BINOMIAL COEFFICIENT–HARMONIC SUM IDENTITIES ASSOCIATED TO SUPERCONGRUENCES

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Abstract

We establish two binomial coefficient–generalized harmonic sum identities using the partial fraction decomposition method. These identities are a key ingredient in the proofs of numerous supercongruences. In particular, in other works of the author, they are used to establish modulo p^k $(k > 1)$ congruences between truncated generalized hypergeometric series, and a function which extends Greene's hypergeometric function over finite fields to the p-adic setting. A specialization of one of these congruences is used to prove an outstanding conjecture of Rodriguez-Villegas which relates a truncated generalized hypergeometric series to the p-th Fourier coefficient of a particular modular form.

1. Introduction and Statement of Results

For non-negative integers i and n, we define the generalized harmonic sum, $H_n^{(i)}$, by

$$H_n^{(i)} := \sum_{j=1}^{n} \frac{1}{j^i}$$

and $H_0^{(i)} := 0$. In [3] Chu proves the following binomial coefficient-generalized harmonic sum identity using the partial fraction decomposition method. If n is a positive integer, then

$$\sum_{k=1}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \left[1 + 2kH_{n+k}^{(1)} + 2kH_{n-k}^{(1)} - 4kH_k^{(1)}\right] = 0. \tag{1}$$

This identity had previously been established using the WZ method [1] and was used by Ahlgren and Ono in proving the Apéry number supercongruence [2].

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In [4], [5] the author establishes various supercongruences between truncated generalized hypergeometric series, and a function which extends Greene's hypergeometric function over finite fields to the p-adic setting. Specifically, let p be an odd prime and let $n \in \mathbb{Z}^+$. For $1 \le i \le n+1$, let $\frac{m_i}{d_i} \in \mathbb{Q} \cap \mathbb{Z}_p$ such that $0 < \frac{m_i}{d_i} < 1$. Let $\Gamma_p(\cdot)$ denote Morita's p-adic gamma function. Then define

$$
{}_{n+1}G\left(\tfrac{m_1}{d_1}, \tfrac{m_2}{d_2}, \dots, \tfrac{m_{n+1}}{d_{n+1}}\right)_p := \frac{-1}{p-1} \sum_{j=0}^{p-2} \left((-1)^j \Gamma_p\left(\tfrac{j}{p-1}\right)\right)^{n+1} \prod_{i=1}^{n+1} \frac{\Gamma_p\left(\langle \frac{m_i}{d_i} - \frac{j}{p-1} \rangle\right)}{\Gamma_p\left(\frac{m_i}{d_i}\right)} (-p)^{-\lfloor \frac{m_i}{d_i} - \frac{j}{p-1} \rfloor}.
$$

Note that when $p \equiv 1 \pmod{d_i}$ this function recovers Greene's hypergeometric function over finite fields. For a complex number a and a non-negative integer n let $(a)_n$ denote the rising factorial defined by

$$
(a)_0 := 1 \quad \text{and} \quad (a)_n := a(a+1)(a+2)\cdots(a+n-1) \text{ for } n > 0.
$$

Then, for complex numbers a_i, b_j and z, with none of the b_j being negative integers or zero, we define the truncated generalized hypergeometric series

$$
{}_pF_q\left[\begin{matrix} a_1, & a_2, & a_3, & \dots, & a_p \\ & b_1, & b_2, & \dots, & b_q \end{matrix} \,\middle|\, z\right]_m := \sum_{n=0}^{m} \frac{(a_1)_n (a_2)_n (a_3)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}.
$$

An example of one the supercongruence results from [5] is the following theorem.

