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A NOTE ON THE EXACT EXPECTED LENGTH OF THE KTH PART OF A RANDOM PARTITION

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Abstract

Kessler and Livingstone proved an asymptotic formula for the expected length of the largest part of a partition drawn uniformly at random. As a first step they gave an exact formula expressed as a weighted sum of Euler's partition function. Here we give a short bijective proof of a generalization of this exact formula to the expected length of the kth part.

1. Results

By $\lambda \vdash n$ we will mean that λ is a partition of n. This means that λ is a finite non-increasing sequence of positive integers, $\lambda_1 \geq \cdots \geq \lambda_N > 0$, which sums to n. The number of partitions of n is Euler's famous partition function $p(n)$, with $p(0) = 1$ by convention.

Corteel *et al.* [1] mention a well-known partition identity attributed to Stanley: The expected number of different part sizes of a uniformly drawn partition $\lambda \vdash n$ is

$$\frac{1}{p(n)} \sum_{\ell \geq 1} \ell \cdot p_\delta(n, \ell) = \frac{1}{p(n)} \sum_{m=0}^{n-1} p(m). \tag{1}$$

Here, $p_\delta(n, \ell)$ denotes the number of partitions of n with exactly ℓ different part sizes. The combinatorial proof in [1] is very simple: For any partition of $m = 0, 1, \ldots, n-1$, create a partition of n by adjoining a part of size $n-m$. In so doing, any given partition of n is created in as many copies as it has different part sizes.

First observe that this proof immediately generalizes to give a formula for the expected number of different part sizes $\geq k$ (that is, not counting any parts of size less than k):

$$\frac{1}{p(n)} \sum_{\ell \geq 1} \ell \cdot p_\delta(n, \ell, k) = \frac{1}{p(n)} \sum_{m=0}^{n-k} p(m), \tag{2}$$

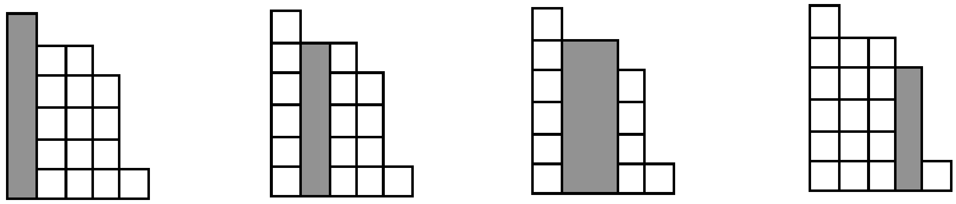

Figure 1: The $\lambda_2 = 4$ ways of obtaining partitions by removing a rectangle of height $d \geq 2$ from the Young diagram of partition $\lambda = (5, 4, 4, 4, 3, 1)$.

where $p_\delta(n, \ell, k)$ denotes the number of partitions of n with exactly ℓ different part sizes $\geq k$.

In this note we will make a similar generalization, with a combinatorial proof of the same flavor as above, of a formula of Kessler and Livingstone [3] for the expected length of the largest part λ_1 (or, equivalently, the number of parts) of a partition $\lambda \vdash n$ drawn uniformly at random:

$$E(\lambda_1) = \frac{1}{p(n)} \sum_{\lambda \vdash n} \lambda_1 = \frac{1}{p(n)} \sum_{m=1}^{n} p(n-m) \cdot \#\{d|m\}, \tag{3}$$

where $\#\{d|m\}$ denotes the number of divisors of m. Kessler and Livingstone used generating functions to prove (3). They then used this formula as a stepping stone toward an asympotic formula for $E(\lambda_1)$. For the large and interesting literature on asymptotic formulas for parts of integer partitions, we refer to Fristedt [2] and Pittel [4]. Here we focus on the finite formula (3). We shall present a simple combinatorial proof that immediately generalizes to the expected length of the kth longest part, λ_k:

$$E(\lambda_k) = \frac{1}{p(n)} \sum_{\lambda \vdash n} \lambda_k = \frac{1}{p(n)} \sum_{m=1}^{n} p(n-m) \cdot \#\{d|m : d \geq k\}. \tag{4}$$

Lemma 1 *Let λ be any integer partition with kth part $\lambda_k > 0$. Then λ_k is also the number of pairs of integers $r \geq 1$ and $d \geq k$ such that subtracting r from each of the d largest parts of λ results in a new partition.*

Proof. Let N be the number of parts of λ, and temporarily define $\lambda_{N+1} = 0$. After subtracting r from each of the d largest parts of λ, what remains is a partition if and only if $\lambda_d - r \geq \lambda_{d+1}$. Thus for each value of $d \geq k$ we have $\lambda_d - \lambda_{d+1}$ possible values of r. The total number of possibilities is

$$(\lambda_k - \lambda_{k+1}) + (\lambda_{k+1} - \lambda_{k+2}) + \cdots + (\lambda_N - \lambda_{N+1}),$$

which simplifies to $\lambda_k - \lambda_{N+1} = \lambda_k$. $\square$

Figure 1 illustrates the lemma.

Proof of (4). For any partition of $n - m$, with $m = 1, \ldots, n$, and any divisor $d \geq k$ of m, create a partition of n by adding the integer $r = m/d \geq 1$ to each of the d largest parts. In so doing, any given partition λ of n is created in exactly λ_k copies according to the lemma. □

Acknowledgments

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