COUNTING DEPTH ZERO PATTERNS IN BALLOT PATHS

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Abstract

The purpose of this work is to extend the theory of finite operator calculus to the multivariate setting, and apply it to the enumeration of certain lattice paths. The lattice paths we consider are ballot paths. A ballot path is a path that stays weakly above the diagonal $y = x$, starts at the origin, and takes steps from the set $\{\uparrow, \rightarrow\} = \{u, r\}$. Given a string p from the set $\{u, r\}^*$, we want to count the ballot paths with a given number of occurrences of p. In order to use finite operator calculus, we must put some restrictions on the string p we wish to keep track of. A ballot path ending on the diagonal can be viewed as a Dyck path, thus all of our results also apply to the enumeration of Dyck paths with a given number of occurrences of p. Finally, we give an example of counting ballot paths with a given number of occurrences of two patterns.

1. Introduction

L. J. Guibas and A. M. Odlyzko [3] derived generating functions for the number of strings over an alphabet that avoid given patterns. Their main tool is the "correlation function" among patterns. This basically extracts the same information from a pattern as the (multiple) bifixes introduced in Section 6. Our work differs in that we consider ballot paths, i.e., a *restricted* alphabet of size two, where the restriction observes how many symbols of one kind occur before the other kind. We can generalize this to a larger alphabet (Motzkin path) and different restrictions, but we are more interested in the approach itself, the finite operator calculus (Rota, Kahaner, and Odlyzko [10]). The finite operator calculus produces explicit results (polynomials), but in some cases, generating functions can also be obtained. So we need another condition on patterns, the depth, to make sure the solutions are polynomials. We systematically discuss avoiding only one pattern, but in the last section we finally give an example for avoiding two patterns.

A second aspect where our work differs from Guibas and Odlyzko is the enumeration of ballot paths with a *given number* of occurrences of some pattern. In a paper by Sapounakis, Tasoulas, and Tsikouras [11], the authors do exactly this for all patterns of length four, but only for ballot paths ending on the diagonal (Dyck paths). We show that it is not the length of the pattern that matters, but its "complexity", its autocorrelation function in the sense of Guibas and Odlyzko.

A ballot path stays weakly above the diagonal $y = x$, starts at the origin, and takes steps from the set $\{\uparrow, \rightarrow\} = \{u, r\}$. A pattern is a finite string made from the same step set; it is also a path. Notice that a ballot path ending at a point along the diagonal is a Dyck path.

Definition 1 *Let $d(p)$ be the number of u's minus the number of r's in the pattern p. The depth of p is* $\max\{d(p') \mid p = qp', q \in \{u, r\}^*\}$.

The patterns we count can be any length, but the patterns we count in this paper have zero depth. We call these patterns depth-zero. An intuitive interpretation of a depth-zero pattern p, is that the reverse pattern $\tilde{p}$ is a ballot path. For example, the reverse pattern of $p = uururrr$ is $\tilde{p} = uuururr$. Since $uuururr$ is a ballot path, $uururrr$ is depth-zero.

Below is a table for the number of ballot paths with 0,1, and 2 occurrences of the pattern rur. We use finite operator calculus to enumerate these paths. For this, we need recursions describing the enumeration. We must consider only two more properties of these patterns to develop the recursions, given in the following definitions.

Definition 2 *The bifix index of a pattern p is the number of distinct non-empty patterns $o \neq p$ such that p that can be written in the form $p = op'$ and $p = p''o$ for $o, p', p'' \in \{u, r\}^*$. If a pattern has bifix index 0, then we call it bifix-free.*

Definition 3 *If a is the number of r's in p and c is the number of u's, then we say p has dimensions $a \times c$.*

If we denote the number of ballot paths reaching (n, m) containing the pattern rur exactly k times by $s_{n,k}(m)$, then we will see that

$$\begin{aligned} s_{n,k}(m) &= s_{n-1,k}(m) + s_{n,k}(m-1) - s_{n-1,k}(m-1) + s_{n-1,k}(m-2) \\ &\quad + s_{n-1,k-1}(m-1) - s_{n-1,k-1}(m-2). \end{aligned}$$

We will prove a general recurrence counting any pattern (Theorem 14). If the depth is zero, we see that each column consists of the values of a polynomial sequence, the objects of finite operator calculus. With these definitions and the recursions we obtain from them, we can use Finite Operator Calculus to find formulas

m	1	8	28	62	105	7	42	120	236	6	45	144	300
7	1	7	21	40	59	6	30	72	120	5	30	78	130
6	1	6	15	24	30	5	20	39	52	4	18	36	40
5	1	5	10	13	13	4	12	18	16	3	9	12	0
4	1	4	6	6	4	3	6	6	0	2	3	0	
3	1	3	3	2	0	2	2	0		1	0		
2	1	2	1	0		1	0			0			
1	1	1	0			0							
0	1	0											
	0	1	2	3	n	2	3	4	5	3	4	5	6
			$k=0$				$k=1$				$k=2$		

Table 1: The number of ballot paths containing rur exactly $k = 0, 1, 2$ times.

that enumerate the ballot paths with a given number of occurrences of a depth-zero pattern. In [7] and [8], we counted the case $k = 0$, the avoidance of a depth-zero pattern. This can be done using ordinary finite operator calculus. For $k > 0$, we need the bivariate extension of this theory. We first briefly introduce the concepts of finite operator calculus.

2. Main Tools

In this section we will present the main tools from finite operator calculus [10] that will be used to solve these enumeration problems. We say a sequence of polynomials $s_n(x)$, where s_n is degree n, is a Sheffer sequence if its generating function is of the form

$$\sum_{n\geq 0} s_n(x)t^n = \sigma(t)e^{x\beta(t)},$$

where $\sigma(t)$ has a multiplicative inverse $\sigma(t)^{-1}$and $\beta(t)$ is of order 1, and thus has a compositional inverse $\beta^{-1}(t)$. Every Sheffer sequence is associated to a basis sequence, usually denoted $b_n(x)$, and its generating function is of the form

$$\sum_{n\geq 0} b_n(x)t^n = e^{x\beta(t)}.$$

The Sheffer operator $B : s_n \to s_{n-1}$ and the shift operator $E^a : p(x) \to p(x + a)$ can be written as power series in the derivative operator $D := \frac{d}{dx}$,

$$B = \beta^{-1}(D), \qquad E^a = e^{aD} = \sum_{n\geq 0} \frac{(aD)^n}{n!}.$$

The second formula above for the shift operator is a restatement of Taylor's Theorem. We need not worry about convergence here since the operators act on a polynomial ring, and thus only a finite number of the terms in the power series are needed for a given polynomial. This is the reason for the name finite operator calculus.