Theorem 1. (Theorem 2.6 in [5]) *Let $r, d \in \mathbb{Z}$ such that $2 \le r \le d-2$ and $\gcd(r,d) = 1$. Let p be an odd prime such that $p \equiv \pm 1 \pmod d$ or $p \equiv \pm r \pmod d$ with $r^2 \equiv \pm 1 \pmod d$. If $s(p) := \Gamma_p\left(\frac{1}{d}\right)\Gamma_p\left(\frac{r}{d}\right)\Gamma_p\left(\frac{d-r}{d}\right)\Gamma_p\left(\frac{d-1}{d}\right)$, then*

$$
{}_4G\left(\tfrac{1}{d}, \tfrac{r}{d}, 1-\tfrac{r}{d}, 1-\tfrac{1}{d}\right)_p \equiv {}_4F_3\left[\begin{matrix} \frac{1}{d}, & \frac{r}{d}, & 1-\frac{r}{d}, & 1-\frac{1}{d} \\ & 1, & 1, & 1 \end{matrix} \,\middle|\, 1\right]_{p-1} + s(p)p \pmod{p^3}.
$$

A specialization of this congruence is used to prove an outstanding supercongruence conjecture of Rodriguez-Villegas, which relates a truncated generalized hypergeometric series to the p-th Fourier coefficient of a particular modular form [4],[6]. Similar results to Theorem 1 exist for ${}_4G$ with other parameters, and also ${}_2G$ and ${}_3G$.

The main results of the current paper, Theorems 2 and 3 below, are two binomial coefficient–generalized harmonic sum identities which factor heavily into the proofs of all the ${}_4G$ congruences. Taking particular values for n, m, l, c_1 and c_2 in these identities allows the vanishing of certain terms in the proofs. Note that letting $m = n$ in Theorem 2 recovers (1).

Theorem 2. *Let m, n be positive integers with $m \geq n$. Then*

$$\sum_{k=0}^{n} \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k}\left[1+k\left(H_{m+k}^{(1)}+H_{m-k}^{(1)}+H_{n+k}^{(1)}+H_{n-k}^{(1)}-4H_{k}^{(1)}\right)\right]$$
$$+\sum_{k=n+1}^{m}(-1)^{k-n}\binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\Big/\binom{k-1}{n}=(-1)^{m+n}.$$

Theorem 3. *Let l, m, n be positive integers with $l > m \geq n \geq \frac{l}{2}$ and $c_1, c_2 \in \mathbb{Q}$ some constants. Then*

$$\sum_{k=0}^{n} \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k}\left\{\left[1+k\left(H_{m+k}^{(1)}+H_{m-k}^{(1)}+H_{n+k}^{(1)}+H_{n-k}^{(1)}-4H_{k}^{(1)}\right)\right]\right.$$
$$\cdot\left[c_1\left(H_{k+n}^{(1)}-H_{k+l-n-1}^{(1)}\right)+c_2\left(H_{k+m}^{(1)}-H_{k+l-m-1}^{(1)}\right)\right]-k\left[c_1\left(H_{k+n}^{(2)}-H_{k+l-n-1}^{(2)}\right)\right.$$
$$\left.\left.+c_2\left(H_{k+m}^{(2)}-H_{k+l-m-1}^{(2)}\right)\right]\right\}+\sum_{k=n+1}^{m}(-1)^{k-n}\binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\Big/\binom{k-1}{n}$$
$$\cdot\left[c_1\left(H_{k+n}^{(1)}-H_{k+l-n-1}^{(1)}\right)+c_2\left(H_{k+m}^{(1)}-H_{k+l-m-1}^{(1)}\right)\right]=0.$$

The remainder of this paper is spent proving Theorems 2 and 3.

2. Proofs

We first develop two algebraic identities of which the binomial coefficient–harmonic sum identities are limiting cases.

