frac{n!}{(k-1)j_{k-2}(n-j_{k-2}) \cdot j_{k-3} \cdot j_{k-4} \cdots j_1}.$$



Application of partition of  $n$  into  $(n - j_{k-2}) + j_{k-2}$  yields

$$S \begin{bmatrix} n \\ k \end{bmatrix} = \frac{k-2}{k-1} \cdot \sum_{j_1 < j_2 < \dots < j_{k-2}}^{n-1} \frac{(n-1)!}{(k-2)j_1 \cdot j_2 \cdots j_{k-3}(n-j_{k-2})} + \frac{1}{k-1} \cdot \sum_{j_1 < j_2 < \dots < j_{k-2}}^{n-1} \frac{(n-1)!}{j_1 \cdot j_2 \cdots j_{k-2}}.$$

Both the first and the second sum on the right-hand-side are equal to  $\begin{bmatrix} n \\ k-1 \end{bmatrix}$ , the former by Equation (20) and the latter by Equation (4).  $\square$

### 3.6. Identities With a Single Pair of Indices

In this section we consider expressions of the form

$$\sum_{j_1 < j_2 < \dots < j_{k-1}} \frac{(n-1)!}{(j_p - j_q)(j_*) \cdot Q''}, \text{ where } Q'' = \prod_{\substack{i=1 \\ i \neq p; i \neq q}}^{k-1} j_i,$$

and where  $j_*$  is either  $j_p$  or  $j_q$ .

Let

$$\begin{bmatrix} n \\ k \end{bmatrix}_\gamma = \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{(n-1)!}{(k-1) \cdot j_1 \cdot j_2 \cdots j_{k-2}(j_{k-1} - j_{k-2})}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_\delta = \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{n!}{(k-2) \cdot j_1 \cdot j_2 \cdots j_{k-3} \cdot j_{k-1}(j_{k-1} - j_{k-2})}.$$

**Theorem 14.** *We have  $\begin{bmatrix} n \\ k \end{bmatrix}_\gamma = \begin{bmatrix} n \\ k \end{bmatrix}_\delta = \begin{bmatrix} n \\ k \end{bmatrix}$ .*

*Proof.* We first verify that the Theorem holds for  $n = k$ . Fixing  $k$  (arbitrarily) we assume that the Theorem holds for  $n$  and show that it holds for  $n + 1$ . Write  $\begin{bmatrix} n+1 \\ k \end{bmatrix}_\gamma$  and separate the sum over  $j_{k-1}$  into a sum from 1 to  $n - 1$  and a term with  $j_{k-1} = n$ . Obtain,

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_\gamma = n \begin{bmatrix} n \\ k \end{bmatrix}_\gamma + \begin{bmatrix} n \\ k-1 \end{bmatrix}_\beta.$$

The second term on the right-hand side equals  $\begin{bmatrix} n \\ k-1 \end{bmatrix}$  by Theorem 6, while the first term equals  $n \begin{bmatrix} n \\ k \end{bmatrix}$  by the induction hypothesis. Hence,  $\begin{bmatrix} n+1 \\ k \end{bmatrix}_\gamma = \begin{bmatrix} n+1 \\ k \end{bmatrix}$ . A similar argument, using the other part of Theorem 6, proves the theorem for  $\begin{bmatrix} n \\ k \end{bmatrix}_\delta$ .  $\square$

We now consider two extensions of the theorem. We first show that it does not hold if the indices in the single pair are not consecutive.

Consider

$$\left[ \begin{matrix} n \\ k \end{matrix} \right]_{\epsilon} = \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{2 \cdot (n-1)!}{(k-1) \cdot j_1 \cdot j_2 \cdots j_{k-2} (j_{k-1} - j_{k-3})}.$$

Note that  $\left[ \begin{matrix} k \\ k \end{matrix} \right]_{\epsilon} = 1$  and  $\left[ \begin{matrix} k+1 \\ k \end{matrix} \right]_{\epsilon} = k$ . Since  $\left[ \begin{matrix} k \\ k \end{matrix} \right] = 1$  and  $\left[ \begin{matrix} k+1 \\ k \end{matrix} \right] = \frac{k(k+1)}{2}$  it is clear that  $\left[ \begin{matrix} n \\ k \end{matrix} \right]_{\epsilon}$  is not a Stirling number.