In our previous papers, we used the theorems in finite operator calculus to count the number of ballot paths *avoiding* a given pattern. From the above example, we see that we have a sequence of polynomial sequences, and so we will need a bivariate form of finite operator calculus. Much of the definitions and theorems are similar, and so we will only present the needed theorems in the bivariate finite operator calculus.

3. Bivariate Operators and Polynomials

The objects in bivariate finite operator calculus are polynomials in $k[u, v]$ and the shift-invariant operators belong to $k[[D_u, D_v]]$, where D_u and D_v are the partial derivatives with respect to u and v, respectively. For a detailed study of multivariate finite operator calculus, see [12].

Every univariate delta series has a compositional inverse, but how do we generalize the concept of a compositional inverse for bivariate formal power series? Given a pair of formal bivariate power series (β_1, β_2) in $k[[s, t]]^2$, we say (γ_1, γ_2) is the inverse pair for (β_1, β_2) if $(\beta_1(\gamma_1, \gamma_2), \beta_2(\gamma_1, \gamma_2)) = (s, t)$. We also use the notation $(\beta_1^{-1}, \beta_2^{-1})$ for the inverse of the pair (β_1, β_2).

We will need to find the compositional inverse of a pair of bivariate power series. The Lagrange-Good inversion formula tells us that a pair of power series has an inverse pair if it is a *delta pair*, that is $(\beta_1, \beta_2) = (s\phi_1, t\phi_2)$ is a delta pair where ϕ_1 and ϕ_2 have multiplicative inverses. We present a form of the bivariate Lagrange-Good inversion formula [4].

Theorem 4 *If $(\gamma_1, \gamma_2) = (s/\epsilon_1, t/\epsilon_2)$ is a delta pair with inverse pair (β_1, β_2), then*

$$\left[\beta_1(s, t)^k \beta_2(s, t)^l\right]_{m,n} = \left[\epsilon_1(s, t)^{m+1} \epsilon_2(s, t)^{n+1} \mathcal{J}\gamma\right]_{m-k,n-l},$$

where $\mathcal{J}\gamma$ stands for the Jacobian

$$\mathcal{J}\gamma = \left|\frac{\partial(\gamma_1, \gamma_2)}{\partial(s, t)}\right| = \begin{vmatrix} \frac{\partial\gamma_1}{\partial s} & \frac{\partial\gamma_2}{\partial s} \\ \frac{\partial\gamma_1}{\partial t} & \frac{\partial\gamma_2}{\partial t} \end{vmatrix}.$$

Since (β_1, β_2) is also a delta pair, we could write $(\beta_1, \beta_2) = (s/\phi_1, t/\phi_2)$, and thus

$$\left[\phi_1(s,t)^k \phi_2(s,t)^l\right]_{m,n} = \left[\epsilon_1(s,t)^{m+1+k}\epsilon_2(s,t)^{n+1+l}\mathcal{J}\gamma\right]_{m,n}.$$

As in the univariate finite operator calculus, we will associate linear operators in $k[[D_u, D_v]]$ with the bivariate formal power series in $k[[s,t]]$. The operators associated with delta pairs will also be associated with the Sheffer sequences in bivariate finite operator calculus.

4. Bivariate Sheffer Sequences

We say a bivariate polynomial sequence $s_{m,n}(u,v)$ is a Sheffer sequence for a delta pair (B_1, B_2) if $B_1 : s_{m,n}(u,v) \to s_{m-1,n}(u,v)$ and $B_2 : s_{m,n}(u,v) \to s_{m,n-1}(u,v)$. Here $s_{m,n}$ has degree m as a polynomial in u and degree n as a polynomial in v. The sequence $b_{m,n}(u,v)$ is the basic sequence for (B_1, B_2) if it is a Sheffer sequence and satisfies the initial values $b_{m,n}(0,0) = \delta_{m,0}\delta_{n,0}$. We have the following theorem that categorizes Sheffer sequences with their generating function. Clearly, this is analogous to the univariate case.

Theorem 5 *The following are equivalent:*
(i) $(s_{m,n})$ *is a Sheffer sequence for the delta pair* (B_1, B_2).
(ii) *There exists a power series* $\sigma(s,t)$ *and a delta pair* $(\beta_1(s,t), \beta_2(s,t))$ *such that the generating function for the polynomial sequence* $(s_{m,n})$ *can be written*

$$\sum_{m,n\geq 0} s_{m,n}(u,v)s^m t^n = \sigma(s,t)e^{u\beta_1(s,t)+v\beta_2(s,t)},$$

where $\sigma(0,0) \neq 0$ *and* $(B_1, B_2) = (\beta_1^{-1}(D_u, D_v), \beta_2^{-1}(D_u, D_v))$.
(iii) $s_{m,n}(u+x, v+y) = \sum_{l=0}^{m}\sum_{k=0}^{n} s_{l,k}(u,v)b_{m-l,n-k}(x,y)$, *where* $(b_{m,n})$ *is the basic sequence for* (B_1, B_2) *with generating function*

$$\sum_{m,n\geq 0} b_{m,n}(u,v)s^m t^n = e^{u\beta_1(s,t)+v\beta_2(s,t)}.$$

From the binomial theorem for Sheffer sequences (Theorem 5 *(iii)*), we have an important corollary that will help in our later applications.

Corollary 6 *We have*

$$b_{m,n}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{i,j}(u,0)b_{m-i,n-j}(0,v).$$

We call the polynomial sequences $b_{m,n}(u,0)$ and $b_{m,n}(0,v)$ *partial* basic sequences. In the general multivariate setting we get a similar result and the partial sequences are obtained by setting all but one of the variables equal to 0. Notice that the basic sequence can be recovered from these partial sequences.

We will see that the objects that count ballot paths with a given number of occurrences of a pattern are these partial sequences. Also, the transfer formulae become much more usable when dealing with partial sequences. We now turn to the transfer formulae for the bivariate basic sequences.

5. Bivariate Transfer Formulae

For the transfer formulae, we must define umbral shifts for multivariate basic sequences and the Pincherle derivative for the corresponding operators.

We define the umbral shifts ϕ and ψ as the multiplication by u and v respectively, so the *partial* Pincherle derivatives are

$$\frac{\partial T}{\partial D_u} = T\phi - \phi T \qquad \text{and} \qquad \frac{\partial T}{\partial D_v} = T\psi - \psi T.$$

In the univariate case, we use the Pincherle derivative on a delta operator in order to find an expression for its basic sequence. It would seem natural to define the bivariate Pincherle derivative as the Jacobian of a pair of operators:

$$\mathcal{J}(T_1, T_2) = \left| \frac{\partial(T_1, T_2)}{\partial(D_u, D_v)} \right|,$$

which can be written in terms of the partial Pincherle derivatives.