Theorem 4. *Let x be an indeterminate and let m, n positive integers with $m \geq n$. Then*

$$\sum_{k=0}^{n} \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k}$$
$$\cdot\left\{\frac{-k}{(x+k)^2}+\frac{1+k\left(H_{m+k}^{(1)}+H_{m-k}^{(1)}+H_{n+k}^{(1)}+H_{n-k}^{(1)}-4H_{k}^{(1)}\right)}{x+k}\right\}$$
$$+\sum_{k=n+1}^{m}\frac{(-1)^{k-n}}{x+k}\binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\Big/\binom{k-1}{n}=\frac{x(1-x)_n(1-x)_m}{(x)_{n+1}(x)_{m+1}}. \qquad (2)$$

Proof. Using partial fraction decomposition we can write

$$f(x) := \frac{x(1-x)_n(1-x)_m}{(x)_{n+1}(x)_{m+1}} = \frac{A}{x} + \sum_{k=1}^{n}\left\{\frac{B_k}{(x+k)^2} + \frac{C_k}{x+k}\right\} + \sum_{k=n+1}^{m}\frac{D_k}{x+k}$$

for some A, B_k, C_k and $D_k \in \mathbb{Q}$. We now isolate these coefficients by taking various limits of $f(x)$ as follows.

$$A = \lim_{x\to 0} xf(x) = \lim_{x\to 0}\frac{(1-x)_n(1-x)_m}{(1+x)_n(1+x)_m} = 1.$$

For $1 \le k \le n$,

$$\begin{aligned}
B_k &= \lim_{x\to -k}(x+k)^2 f(x)\\
&= \lim_{x\to -k}\frac{x(1-x)_n(1-x)_m}{(x)_k^2(x+k+1)_{n-k}(x+k+1)_{m-k}}\\
&= \frac{-k(k+1)_n(k+1)_m}{(-k)_k^2(1)_{n-k}(1)_{m-k}}\\
&= \frac{-k(k+1)_n(k+1)_m}{(-1)^{2k}k!^2(n-k)!(m-k)!}\\
&= -k\binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k},
\end{aligned}$$

and, using L'Hôspital's rule,

$$\begin{aligned}
C_k &= \lim_{x\to -k}\frac{(x+k)^2 f(x) - B_k}{x+k}\\
&= \lim_{x\to -k}\frac{d}{dx}\left[(x+k)^2 f(x)\right]\\
&= \lim_{x\to -k}\frac{d}{dx}\left[\frac{x(1-x)_n(1-x)_m}{(x)_k^2(x+k+1)_{n-k}(x+k+1)_{m-k}}\right]
\end{aligned}$$

$$= \lim_{x\to -k}\left\{\left[\frac{(1-x)_n(1-x)_m}{(x)_k^2(x+k+1)_{n-k}(x+k+1)_{m-k}}\right]\left[1-x\left(\sum_{s=1}^{n}(-x+s)^{-1}\right.\right.\right.$$

$$\left.\left.\left.+\sum_{s=1}^{m}(-x+s)^{-1}+\sum_{s=1}^{n-k}(x+k+s)^{-1}+\sum_{s=1}^{m-k}(x+k+s)^{-1}+2\sum_{s=0}^{k-1}(x+s)^{-1}\right)\right]\right\}$$

$$= \left[\frac{(1+k)_n(1+k)_m}{(-k)_k^2(1)_{n-k}(1)_{m-k}}\right]\left[1+k\left(\sum_{s=1}^{n}(k+s)^{-1}+\sum_{s=1}^{m}(k+s)^{-1}+\sum_{s=1}^{n-k}(s)^{-1}\right.\right.$$

$$\left.\left.+\sum_{s=1}^{m-k}(s)^{-1}+2\sum_{s=0}^{k-1}(-k+s)^{-1}\right)\right]$$

$$= \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k}\left[1+k\left(H_{m+k}^{(1)}+H_{m-k}^{(1)}+H_{n+k}^{(1)}+H_{n-k}^{(1)}-4H_k^{(1)}\right)\right].$$

