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EULER'S PENTAGON NUMBER THEOREM IMPLIES THE JACOBI TRIPLE PRODUCT IDENTITY

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Abstract

By means of Liouville's theorem, we show that Euler's pentagon number theorem implies the Jacobi triple product identity.

1. The Result

For two complex numbers x and q, define the q-shifted factorial by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = \prod_{i=0}^{n-1}(1-q^i x) \quad \text{for} \quad n \in \mathbb{N}.$$

When $|q| < 1$, the following product of infinite order is well-defined:

$$(x;q)_\infty = \prod_{i=0}^{\infty}(1-q^i x).$$

Then Euler's pentagon number theorem(cf. Andrews, Askey and Roy [2, Section 10.4]) and the Jacobi triple product identity(cf. Jacobi [4]) can be stated, respectively, as follows:

$$\sum_{k=-\infty}^{+\infty}(-1)^k q^{\frac{k}{2}(3k+1)} = (q;q)_\infty, \tag{1}$$

$$\sum_{k=-\infty}^{+\infty}(-1)^k q^{\binom{k}{2}} x^k = (q;q)_\infty (x;q)_\infty (q/x;q)_\infty \quad \text{where } x \neq 0. \tag{2}$$

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It is well-known that (2) contains (1) as a special case. We shall prove that (1) implies (2) by means of Liouville's theorem: *every bounded entire function must be a constant.* It is a surprise that our proof for (2), which will be displayed, does not require expanding the expression $(q;q)_\infty(x;q)_\infty(q/x;q)_\infty$ as a power series in x.

For facilitating the use of Liouville's theorem, Chu and Yan [1] gave the following statement.

Lemma 1. *Let f be a holomorphic function on $\mathbb{C}\setminus\{0\}$ satisfying the functional equation $f(z)=f(qz)$ where $0<|q|<1$. Then f is a constant.*

Proof of the Jacobi triple product identity. Define $F(x)=U(x)/V(x)$, where $U(x)$ and $V(x)$ stand respectively for

$$\begin{aligned} U(x) &= \sum_{k=-\infty}^{+\infty}(-1)^k q^{\binom{k}{2}}x^k, \\ V(x) &= (q;q)_\infty(x;q)_\infty(q/x;q)_\infty. \end{aligned}$$

It is not difficult to check the two equations:

$$U(x)=-xU(qx) \quad \text{and} \quad V(x)=-xV(qx),$$

which lead consequently to the following relation: $F(x)=F(qx)=F(q^2x)=\cdots$. Observe that the possible poles of $F(x)$ are given by the zeros of $V(x)$, which consist of $x=q^n$ with $n\in\mathbb{Z}$ and are all simple. However, $U(q^n)=0$ for $n\in\mathbb{Z}$, which is justified as follows. Shifting the summation index $k\to k-n$ for $U(q^n)$, we obtain the equation:

$$\begin{aligned} U(q^n) &= \sum_{k=-\infty}^{+\infty}(-1)^k q^{\binom{k}{2}+nk} = \sum_{k=-\infty}^{+\infty}(-1)^{k-n} q^{\binom{k-n}{2}+n(k-n)} \\ &= (-1)^n q^{-\binom{n}{2}}\sum_{k=-\infty}^{+\infty}(-1)^k q^{\binom{k}{2}}. \end{aligned}$$

Splitting the last sum into two parts and performing the replacement $k\to 1-k$ for the second sum, we have

$$\begin{aligned} U(q^n) &= (-1)^n q^{-\binom{n}{2}}\left\{\sum_{k=1}^{+\infty}(-1)^k q^{\binom{k}{2}} + \sum_{k=-\infty}^{0}(-1)^k q^{\binom{k}{2}}\right\} \\ &= (-1)^n q^{-\binom{n}{2}}\left\{\sum_{k=1}^{+\infty}(-1)^k q^{\binom{k}{2}} - \sum_{k=1}^{+\infty}(-1)^k q^{\binom{k}{2}}\right\} \\ &= 0. \end{aligned}$$

Therefore, $F(x)$ is a holomorphic function on $\mathbb{C} \setminus \{0\}$ and must be a constant thanks to Lemma 1. It remains to be shown that this constant is one. Denote by $\omega = exp(2\pi/3)$ the cubic root of unity. Then we get the equation:

$$U(\omega) = \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{3k}{2}} - \omega \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{1+3k}{2}} + \omega^2 \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{2+3k}{2}}.$$

According to Euler's pentagon number theorem (1), we can check, without difficulty, that

$$\sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{3k}{2}} = \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{1+3k}{2}} = (q^3; q^3)_\infty.$$

Combining the last identity with the derivation

$$\begin{aligned}
\sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{2+3k}{2}} &= \sum_{k=0}^{+\infty}(-1)^k q^{\binom{2+3k}{2}} + \sum_{k=-\infty}^{-1}(-1)^k q^{\binom{2+3k}{2}} \\
&= \sum_{k=0}^{+\infty}(-1)^k q^{\binom{2+3k}{2}} - \sum_{k=0}^{+\infty}(-1)^k q^{\binom{2+3k}{2}} \\
&= 0,
\end{aligned}$$

we achieve the following relation: $U(\omega) = (1 - w)(q^3; q^3)_\infty = V(\omega)$, which leads to

$$F(x) = F(\omega) = U(\omega)/V(\omega) = 1.$$

This proves the Jacobi triple product identity (2). □

Remark: One can also show that Euler's pentagon number theorem implies the quintuple product identity(cf. Gasper and Rahman [3, Section 1.6]) in the same method. The details will not be reproduced here.

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