We would also like an expression for the umbral shift associated to the delta pair (B_1, B_2), that is the operators, θ_{B_1} and θ_{B_2} such that $\theta_{B_1} b_{m,n}(u,v) = (m+1)b_{m+1,n}(u,v)$ and $\theta_{B_2} b_{m,n}(u,v) = (n+1)b_{m,n+1}(u,v)$. We have the following lemma concerning these umbral shifts.

Lemma 7 *If θ_{B_ρ} are the umbral shifts associated to the delta pair (B_1, B_2) with basic sequence $(b_{m,n})$, then, for $\rho = 1, 2$, we have*

$$\theta_{B_\rho} = \phi \frac{dD_u}{dB_\rho} + \psi \frac{dD_v}{dB_\rho}.$$

Proof. We prove these in a way similar to the univariate case. We know that $\sum_{n\geq 0} b_{m,n}(u,v) s^m t^n = e^{u\beta_1(s,t)+v\beta_2(s,t)}$, where (β_1, β_2) is a delta pair,

$B_1 = \beta_1^{-1}(D_u, D_v)$, and $B_2 = \beta_2^{-1}(D_u, D_v)$. Then

$$\begin{aligned}\theta_{B_1} \sum_{m,n\geq 0} b_{m,n}(u,v)s^m t^n &= \sum_{m,n\geq 0}(m+1)b_{m+1,n}(u,v)s^m t^n \\ &= D_s \sum_{m,n\geq 0} b_{m,n}(u,v)s^m t^n \\ &= \left(u\frac{\partial}{\partial s}\beta_1(s,t) + v\frac{\partial}{\partial s}\beta_2(s,t)\right) e^{u\beta_1(s,t)+v\beta_2(s,t)}.\end{aligned}$$

We also know that $f(B_1, B_2)e^{u\beta_1(s,t)+v\beta_2(s,t)} = f(s,t)e^{u\beta_1(s,t)+v\beta_2(s,t)}$ for any power series f. Thus,

$$\theta_{B_1} = u\frac{\partial}{\partial B_1}\beta_1(B_1,B_2) + v\frac{\partial}{\partial B_1}\beta_2(B_1,B_2) = \phi\frac{\partial D_u}{\partial B_1} + \psi\frac{\partial D_v}{\partial B_1},$$

since $D_u = \beta_1(B_1, B_2)$ and $D_v = \beta_2(B_1, B_2)$. A similar argument shows $\theta_{B_2} = \psi\frac{\partial D_v}{\partial B_2} + \phi\frac{\partial D_u}{\partial B_2}$. □

This is very similar to the univariate case. For the Pincherle derivative we get

$$\left|\frac{\partial(T_1,T_2)}{\partial(B_1,B_2)}\right| = \begin{vmatrix} T_1\theta_{B_1} - \theta_{B_1}T_1 & T_2\theta_{B_1} - \theta_{B_1}T_2 \\ T_1\theta_{B_2} - \theta_{B_2}T_1 & T_2\theta_{B_2} - \theta_{B_2}T_2 \end{vmatrix}.$$

Because each expansion is similar, we show the top left:

$$\begin{aligned}T_1\theta_{B_1} - \theta_{B_1}T_1 &= T_1\left(\phi\frac{\partial D_u}{\partial B_1} + \psi\frac{\partial D_v}{\partial B_1}\right) - \left(\phi\frac{\partial D_u}{\partial B_1} + \psi\frac{\partial D_v}{\partial B_1}\right)T_1 \\ &= (T_1\phi - \phi T_1)\frac{\partial D_u}{\partial B_1} + (T_1\psi - \psi T_1)\frac{\partial D_v}{\partial B_1} \\ &= \frac{\partial T_1}{\partial B_1}\end{aligned}$$

by the chain rule for partial derivatives. We are now ready to present the bivariate transfer formula. (Equation (1) is shown in [12, Theorem 1.3.6].)

Theorem 8 *Suppose $(B_1, B_2) = (D_u P_1^{-1}, D_v P_2^{-1})$ is a delta pair, then*

$$\begin{aligned}b_{m,n}(u,v) &= P_1^{m+1}P_2^{n+1}\mathcal{J}(B_1,B_2)\frac{u^m v^n}{m!n!} &(1)\\ &= \left(uP_1^m vP_2^n + vP_2^n uP_1^m - uvP_1^m P_2^n\right)\frac{u^{m-1}v^{n-1}}{m!n!} &(2)\end{aligned}$$

is the associated basic sequence.

Proof. We begin by showing the equivalence of the two forms. We have a similar simplification as in the univariate transfer formula;

$$P_1^{m+1}P_2^{n+1}\mathcal{J}(B_1,B_2)\frac{u^m v^n}{m!n!} = \begin{vmatrix} P_1^m - \frac{D_u}{m}\frac{\partial P_1^m}{\partial D_u} & -\frac{D_u}{m}\frac{\partial P_1^m}{\partial D_v} \\ -\frac{D_v}{n}\frac{\partial P_2^n}{\partial D_u} & P_2^n - \frac{D_v}{n}\frac{\partial P_2^n}{\partial D_v} \end{vmatrix}\frac{u^m v^n}{m!n!}. \quad (3)$$

When we expand the determinant, we apply D_u and D_v to $\frac{u^m v^n}{m!n!}$, giving us the following operator on $\frac{u^{m-1}v^{n-1}}{m!n!}$, where, for elegance, we will denote P_1^m by Q_1 and P_2^n by Q_2.

$$Q_1 Q_2 uv - Q_1 \frac{\partial Q_2}{\partial D_v} u - Q_2 \frac{\partial Q_1}{\partial D_u} v + \mathcal{J}(Q_1, Q_2).$$

To simplify, we expand the Jacobian as follows:

$$\begin{aligned}
\mathcal{J}(Q_1, Q_2) &= \frac{\partial Q_1}{\partial D_u}\frac{\partial Q_2}{\partial D_v} - \frac{\partial Q_1}{\partial D_v}\frac{\partial Q_2}{\partial D_u} \\
&= \frac{\partial Q_1}{\partial D_u}(Q_2 v - v Q_2) - (Q_1 v - v Q_1)\frac{\partial Q_2}{\partial D_u} \\
&= Q_2 \frac{\partial Q_1}{\partial D_u} v - \frac{\partial Q_1}{\partial D_u} v Q_2 - Q_1 v \frac{\partial Q_2}{\partial D_u} + v \frac{\partial Q_2}{\partial D_u} Q_1.
\end{aligned}$$