Similarly, for $n+1 \leq k \leq m$,

$$\begin{aligned} D_k &= \lim_{x\to -k}(x+k)f(x) \\ &= \lim_{x\to -k}\frac{x(1-x)_n(1-x)_m}{(x)_{n+1}(x)_k(x+k+1)_{m-k}} \\ &= \frac{-k(k+1)_n(k+1)_m}{(-k)_{n+1}(-k)_k(1)_{m-k}} \\ &= (-1)^{k-n}\binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\Big/\binom{k-1}{n}. \end{aligned}$$

□

Theorem 5. *Let x be an indeterminate and let l, m, n be positive integers with $l > m \geq n \geq \frac{l}{2}$ and $c_1, c_2 \in \mathbb{Q}$ some constants. Then*

$$\sum_{k=0}^{n}\frac{1}{x+k}\binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k}\left\{\left[c_1\left(H_{k+n}^{(1)}-H_{k+l-n-1}^{(1)}\right)\right.\right.$$

$$\left.+c_2\left(H_{k+m}^{(1)}-H_{k+l-m-1}^{(1)}\right)\right]\cdot\left[\frac{x}{x+k}+k\left(H_{m+k}^{(1)}+H_{m-k}^{(1)}+H_{n+k}^{(1)}+H_{n-k}^{(1)}-4H_k^{(1)}\right)\right]$$

$$\left.-k\left[c_1\left(H_{k+n}^{(2)}-H_{k+l-n-1}^{(2)}\right)+c_2\left(H_{k+m}^{(2)}-H_{k+l-m-1}^{(2)}\right)\right]\right\}$$

$$+\sum_{k=n+1}^{m} \frac{(-1)^{k-n}}{x+k} \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k} \Big/ \binom{k-1}{n}$$

$$\times \left[c_1 \left(H_{k+n}^{(1)} - H_{k+l-n-1}^{(1)} \right) + c_2 \left(H_{k+m}^{(1)} - H_{k+l-m-1}^{(1)} \right) \right]$$

$$= \frac{x(1-x)_n (1-x)_m}{(x)_{n+1}(x)_{m+1}} \left[c_1 \sum_{s=l-n}^{n} (-x+s)^{-1} + c_2 \sum_{s=l-m}^{m} (-x+s)^{-1} \right]. \quad (3)$$

Proof. Using partial fraction decomposition we can write

$$f(x) := \frac{x(1-x)_n (1-x)_m}{(x)_{n+1}(x)_{m+1}} \left[c_1 \sum_{s=l-n}^{n} (-x+s)^{-1} + c_2 \sum_{s=l-m}^{m} (-x+s)^{-1} \right]$$

$$= \frac{A}{x} + \sum_{k=1}^{n} \left\{ \frac{B_k}{(x+k)^2} + \frac{C_k}{x+k} \right\} + \sum_{k=n+1}^{m} \frac{D_k}{x+k}$$

for some A, B_k, C_k and $D_k \in \mathbb{Q}$. As in the proof of Theorem 4, we isolate the coefficients A, B_k, C_k and D_k by taking various limits of $f(x)$. For brevity, we first let

$$T_a^{(r)} := c_1 \sum_{s=l-n}^{n} (a+s)^{-r} + c_2 \sum_{s=l-m}^{m} (a+s)^{-r}$$

and

$$U^{(r)} := c_1 \left(H_{k+n}^{(r)} - H_{k+l-n-1}^{(r)} \right) + c_2 \left(H_{k+m}^{(r)} - H_{k+l-m-1}^{(r)} \right).$$

Then we have

$$A = \lim_{x \to 0} x f(x)$$

$$= c_1 \lim_{x\to 0} \sum_{s=l-n}^{n} \frac{(1-x)_n (1-x)_m}{(1+x)_n (1+x)_m (s-x)} + c_2 \lim_{x \to 0} \sum_{s=l-m}^{m} \frac{(1-x)_n (1-x)_m}{(1+x)_n (1+x)_m (s-x)}$$

$$= c_1 \sum_{s=l-n}^{n} s^{-1} + c_2 \sum_{s=l-m}^{m} s^{-1}$$

$$= c_1 \left(H_n^{(1)} - H_{l-n-1}^{(1)} \right) + c_2 \left(H_m^{(1)} - H_{l-m-1}^{(1)} \right).$$