The second extension we consider retains the consecutive indices in the single pair, but allows them not to be at the top. Since application of  $S$  a sufficient number of times reduces the expression to one with the pair at the top, we obtain

**Theorem 15.** *We have*

$$\begin{aligned} \left[ \begin{matrix} n \\ k \end{matrix} \right] &= \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{(n-1)!}{(\ell+1) \cdot j_1 \cdot j_2 \cdots j_{\ell} (j_{\ell+1} - j_{\ell}) j_{\ell+2} \cdots j_{k-1}} \\ &= \sum_{j_1 < j_2 < \dots < j_{k-1}}^{n-1} \frac{(n-1)!}{\ell \cdot j_1 \cdot j_2 \cdots j_{\ell-1} (j_{\ell+1} - j_{\ell}) j_{\ell+1} \cdots j_{k-1}} \end{aligned}$$

### 3.7. A Conjecture

**Conjecture 16.** *The following holds:*

$$\left[ \begin{matrix} n \\ 2k+1 \end{matrix} \right] = \sum_{j_1 < j_2 < \dots < j_{2k}}^{n-1} \frac{(n-1)!}{2j_1 j_2 \cdots j_{2k-1} (j_{2k} - j_{2k-1} + j_{2k-2} - \dots + j_2 - j_1)}.$$

## 4. Comprehensive Exploration For Low k Values

A comprehensive search for all minimal homogeneous  $Q$ 's that consist of a product of  $k - 2$  differences of pairs of indices and one single index, for  $k \leq 6$ , was carried out. The corresponding expressions were normalized at  $n = k$  and then evaluated for higher  $n$  to discard those that are non-Stirling. Most of the expressions that pass this test are special cases of expressions that were shown to be Stirling above, or can be related to such expressions by means of appropriate transformations. The remaining expressions can be derived assuming that Conjecture 16 holds.

Of course, one should expect that for higher  $k$  the theorems derived above, along with Conjecture 16, will not be sufficient to establish the Stirling property for all possible candidates.

**4.1. The Cases  $k = 1, k = 2,$  and  $k = 3$**

These cases are mentioned for the sake of completeness. For  $k = 1$  one obtains from either the recurrence relation, Equation (4), Equation (5) or the combinatorial meaning of the Stirling numbers of the first kind that  $\left[ \begin{matrix} n \\ 1 \end{matrix} \right] = (n - 1)!$ .

For  $k = 2$  one obtains  $\left[ \begin{matrix} n \\ 2 \end{matrix} \right] = \sum_{j_1=1}^{n-1} \frac{(n-1)!}{j_1}$ , which follows from both Equation (4) and Equation (7).

The minimal identities corresponding to  $k = 3$  were considered in section 2.3. Six identities with the shadow (1, 2) and three with the shadow (1, 1) were presented in Equations (10) and (11), respectively.

**4.2.  $k = 4$**

In this case each normalization coefficient still corresponds to a unique shadow. The orbit under the  $T$ -transformations (the  $T$ -orbit) is presented for each normalization coefficient, and the Stirling property is established using the general results presented in the previous section.

**4.2.1.  $\alpha = 6$**

The expression involving  $j_1(j_2 - j_1)(j_3 - j_2)$  is a special case of Equation (7). It is invariant under both  $t_1$  and  $t_2$ .

**4.2.2.  $\alpha = 3$**

This case gives rise to the  $T$ -orbit

$$\begin{array}{ccc}
 j_1 j_2 (j_3 - j_2) & \xleftrightarrow{t_1} & (j_3 - j_2)(j_2 - j_1) j_2 \\
 t_2 \downarrow & & \downarrow t_2 \\
 t_1 \downarrow & & \downarrow t_1 \\
 j_1 (j_2 - j_1)(j_3 - j_1) & \xleftrightarrow{t_2} & j_1 (j_3 - j_2)(j_3 - j_1)
 \end{array} \tag{33}$$

where we have skipped two intermediate expressions of a more complicated form. The expression on the right-hand-side of the top row is a special case of Theorem 11, and that on the left-hand-side of the top row is a special case of Theorem 15. The expressions on the left- and right-hand-sides of the bottom row are special cases of Lemmas 12 and 13, respectively.

**4.2.3.  $\alpha = 2$**

The sequence

$$t_2 \subset (j_2 - j_1)(j_3 - j_2)j_3 \xleftrightarrow{t_1} j_1(j_3 - j_2)j_3 \xleftrightarrow{t_2} j_1(j_2 - j_1)j_3 \supset t_1 \quad (34)$$

is closed because the leftmost member is invariant under  $t_2$  and the rightmost member is invariant under  $t_1$ . The intermediate as well as the rightmost expressions are Stirling by Theorem 15, and the leftmost expression is Stirling by Lemma 9.