We expand the remaining derivatives and, upon cancellation, we get

$$u Q_1 v Q_2 + v Q_2 u Q_1 - uv Q_1 Q_2.$$

Because of the lack of commutativity, there are many forms for the transfer formula. This one has the least amount of terms while retaining symmetry. We need to show that this is the basic sequence for the delta pair (B_1, B_2). In the first form we can show that

$$B_1 b_{m,n}(u,v) = P_1^m P_2^{n+1} \mathcal{J}(B_1, B_2) \frac{u^{m-1}v^n}{(m-1)!n!} = b_{m-1,n}(u,v),$$

and a similar result for B_2. What remains is to show $b_{m,n}(0,0) = \delta_{m,0}\delta_{n,0}$. The second form only holds for positive values of m and n. The following forms show that $b_{m,n}(0,0) = 0$ when $(m,n) \neq (0,0)$;

$$\begin{aligned}
b_{m,n}(u,v) &= \left(u \frac{\partial B_2}{\partial D_v} - v \frac{\partial B_2}{\partial D_u}\right) P_1^m P_2^{n+1} \frac{u^{m-1}v^n}{m!n!} \\
&= \left(v \frac{\partial B_1}{\partial D_u} - u \frac{\partial B_1}{\partial D_v}\right) P_1^{m+1} P_2^n \frac{u^m v^{n-1}}{m!n!}.
\end{aligned}$$

We prove the first one; again we denote P_1^m by Q_1 and P_2^n by Q_2. Expanding the partial derivatives as

$$\frac{\partial B_2}{\partial D_v} = P_2^{-1}\left(1 - P_2^{-1} D_v \frac{\partial P_2}{\partial D_v}\right) \quad \text{and} \quad \frac{\partial B_2}{\partial D_u} = -P_2^{-2} D_v \frac{\partial P_2}{\partial D_u},$$

we get the following operator on $\frac{u^{m-1}v^{n-1}}{m!n!}$:

$$\begin{aligned} & uQ_1Q_2v - uQ_1\frac{\partial Q_2}{\partial D_v} + vQ_1\frac{\partial Q_2}{\partial D_u} \\ = \;& uQ_1Q_2v - uQ_1(Q_2v - vQ_2) + v\frac{\partial Q_2}{\partial D_u}Q_1 \\ = \;& uQ_1vQ_2 + v(Q_2u - uQ_2)Q_1 \\ = \;& uQ_1vQ_2 + vQ_2uQ_1 - uvQ_1Q_2. \end{aligned}$$

Finally, evaluating (3) at $m = n = 0$ shows us that $b_{0,0}(u, v) = 1$, which completes the proof. □

Corollary 9 *If (A_1, A_2) and (B_1, B_2) are delta pairs with basic sequences $(a_{m,n})$ and $(b_{m,n})$ respectively, then*

$$b_{m,n}(u, v) = V_1^{m+1}V_2^{n+1}\left|\frac{\partial(B_1, B_2)}{\partial(A_1, A_2)}\right| a_{m,n}(u, v)$$

or

$$b_{m,n}(u, v) = \frac{1}{mn}\left(\theta_{A_1}V_1^m\theta_{A_2}V_2^n + \theta_{A_2}V_2^n\theta_{A_1}V_1^m - \theta_{A_1}\theta_{A_2}V_1^mV_2^n\right) a_{m-1,n-1}(u, v),$$

where $A_i = V_iB_i$ and $\left|\frac{\partial(B_1, B_2)}{\partial(A_1, A_2)}\right|$ is the Jacobian with respect to A_1 and A_2.

The proof of the corollary is analogous to that of the theorem. We present an important special case to this corollary by letting $A_2 = B_2$.

Corollary 10 *If (A_1, A_2) and (B_1, B_2) are delta pairs with basic sequences $(a_{m,n})$ and $(b_{m,n})$ respectively, and $A_2 = B_2$, then*

$$b_{m,n}(u, v) = \frac{1}{m}\theta_{A_1}V_1^m a_{m-1,n}(u, v).$$

In order to use these transfer formulae, we need to expand the V_i in terms of the A_i. For this we use the Lagrange-Good Inversion to get the following corollary.

Corollary 11 *If (A_1, A_2) and (B_1, B_2) are delta pairs with basic sequences $(a_{m,n})$ and $(b_{m,n})$ respectively, then*

$$\begin{aligned} V_1^mV_2^n &= \sum_{i\geq 0}\sum_{j\geq 0}\left[\tau_1^{m-1-i}\tau_2^{n-1-j}\left|\frac{\partial(\tau_1, \tau_2)}{\partial(s, t)}\right|\right]_{m-1,n-1} A_1^iA_2^j \\ &= \sum_{i\geq 0}\sum_{j\geq 0}\left[\epsilon_1^{i+1-m}\epsilon_2^{j+1-n}\left|\frac{\partial(\tau_1, \tau_2)}{\partial(s, t)}\right|\right]_{i,j} A_1^iA_2^j, \end{aligned}$$

where $A_i = V_iB_i = \tau_i(B_1, B_2) = B_i/\epsilon_i(B_1, B_2)$ for $i = 1, 2$.

Note that the bivariate series τ_i in this corollary may contain linear operators as coefficients. The following is a technical lemma that will be used in our applications.

Lemma 12 *Suppose $b_{m,n}(u,v)$ is a bivariate basic sequence for the delta pair (B_1, B_2). Then*

$$\theta_{B_1} b_{m,n}(u+c,v)|_{v=0} = (m+1)\frac{u}{u+c} b_{m+1,n}(u+c,0).$$

Proof. We first recall that $\theta_{B_1} = \phi\frac{dD_u}{dB_1} + \psi\frac{dD_v}{dB_1}$. We have

$$\begin{aligned}
\theta_{B_1} E_u^c &= E_u^c \theta_{B_1} - \frac{\partial E_u^c}{\partial B_1} \\
&= E_u^c \theta_{B_1} - \left(\frac{\partial E_u^c}{\partial D_u}\frac{\partial D_u}{\partial B_1} + \frac{\partial E_u^c}{\partial D_v}\frac{\partial D_v}{\partial B_1}\right) \\
&= E_u^c \theta_{B_1} - cE_u^c \frac{\partial D_u}{\partial B_1} \\
&= E_u^c \theta_{B_1} - cE_u^c \phi^- \left(\theta_{B_1} - \psi\frac{\partial D_v}{\partial B_1}\right) \\
&= E_u^c \left(I - c\phi^-\right)\theta_{B_1} + cE_u^c \phi^- \psi \frac{\partial D_v}{\partial B_1},
\end{aligned}$$

where ϕ^- is the left inverse of ϕ. The second term vanishes when $v = 0$. So we have

$$\theta_{B_1} E_u^c b_{m,n}(u,v)|_{v=0} = E_u^c \left(I - c\phi^-\right)\theta_{B_1} b_{m,n}(u,v)\Big|_{v=0}.$$