For $1 \leq k \leq n$,

$$\begin{aligned}
B_k &= \lim_{x\to -k} (x+k)^2 f(x) \\
&= \lim_{x\to -k} \frac{x(1-x)_n(1-x)_m}{(x)_k^2(x+k+1)_{n-k}(x+k+1)_{m-k}}\, T_{-x}^{(1)} \\
&= \frac{-k(k+1)_n(k+1)_m}{(-k)_k^2(1)_{n-k}(1)_{m-k}}\, T_k^{(1)} \\
&= -k\binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k} U^{(1)}
\end{aligned}$$

and

$$\begin{aligned}
C_k &= \lim_{x\to -k} \frac{d}{dx}\left[(x+k)^2 f(x)\right] \\
&= \lim_{x\to -k} \frac{d}{dx}\left[\frac{x(1-x)_n(1-x)_m}{(x)_k^2(x+k+1)_{n-k}(x+k+1)_{m-k}}\, T_{-x}^{(1)}\right] \\
&= \lim_{x\to -k} \Bigg\{\left[\frac{(1-x)_n(1-x)_m}{(x)_k^2(x+k+1)_{n-k}(x+k+1)_{m-k}}\right]\Bigg[x\, T_{-x}^{(2)} + T_{-x}^{(1)} - x\, T_{-x}^{(1)} \\
&\qquad \cdot \Bigg(\sum_{s=1}^{n}(-x+s)^{-1} + \sum_{s=1}^{m}(-x+s)^{-1} + \sum_{s=1}^{n-k}(x+k+s)^{-1} \\
&\qquad\qquad + \sum_{s=1}^{m-k}(x+k+s)^{-1} + 2\sum_{s=0}^{k-1}(x+s)^{-1}\Bigg)\Bigg]\Bigg\} \\
&= \left[\frac{(1+k)_n(1+k)_m}{(-k)_k^2(1)_{n-k}(1)_{m-k}}\right]\Bigg[-kT_k^{(2)} + T_k^{(1)}\Bigg(1 + k\Bigg(\sum_{s=1}^{n}(k+s)^{-1} + \sum_{s=1}^{m}(k+s)^{-1} \\
&\qquad + \sum_{s=1}^{n-k}(s)^{-1} + \sum_{s=1}^{m-k}(s)^{-1} + 2\sum_{s=0}^{k-1}(-k+s)^{-1}\Bigg)\Bigg)\Bigg] \\
&= \binom{m+k}{k}\binom{m}{k}\binom{n+k}{k}\binom{n}{k} \\
&\qquad \cdot \left[-k\, U^{(2)} + \left(1 + k\left(H_{m+k}^{(1)} + H_{m-k}^{(1)} + H_{n+k}^{(1)} + H_{n-k}^{(1)} - 4H_k^{(1)}\right)\right)U^{(1)}\right].
\end{aligned}$$

For $n+1 \le k \le m$,

$$D_k = \lim_{x \to -k} (x+k) f(x) = \lim_{x \to -k} \frac{x(1-x)_n (1-x)_m}{(x)_{n+1} (x)_k (x+k+1)_{m-k}} \, T_{-x}^{(1)}$$

$$= \frac{-k(k+1)_n (k+1)_m}{(-k)_{n+1} (-k)_k (1)_{m-k}} \, T_k^{(1)}$$

$$= (-1)^{k-n} \, U^{(1)} \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} \Big/ \binom{k-1}{n}.$$

□

Proofs of Theorems 2 and 3. Multiply both sides of (2) and (3) respectively by x and take the limit as $x \to \infty$. □

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