**4.2.4.  $\alpha = 1$**

The expressions corresponding to this normalization coefficient form the  $T$ -orbit

$$\begin{array}{ccc} j_1j_2j_3 & \xleftrightarrow{t_1} & (j_2 - j_1)j_2j_3 \\ t_2 \downarrow & & \uparrow t_2 \\ t_1 \downarrow & & \uparrow t_1 \\ (j_2 - j_1)(j_3 - j_1)j_3 & \xleftrightarrow{t_2} & (j_3 - j_1)(j_3 - j_2)j_3 \end{array} \quad (35)$$

The expressions on the left- and right-hand-sides of the top row are special cases of Equation (4) and Theorem 15, respectively. Those on the left- and right-hand-sides of the bottom row are Stirling by Equations (14) and (15), respectively.

**4.3.  $k = 5$**

Here we encounter the first suspected Stirling identity that does not follow from any of the results presented above, requiring the application of Conjecture 16. We also encounter, for the first time, examples in which the same normalization coefficient is obtained for two different shadows. The notation  $s : Q$  will be used to specify a shadow  $s$  and a representative expression  $Q$ . Wherever possible, this will be followed by a brief indication of the reason for the specified  $Q$  to be Stirling. Orbits under multiset-automorphisms are not examined.

**4.3.1.  $\alpha = 24$**

The expression  $j_1(j_2 - j_1)(j_3 - j_2)(j_4 - j_3)$  is Stirling by Equation (7).

**4.3.2.  $\alpha = 12$**

The expression  $j_2(j_2 - j_1)(j_3 - j_2)(j_4 - j_3)$  is Stirling by Theorem 11.

**4.3.3.  $\alpha=8$**

The expression  $j_3(j_2 - j_1)(j_3 - j_2)(j_4 - j_3)$  is a special case of Theorem 11.

**4.3.4.  $\alpha=6$ . Two Different Shadows**

$$\begin{aligned} (1, 1, 1, 4) : & \quad (j_2 - j_1)(j_3 - j_2)(j_4 - j_3)j_4 \text{ is Stirling by Lemma 9.} \\ (1, 1, 2, 2) : & \quad j_2(j_2 - j_1)(j_4 - j_2)(j_4 - j_3) \xrightarrow{t_1} j_1j_2(j_4 - j_2)(j_4 - j_3), \\ & \quad \text{is Stirling by Theorem 8.} \end{aligned}$$

**4.3.5.  $\alpha=4$**

$$(1, 1, 2, 3) : \quad j_1(j_2 - j_1)(j_3 - j_1)(j_4 - j_1) \xrightarrow{t_3t_2t_1} j_1j_2j_3(j_4 - j_3), \text{ Stirling by Theorem 15.}$$

**4.3.6.  $\alpha=3$ . Two Different Shadows**

The shadow  $(1, 2, 2, 2)$  is represented by  $j_2(j_2 - j_1)(j_3 - j_1)(j_4 - j_2)$ . To establish the fact that this expression is Stirling we apply  $S$ , partition of  $n$  into  $j_2 + (n - j_2)$ , followed by application of  $I_3$  to the resulting inhomogeneous expression. The two expressions obtained are  $\sum_{j_1 < j_2 < j_3}^{n-1} \frac{(n-1)!}{3j_2(j_3 - j_1)(j_3 - j_2)}$  and  $\sum_{j_1 < j_2 < j_3}^{n-1} \frac{(n-1)!}{3j_2(j_3 - j_1)(j_2 - j_1)}$ . None of these is Stirling, but summing their summands we obtain

$$\sum_{j_1 < j_2 < j_3}^{n-1} \frac{(n-1)!}{3j_2(j_3 - j_2)(j_2 - j_1)},$$

which is Stirling by Theorem 11.

The second shadow,  $(1, 1, 2, 4)$ , gives rise to

$$j_4(j_2 - j_1)(j_3 - j_1)(j_4 - j_3) \xrightarrow{t_2t_1} j_1j_2j_4(j_4 - j_3), \text{ Stirling by Theorem 14.}$$

**4.3.7.  $\alpha=2$ . Two Different Shadows, One of Which Depends on a Conjecture**

$$\begin{aligned} (1, 2, 2, 3) : & \quad j_2(j_3 - j_1)(j_4 - j_1)(j_3 - j_2) \xrightarrow{t_1t_2t_3t_1} j_1j_2j_3(j_4 - j_3 + j_2 - j_1), \\ & \quad \text{is Stirling if Conjecture 16 holds.} \\ (1, 1, 3, 4) : & \quad t_2 \subset j_4(j_2 - j_1)(j_4 - j_1)(j_3 - j_2) \xrightarrow{t_3t_2t_1} j_1j_3j_4(j_2 - j_1), \\ & \quad \text{is Stirling by Theorem 15.} \end{aligned}$$

**4.3.8.  $\alpha = 1$**

This is a special case of Equation (12).