Expanding the right-hand side simplifies to the right-hand side of the lemma. □

The last transfer formula is a special case when $A_i = \tau_i(B_1, B_2)$ and $\tau_i \in k[[s,t]]$, that is, τ_i does not contain operator coefficients. If $(a_{m,n})$ is basic for (A_1, A_2) and $(b_{m,n})$ is basic for (B_1, B_2), then

$$\begin{aligned}
\sum_{m,n\geq 0} b_{m,n}(u,v)s^m t^n &= e^{u\beta_1(s,t)+v\beta_2(s,t)} = e^{u\alpha_1(\tau_1,\tau_2)+v\alpha_2(\tau_1,\tau_2)} \qquad (4)\\
&= \sum_{m,n\geq 0} a_{m,n}(u,v)\tau_1^m \tau_2^n,
\end{aligned}$$

where $A_i = \alpha_i^{-1}(D_u, D_v)$ and $b_i = \beta_i^{-1}(D_u, D_v)$, for $i = 1, 2$. We have proven the following theorem.

Theorem 13 *If $A_i = \tau_i(B_1, B_2)$ where $\tau_i \in k[[s,t]]$, $(a_{m,n})$ is basic for (A_1, A_2), and $(b_{m,n})$ is basic for (B_1, B_2), then*

$$b_{m,n}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \left[\tau_1^i \tau_2^j\right]_{m,n} a_{i,j}(u,v).$$

Now that we have all the tools, we need the recurrence for counting strings in ballot paths.

6. General Recurrence

Let $s_{n,k}(m)$ be the number of ballot paths ending at the point (n,m) with k occurrences of the given string p and let $s_{n,k}(m;p)$ be those paths counted by $s_{n,k}(m)$ that end with p. For all $m \geq n$ we have

$$s_{n,k}(m) = s_{n-1,k}(m) + s_{n,k}(m-1) - s_{n,k+1}(m;p) + s_{n,k}(m;p). \tag{5}$$

The first two terms are simply noticing that each path could end with an up step or a right step. Consider each path counted by the first two terms. If we attach the corresponding step to the end of each path, we may be completing the pattern p. The third term takes care of this possibility. Finally, the last term takes into account the paths with k occurrences of p that end in p.

We count a pattern twice if it overlaps with itself. We call the overlaps o_i *bifixes* because they appear at the beginning and end of the pattern. We write the pattern p with bifix o_i as $p''_i o_i = p = o_i p'_i$, where the "right end" p'_i and "left end" p''_i have dimensions $b_i \times d_i$, and p has dimensions $a \times c$. As an example consider the pattern $rurrur$, which has bifixes $o_1 = r$, $p''_1 = rurru$, $p'_1 = urrur$, and $o_2 = rur$, $p''_2 = rur$, $p'_2 = rur$. So, the path $uuururrururrur$ has two occurrences of p overlapping in o_1.

Now we consider the term $s_{n,k+1}(m;p)$, and let us order the bifixes of p by size, i.e. $|o_1| > |o_2| > \cdots > |o_l|$. The pattern at the end can either overlap with o_1, or not. Thus

$$s_{n,k+1}(m;p) = s_{n-b_1,k}(m-d_1;p) + s_{n-b_1,k}(m-d_1;(\neg p''_1)o_1), \tag{6}$$

where $\neg p$ means any pattern except p.

At this point the proof splits into two cases. We say a pattern p is *periodic* if there is some subpattern q such that $p = q_0 q^k = (q')^k q_0$ for some $k > 1$ and possibly empty pattern q_0. For example, $p = r(ur)^k$ is periodic with $q = ur$, $q' = ru$, and $q_0 = r$. We will assume q is the smallest subpattern of p where $p = q_0 q^k$. We continue the proof for the case where p is not periodic.

Choosing the longest overlap o_1, i.e., removing the shortest "left end" p'_1 guarantees that only one occurrence of p is deleted from the end of the path. Now, each bifix is contained in every larger one, i.e. $o_i = o_j x_j$ for every $i > j$ and some nonempty string x_j. In particular, $o_1 = o_2 x_2$, and so the last term either contains paths ending in px_2, or not. Thus

$$\begin{aligned} s_{n,k+1}(m;p) &= s_{n-b_1,k}(m-d_1;p) + s_{n-b_2,k}(m-d_2;p) \\ &\quad + s_{n-b_2,k}(m-d_2;\neg(p''_1 \vee p''_2)o_2), \end{aligned}$$

where $p \vee q$ means p or q. Continuing, all of the bifixes will be exhausted, ending with the last bifix o_l. Hence,

$$\begin{aligned}
s_{n,k+1}(m;p) &= \sum_{i=1}^{l} s_{n-b_i,k}(m-d_i;p) + s_{n-b_l,k}(m-d_l;\neg\left(\bigvee_{i=1}^{l} p_i''\right) o_l) \\
&= \sum_i s_{n-b_i,k}(m-d_i;p) + s_{n-a,k}(m-c;\neg\bigvee_i p_i'') \\
&= \sum_i s_{n-b_i,k}(m-d_i;p) + s_{n-a,k}\,(m-c) - s_{n-a,k}(m-c;\bigvee_i p_i'') \\
&= \sum_i s_{n-b_i,k}(m-d_i;p) + s_{n-a,k}\,(m-c) - \sum_i s_{n-a,k}\,(m-c;p_i'') \\
&= \sum_i s_{n-b_i,k}(m-d_i;p) + s_{n-a,k}\,(m-c) - \sum_i s_{n-b_i,k+1}\,(m-d_i;p_i'' o_i)\,.
\end{aligned}$$

Finally,

$$s_{n,k+1}(m;p) = \sum_i \left(s_{n-b_i,k}(m-d_i;p) - s_{n-b_i,k+1}\,(m-d_i;p)\right) + s_{n-a,k}\,(m-c)\,. \quad (7)$$

The paths ending in $\bigvee_i p_i''$ become a disjoint union because for each such path, there is a unique bifix that will add exactly one more occurrence of the pattern p.

Next, we show that (7) holds when p is periodic. The last term of (6) counts paths that end in $(\neg p_1'')o_1$. This term cannot split if p is periodic using the next bifix. In this case we have

$$s_{n,k+1}(m;p) = s_{n-b_1,k}(m-d_1;p) + s_{n-a,k}(m-c;\neg q').$$

Next, similar to the non-periodic case, we have

$$s_{n,k+1}(m;p) = s_{n-b_1,k}(m-d_1;p) + s_{n-a,k}(m-c) - s_{n-a,k}(m-c;q').$$

Now, to the last term, we append q_0 and as many q's necessary to create exactly one more occurrence of the pattern p. Again, due the periodic nature, the ending pattern cannot have a q' before it with the exception of appending only q_0 (or one q if q_0 is empty). All of this gives

$$\begin{aligned}
s_{n,k+1}(m;p) &= s_{n-b_1,k}(m-d_1;p) + s_{n-a,k}(m-c) - s_{n-b_l,k+1}(m-d_l;p) \qquad (8) \\
&\qquad - \sum_{i=1}^{l-1} s_{n-b_i,k+1}(m-d_i;(\neg q')p) \\
&= s_{n-b_1,k}(m-d_1;p) + s_{n-a,k}(m-c) - s_{n-b_l,k+1}(m-d_l;p) \\
&\qquad - \sum_{i=1}^{l-1} (s_{n-b_i,k+1}(m-d_i;p) - s_{n-b_{i+1},k}(m-d_{i+1};p)),
\end{aligned}$$

which is equivalent to (7).

Next, rewrite each difference in (7) using (5), giving

$$s_{n,k+1}(m;p) = \sum_i \left[s_{n-b_i,k}(m-d_i) - s_{n-b_i-1,k}(m-d_i) - s_{n-b_i,k}(m-d_i-1)\right] \\ +s_{n-a,k}(m-c).$$

Finally, use this to replace the last two terms of (5). This proves the following rheorem.

Theorem 14 *Let $s_{n,k}(m)$ be the number of $\{\uparrow,\rightarrow\}$ lattice paths from the origin to the point (n,m) with k occurrences of the pattern p. Then*

$$\begin{aligned} s_{n,k}(m) = {} & s_{n-1,k}(m) + s_{n,k}(m-1) - s_{n-a,k}(m-c) + s_{n-a,k-1}(m-c) \\ & - \sum_i \left[s_{n-b_i,k}(m-d_i) - s_{n-b_i-1,k}(m-d_i) - s_{n-b_i,k}(m-d_i-1)\right] \\ & + \sum_i \left[s_{n-b_i,k-1}(m-d_i) - s_{n-b_i-1,k-1}(m-d_i) - s_{n-b_i,k-1}(m-d_i-1)\right], \end{aligned}$$

where p has dimensions $a \times c$.

7. Counting Strings in Ballot Paths

We return to our pattern rur as our guiding example. We have seen a table of values for $k = 0, 1, 2$ in the Introduction. With this pattern, the general recurrence simplifies, giving us

$$\begin{aligned} s_{n,k}(m) &= s_{n-1,k}(m) + s_{n,k}(m-1) - s_{n-1,k}(m-1) + s_{n-1,k}(m-2) \\ &\quad +s_{n-1,k-1}(m-1) - s_{n-1,k-1}(m-2). \end{aligned}$$

The first question we answer is about the first nonzero column for each $k > 0$. The following lemma gives a complete description.

Lemma 15 *Let the depth of a pattern p be zero. Given p, for $k > 0$ we have,*

$$s_{n,k}(m) = \begin{cases} 0 & \text{if } n < a + b(k-1) \\ m + 1 - a - b(k-1) & \text{if } n = a + b(k-1), \end{cases}$$

where a is the number of r's in p and $b = \min\{b_i\}$ corresponding to the largest bifix in p, or $b = a$ if p is bifix free. In particular, the first nonzero column is a linear polynomial in m.

Proof. Given $k > 0$, the smallest p can appear k times is overlapping itself $(k-1)$ times using its largest bifix, or concatenating with itself k times if p is bifix free. Let

p_k be the resulting pattern obtained by this construction. The earliest p_k can appear is if it starts on the y-axis, and thus the first nonzero column is at $n = a + b(k-1)$. Clearly, when this column meets the diagonal $y = x$, there can be only one path containing p_k. Moving up this column, a paths containing p_k reaching the point (n, m) can be appended with an up step so they reach $(n, m+1)$, and also one path coming from the left contains p_k. Thus, there is exactly one more path containing p_k reaching $(n, m+1)$ than (n, m), and the proof is complete by induction. □

For each $k > 0$, we have a difference recursion that implies $(s_{n,k})$ is a polynomial sequence [5]. Thus, $\deg s_{n,k} = n - a - b(k-1) + 1$ for $k > 0$ and $n \geq a + b(k-1)$, and we have already seen that $s_{n,0}$ is a polynomial of degree n [7].

Before we can start using the bivariate theory, we need to do two things. First, we must modify our polynomials a little so that they are like basic sequences. Second, $(s_{n,k})$ is not a bivariate polynomial sequence. We can make it into one by choosing our favorite univariate basic sequence, and do a construction similar to Corollary 6.

8. Creating a Bivariate Basic Sequence

For all n and k notice that $s_{n,k}(m+n-1) = 0$ at $m = 0$ except when $n = k = 0$, in which case $s_{0,0}$ is a constant, we get 1. This is still true for $b_{n,k}(m) := s_{n+kb,k}(m + n + kb - 1)$. We define $b_{n,k}$ this way for more elegance in the later equations. With Corollary 6 in mind, we want to use $b_{n,k}$ as one of the partial bivariate sequences, so we pick a univariate basic sequence (a_n) to be the other. Notice that given two univariate basic sequences (p_n) and (q_n), the product $a_{m,n}(u,v) := p_m(u)q_n(v)$ is a basic sequence. We say that the bivariate sequence factors if it can be written this way. The partial bivariate sequences are $a_{m,n}(u,0) = p_m(u)\delta_{n,0}$ and $a_{m,n}(0,v) = q_n(v)\delta_{m,0}$. With this in mind, we define

$$b^{(a)}_{m,n}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{i,j}(u)a_{n-j}(v)\delta_{m-i,0} = \sum_{j=0}^{n} b_{m,j}(u)a_{n-j}(v).$$

Notice that

$$b^{(a)}_{m,n}(0,0) = \sum_{j=0}^{n} b_{m,j}(0)a_{n-j}(0) = b_{m,n}(0) = \delta_{n,0}\delta_{m,0},$$

and $b_{0,0}(u,v) = b_{0,0}(u)a_0(v) = 1$, so it meets some of the requirements for a basic sequence. Suppose $B : s_{n,k} \to s_{n-1,k}$ and $K : s_{n,k} \to s_{n,k-1}$, then $B_1 := BE_u^{-1} : b_{m,n}(u) \to b_{m-1,n}(u)$ and $B_2 := K(BE_u^{-1})^b : b_{m,n}(u) \to b_{m,n-1}(u)$. Since $b^{(a)}_{m,n}(u,v)$ is a linear combination of the $b_{m,n}(u)$, it will have the same recursion. Transforming the general recurrence (Theorem 14) for $s_{n,k}(u)$ into operators, we get

$$\nabla_u = B - \frac{(1-K)B^a E_u^{-c}}{1 + (1-K)\sum_i B^{b_i} E_u^{-d_i}},$$

where $\nabla_u = 1 - E_u^{-1}$. Notice that when $K = 0$, we get the univariate operator equation in [7] for ballot paths avoiding a pattern. Substituting the operators B_1 and B_2 gives

$$\begin{aligned} \nabla_u &= B_1 E_u - \frac{B_1^a E_u^{a-c} - B_2 B_1^{a-b} E_u^{a-c}}{1 + \sum_i B_1^{b_i} E_u^{b_i - d_i} - \sum_i B_2 B_1^{b_i - b} E_u^{b_i - d_i}} \\ &= B_1 E_u - \frac{(B_1^b - B_2) B_1^{a-b} E_u^{a-c}}{1 + (B_1^b - B_2)\sum_i B_1^{b_i - b} E_u^{b_i - d_i}}. \end{aligned}$$

Solving this in general would be quite messy, and not very enlightening.