**4.4.  $k = 6$**

Here we encounter the first instance of a shadow that corresponds to no  $Q$  with the Stirling property.

**4.4.1.  $\alpha = 120, 60, 40, 24$**

Each of these four cases corresponds to a unique shadow. Each has a member of the form  $j_\ell(j_2 - j_1)(j_3 - j_2)(j_4 - j_3)(j_5 - j_4)$  with  $\ell = 1, 2, 3, 5$ , respectively. These expressions are Stirling by Theorem 11.

**4.4.2.  $\alpha = 30$ . Two Different Shadows**

$(1, 1, 1, 1, 4)$  :  $j_4(j_2 - j_1)(j_3 - j_2)(j_4 - j_3)(j_5 - j_4)$  is Stirling by Theorem 11.  
 $(1, 1, 1, 2, 2)$  :  $j_2(j_2 - j_1)(j_3 - j_2)(j_4 - j_3)(j_5 - j_3)$ , upon application of  $S$  followed by partition of  $n$  into  $(n - j_3) + (j_3 - j_2) + j_2$ , yields three expressions. The one obtained by cancellation of  $(n - j_3)$  can be readily identified as Stirling. The one obtained by cancellation of  $j_2$  can be transformed into the first one by application of  $I_4$ . Application of  $t_1$  to the third expression yields  $j_1 j_2 (j_4 - j_3)(n - j_3)$ , which is Stirling by application of  $S$  followed by partition of  $n$ .

**4.4.3.  $\alpha = 20$**

Application of  $S$  on  $(1, 1, 1, 2, 3)$  :  $j_3(j_2 - j_1)(j_3 - j_2)(j_4 - j_3)(j_5 - j_3)$ , followed by partition of  $n$ , yields two expressions, both being Stirling by Theorem 11.

**4.4.4.  $\alpha = 15$ . Two Different Shadows**

Application of  $S$  followed by partition of  $n$  on  $(1, 1, 1, 2, 4)$  :  $j_4(j_2 - j_1)(j_3 - j_2)(j_4 - j_2)(j_5 - j_4)$  yields a term that is Stirling, belonging to the case  $k = 5, \alpha = 3$ , shadow  $(1, 1, 2, 4)$ , and a term that after application of  $I_4$  is Stirling, belonging to the case  $k = 5, \alpha = 12$ .

$(1, 1, 2, 2, 2)$  :  $j_1(j_3 - j_1)(j_3 - j_2)(j_4 - j_2)(j_5 - j_3)$ , upon application of  $S$  followed by  $I_4$ , yields an expression that allows partition of  $n$  into  $(n - j_4) + (j_4 - j_2) + j_2$ . The expression obtained by cancellation of  $(n - j_4)$  is Stirling (see  $k = 5, \alpha = 3$  above). However, neither of the other two expressions is Stirling. We shall show that their sum is Stirling (which is sufficient). We apply  $S$  on the term obtained by cancellation of the factor  $(j_4 - j_2)$ , and we apply  $I_4$ , followed

by  $S$ , followed by  $I_3$ , on the term obtained by cancellation of  $j_2$ . The resulting expressions,  $\frac{1}{n}j_2(j_3 - j_1)(j_3 - j_2)(n - j_3)$  and  $\frac{1}{n}j_2(j_2 - j_1)(j_3 - j_1)(n - j_3)$ , yield sums that can be added, obtaining a standard sum containing the expression  $\frac{1}{n}j_2(j_3 - j_2)(j_2 - j_1)(n - j_3)$ . To show that this sum is Stirling we perform partition of  $n$  to obtain three terms, each of which is Stirling.

**4.4.5.  $\alpha = 12$**

$(1, 1, 1, 2, 5) : j_5(j_2 - j_1)(j_3 - j_2)(j_4 - j_2)(j_5 - j_4) \xrightarrow{S, I_4}$  Stirling by case  $k = 5, \alpha = 12$ .