8.1. Counting *rur* in Ballot Paths

We will now show how the finite operator theory applies to our guiding example *rur*. This example is very basic, and we will solve it in three ways. The pattern *rur* has just one bifix *r*, and so we have $a = 2$ and $b_1 = c = d_1 = 1$. The corresponding operator equation becomes

$$\nabla_u = B_1 E_u - \frac{(B_1 - B_2) B_1 E_u}{1 + (B_1 - B_2)} = \frac{B_1 E_u}{1 + B_1 - B_2}.$$

Since $B_2 = A : a_n \to a_{n-1}$, we can use Corollary 10 to find the solution. Using Corollary 11,

$$V_1^m = \sum_{i \ge 0} \sum_{j \ge 0} \left[\epsilon_1^{i+1-m} \frac{\partial \tau_1}{\partial s} \right]_{i,j} A_1^i A_2^j,$$

where $\tau_1(s,t) = \dfrac{sE_u}{1+s-t}$ and $\epsilon_1(s,t) = E_u^{-1}(1+s-t)$. We get

$$V_1^m = \sum_{i \ge 0} \sum_{j \ge 0} (-1)^i \frac{m}{m-i} \binom{m+j-1}{i,j} E^{m-i} A_1^i A_2^j.$$

Thus, by Corollary 10,

$$b_{m,n}^{(a)}(u,v) = \theta_{A_1} \sum_{i \ge 0} \sum_{j \ge 0} \binom{m+j-1}{i,j} \frac{(-1)^i}{m-i} a_{m-1-i,n-j}(u+m-i, v).$$

Using Lemma 12,

$$b_{m,n}^{(a)}(u,0) = \sum_{i \ge 0} \sum_{j \ge 0} (-1)^i \binom{m+j-1}{i,j} \frac{u}{u+m-i} a_{m-i,n-j}(u+m-i, 0).$$

We know that $b^{(a)}_{m,n}(u,0) = b_{m,n}(u)$ and $(a_{m,n})$ is a bivariate basic sequence for the delta pair (∇_u, A). Since A is a univariate operator, $(a_{m,n})$ factors, that is, $a_{m,n}(u,v) = \binom{u+m-1}{m} a_n(v)$. Thus, $a_{m,n}(u,0) = \binom{u+m-1}{m}\delta_{n,0}$, which implies

$$\begin{aligned} b_{m,n}(u) &= \sum_{i\geq 0}\sum_{j\geq 0}(-1)^i \binom{m+j-1}{i,j}\frac{u}{u+m-i}\binom{u+2m-2i-1}{m-i}\delta_{n-j,0} \\ &= \sum_{i\geq 0}(-1)^i \binom{m+n-1}{i,n}\frac{u}{u+m-i}\binom{u+2m-2i-1}{m-i}. \end{aligned}$$

Finally, since $s_{n,k}(m) = b_{n-k,k}(m-n+1)$, we have

$$s_{n,k}(m) = \sum_{i\geq 0}(-1)^i \binom{n-1}{i,k}\frac{m-n+1}{m+1-k-i}\binom{m+n-2k-2i}{n-k-i}. \tag{9}$$

Notice that we could also write

$$\nabla_u E_u^{-1} = \frac{B_1}{1+B_1-B_2}$$

for the operator equation. The right-hand side is a power series in B_1 and B_2 with no operator coefficients, which means we can use Theorem 13 to find the solution. In this case, we have $A_1 = \nabla_u E_u^{-1}$, which has $\frac{u}{u+m}\binom{u+2m-1}{m}$ as its basic sequence, $\tau_1(s,t) = \frac{s}{1+s-t}$, and $a_{i,j}(u,v) = \frac{u}{u+i}\binom{u+2i-1}{i}a_j(v)$. Using the theorem with $\tau_2(s,t) = t$, we get $b^{(a)}_{m,n}(u,v) =$

$$\begin{aligned} &\sum_{i=0}^{m}\sum_{j=0}^{n}\left[\tau_1^i\right]_{m,n-j}\frac{u}{u+i}\binom{u+2i-1}{i}a_j(v) \\ =\ &\sum_{i=0}^{m}\sum_{j=0}^{n}\left[\sum_{k\geq 0}\binom{-i}{k}(s-t)^k\right]_{m-i,n-j}\frac{u}{u+i}\binom{u+2i-1}{i}a_j(v) \\ =\ &\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^i\binom{m+n-j-1}{i,n-j}\frac{u}{u+m-i}\binom{u+2(m-i)-1}{m-i}a_j(v). \end{aligned}$$

When we let $v = 0$, then the nonzero terms occur when $j = 0$; hence,

$$b_{m,n}(u) = \sum_{i=0}^{m}(-1)^i\binom{m+n-1}{i,n}\frac{u}{u+m-i}\binom{u+2(m-i)-1}{m-i},$$

and finally since $s_{n,k}(m) = b_{n-k,k}(m-n+1)$, we get (9).

$\uparrow m$	1	9	29	49	78	131	171	204	210	154
8	1	8	22	32	50	76	87	90	66	0
7	1	7	16	20	31	40	39	29	0	
6	1	6	11	12	18	18	13	0		
5	1	5	7	7	9	6	0			
4	1	4	4	4	3	0				
3	1	3	2	2	0					
2	1	$2_{\rightarrow}$	1	0						
1	$1_{\rightarrow}$	$1_{\uparrow}$	1							
0	1	0								
	0	1	2	3	4	5	6	7	8	$\rightarrow n$

Table 2: Number of ballot paths to (n, m) avoiding rur and $urru$

Finally, because

$$E_u^{-1} - E_u^{-2} = \frac{B_1}{1 + B_1 - B_2},$$

we can solve for E_u in terms of B_1 and B_2, $E_u = 1 + \tau_1(B_1, B_2)$, say (see (4)). Let $b(s, t; u) = \sum_{m,n \geq 0} b_{m,n}(u) s^m t^n$, and thus

$$b(s, t; u) = (1 + \tau_1(s, t))^u = \left(\frac{1 + s - t - \sqrt{(1 - t - s)^2 - 4s^2}}{2s} \right)^u.$$

We find

$$\begin{aligned} \sum_{n,k \geq 0} s_{n+k,k}(m + n + k) s^n t^k &= b(s, st; m + 1) \\ &= \left(\frac{1 + s - st - \sqrt{(1 - st - s)^2 - 4s^2}}{2s} \right)^{m+1}. \end{aligned}$$

9. Outlook: Ballot Paths Avoiding Two Patterns

We are interested in the number of ballot paths containing several patterns at the same time. Here is an example of a ballot path avoiding the patterns $urru$ and rur (note that $urru$ has depth 1). In addition to avoiding each pattern, $urru$ and rur, we also have to avoid overlaps like $rurru$. Without the benefit of the general recurrence formula (Theorem 14), we have to show the following lemma. Let $s_n(m)$ be the number of paths from $(0, 0)$ to (n, m) avoiding both patterns.

Lemma 16 *We have*

$$\begin{aligned} s_n(m) &= s_n(m-1) + s_{n-1}(m) - s_{n-2}(m-2) + s_{n-3}(m-2) \\ &\quad - s_{n-1}(m-1) + s_{n-1}(m-2). \end{aligned}$$

We will first show a technical result, using the same notation as in Section 6.

Lemma 17 *We have* $s_n(m;ru) + s_{n-2}(m-2;ur) = s_n(m-1) - s_n(m-2)$.

Proof.

$$\begin{aligned} s_n(m;ru) &= s_n(m;rrru) + s_n(m;uuru) \text{ (avoiding urru and rur)} \\ &= s_n(m-1;rrr) + s_n(m-1;uur) \text{ (deleting last up-step)} \\ &= s_n(m-1) - s_n(m-1;urr) - s_n(m-1;u) \text{ (complement)} \\ &= s_n(m-1) - s_{n-2}(m-2;\neg r) - s_n(m-2;\neg urr) \text{ (av. and del.)} \\ &= s_n(m-1) - s_n(m-2) - s_{n-2}(m-2;u) + s_n(m-2;urr) \text{ (comp.)} \\ &= s_n(m-1) - s_n(m-2) - s_{n-2}(m-2;u) + s_{n-2}(m-2;uu) \\ &= s_n(m-1) - s_n(m-2) - s_{n-2}(m-2;ur). \end{aligned}$$

□

Now we can prove Lemma 16.
Proof of Lemma 16. Various steps of pattern avoiding and complement-taking show that

$$\begin{aligned} s_n(m) &= s_n(m-1;\neg urr) + s_{n-1}(m;\neg ru) \\ &= s_n(m-1) - s_n(m-1;urr) + s_{n-1}(m) - s_{n-1}(m;ru) \\ &= s_n(m-1) + s_{n-1}(m) - s_n(m-1;urr) - s_{n-1}(m;ru) \\ &= s_n(m-1) + s_{n-1}(m) - s_{n-2}(m-2;\neg r) - s_{n-1}(m;ru) \\ &= s_n(m-1) + s_{n-1}(m) - s_{n-2}(m-2) + s_{n-2}(m-2;r) - s_{n-1}(m;ru) \\ &= s_n(m-1) + s_{n-1}(m) - s_{n-2}(m-2) + s_{n-3}(m-2) \\ &\quad - s_{n-3}(m-2;ru) - s_{n-1}(m;ru). \end{aligned}$$

It remains to show that $s_{n-2}(m-2;ru) + s_n(m;ru) = s_n(m-1) - s_n(m-2)$, which follows from Lemma 17. □

If we denote by B the operator mapping $s_n(n+m)$ into $s_{n-1}(n-1+m)$, then

$$\nabla = B\left(E + E^{-1} - 1\right) - B^2 + EB^3, \tag{10}$$

and thus$\tau(t) = (E-\nabla)t - t^2 + Et^3$. It follows from the transfer theorem of the univariate finite operator calculus [6] that $b_n(x) =$

$$\begin{aligned} & x\sum_{i=1}^{n}\left[\tau(t)^i\right]_n \frac{1}{x}\binom{i+x-1}{i} \\ = \quad & x\sum_{i=1}^{n}\sum_{j=0}^{i}\binom{i}{i-j,n-i-j}(E-\nabla)^{i-j}E^{n-i-j}(-1)^{n-i}\frac{1}{x}\binom{i+x-1}{i} \\ = \quad & x\sum_{i=1}^{n}\frac{(-1)^{n-i}}{i}\sum_{j=0}^{i}\binom{i}{j,n-2i+j}\sum_{k=0}^{\infty}\binom{j}{k}\binom{n-i+k+x-1}{i-1-k} \end{aligned}$$

is the corresponding basic sequence. Because

$$E = \left(1+B^2+B-\sqrt{1-B(1+B)(3B^2-B+2)}\right)/\left(2B+2B^3\right)$$

in (10), we find

$$\sum_{n\geq 0} b_n(x)t^n = \left(\frac{1+t^2+t-\sqrt{1-t(1+t)(3t^2-t+2)}}{2(t+t^3)}\right)^x.$$

To determine (s_n) we need initial values. Because both patterns can only occur when $n \geq 2$, we need separate initial values for the beginning of the recursion; we see from the table that $s_0(-1) = 1$, $s_1(0) = 0$, $s_2(1) = 1$, and $s_3(2) = 0$. By Lemma 16, $s_n(n) =$

$$s_n(n-1) + s_{n-1}(n) - s_{n-2}(n-2) + s_{n-3}(n-2) - s_{n-1}(n-1) + s_{n-1}(n-2),$$

and hence $s_n(n-1) + s_{n-1}(n-2) = 0$, and thus $s_n(n-1) = 0$ for $n \geq 4$. The binomial theorem for Sheffer sequences shows that

$$s_n(n+x) = \sum_{l=0}^{n} s_l(l-1)\, b_{n-l}(x+1) = b_n(x+1) + b_{n-2}(x+1)$$

and

$$\sum_{n\geq 0} s_n(n+m)t^n = \left(1+t^2\right)\left(\frac{1+t^2+t-\sqrt{1-t(1+t)(3t^2-t+2)}}{2(t+t^3)}\right)^{m+1}.$$

Especially the number of Dyck paths avoiding *uddu* and *dud* has the generating function

$$\sum_{n\geq 0} s_n(n)t^n = \frac{1+t^2+t-\sqrt{1-t(1+t)(3t^2-t+2)}}{2t}.$$

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