**4.4.6.  $\alpha = 10$ . Two Different Shadows, One of Which Depends on a Conjecture**

The expression  $(1, 1, 1, 3, 4) : j_4(j_2 - j_1)(j_4 - j_1)(j_4 - j_3)(j_5 - j_4)$  reduces, after application of  $S$  followed by partition of  $n$ , to a sum of two expressions, one of which is Stirling if Conjecture 16 holds, and the other becomes, after application of  $I_4$ , a member of the case  $k = 5 \alpha = 8$ .

The expression  $(1, 1, 2, 2, 3) : j_3(j_2 - j_1)(j_3 - j_1)(j_4 - j_3)(j_5 - j_3)$  reduces, after application of  $S$ , into a sum of two expressions. One of them belongs to the case  $k = 5 \alpha = 4$ , and the other, after application of  $I_4$ , belongs to the case  $k = 5, \alpha = 6$ .

**4.4.7.  $\alpha = 8$**

$(1, 1, 1, 3, 5) : j_5(j_2 - j_1)(j_4 - j_1)(j_3 - j_2)(j_5 - j_4) \xrightarrow{S, I_4}$  Stirling by case  $k = 5, \alpha = 8$ .

**4.4.8.  $\alpha = 6$ . Two Different Shadows**

$(1, 1, 1, 4, 5) : j_5(j_2 - j_1)(j_5 - j_1)(j_3 - j_2)(j_4 - j_3) \xrightarrow{S, I_4}$  Stirling by Theorem 11.

$(1, 1, 2, 2, 5) : j_5(j_2 - j_1)(j_3 - j_1)(j_4 - j_3)(j_5 - j_3) \xrightarrow{S, I_4}$  Stirling by case  $k = 5, \alpha = 6$ .

**4.4.9.  $\alpha = 5$ . Two Different Shadows, One of Which Yields No Stirling Expression**

$(1, 1, 2, 3, 4)$ :  $j_1(j_5 - j_1)(j_5 - j_2)(j_5 - j_3)(j_5 - j_4)$  is Stirling by Equation (32).

The shadow  $(1, 2, 2, 2, 3)$  does not correspond to any expression that is Stirling. This is the first case encountered of such a shadow.

**4.4.10.  $\alpha = 4$**

$(1, 1, 2, 3, 5)$ :  $j_5(j_4 - j_1)(j_4 - j_2)(j_4 - j_3)(j_5 - j_4) \xrightarrow{S, I_4}$  Stirling by Lemma 12.

**4.4.11.  $\alpha = 3$ . Two Different Shadows**

$(1, 1, 2, 4, 5)$ :  $j_5(j_3 - j_1)(j_5 - j_1)(j_3 - j_2)(j_4 - j_3) \xrightarrow{S, I_4}$  Stirling by case  $k = 5$ ,  $\alpha = 3$ .

$(1, 2, 2, 2, 5)$ :  $j_5(j_3 - j_1)(j_3 - j_2)(j_4 - j_2)(j_5 - j_3) \xrightarrow{S, I_4}$  Stirling by case  $k = 5$ ,  $\alpha = 3$ .

**4.4.12.  $\alpha = 2$ . Two Different Shadows, One of Which Depends on a Conjecture**

$(1, 1, 3, 4, 5)$ :  $j_5(j_5 - j_1)(j_3 - j_2)(j_5 - j_2)(j_4 - j_3) \xrightarrow{T} j_1(j_2 - j_1)j_3j_4j_5$ , where  $T = t_3t_2t_1t_2t_4t_3t_2t_1$ . The rightmost expression is Stirling by Theorem 15.

The shadow  $(1, 2, 2, 3, 5)$  corresponds to expressions that are Stirling if Conjecture 16 holds.

**4.4.13.  $\alpha = 1$**

This is a special case of Equation (12).

**5. Concluding Remarks**

The multitude of expressions for the Stirling numbers of the first kind presented in this article reveal a new facet of these time-honored entities. The results presented are certainly incomplete. The conjectures and the open problem stated, as well as the highly likely existence of additional expressions whose Stirling property can be established, call for further work. The issue of  $q$ -analogues appears to be far from trivial.

**Acknowledgement.** Helpful discussions with Professor Johann Cigler are gratefully acknowledged.



**References**

- [1] R. P. Stanley, *Enumerative Combinatorics, Vol. I*, Cambridge University Press, Cambridge 1997.
- [2] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht 1974.
- [3] V. Adamchik, On Stirling numbers and Euler sums, *J. Comput. Appl. Math.* **79** (1997) 119-130.