ON THE MAXIMAL CROSS NUMBER OF UNIQUE FACTORIZATION ZERO-SUM SEQUENCES OVER A FINITE ABELIAN GROUP

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Abstract

Let $S = (g_1, \cdots, g_l)$ be a sequence of elements from an additive finite abelian group G, and let

$$k(S) = \sum_{i=1}^{l} \frac{1}{\mathrm{ord}(g_i)}$$

denote the cross number of S. A zero-sum sequence S of nonzero elements from G is called *a unique factorization sequence* if S can be written in the form $S = S_1 \cdots S_r$ uniquely, where all S_i are minimal zero-sum subsequences of S. In this short note we investigate the following invariant of G concerning the cross number of unique factorization sequences. Define

$$K_1(G) = \max\{k(S) | S \text{ is a unique factorization sequence over } G \setminus \{0\}\},$$

where the maximum is taken when S runs over all unique factorization sequences over $G \setminus \{0\}$. We determine $K_1(G)$ for some special groups including the cyclic groups of prime power order.

1. Introduction and Main Results

Let $\mathbb{Z}$ denote the set of integers, and let $\mathbb{N}$ denote the set of positive integers. For real numbers $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{Z} | a \le x \le b\}$. Let G be an additive finite abelian group. We denote by $|G|$ the *order* of G. A sequence $S = (g_1, \cdots, g_l)$ of elements (repetition allowed) from G will be called a sequence over G. For convenience, we often write S in the form $S = g_1 \cdot \ldots \cdot g_l$. We call $|S| = l$ the *length* of S. If $g_1 = \cdots = g_l = g$ then we can simply write S in the form $S = g^l$.

For every $g \in G$, let $v_g(S)$ denote the number of the times that g occurs in S. Let $T = g_{i_1} \cdots g_{i_t}$ be a subsequence of S. We call $I_T \stackrel{\text{def}}{=} \{i_1, \cdots, i_t\}$ the *index set* of T. We denote by ST^{-1} the subsequence of S with the index set $\{1, \cdots, l\} \setminus I_T$. Let T_1 and T_2 be two subsequences of S. By $T_1 \cap T_2$ we denote the sequence with the index set $I_{T_1} \cap I_{T_2}$. We say T_1 and T_2 are disjoint if $I_{T_1} \cap I_{T_2} = \emptyset$, and denote by T_1T_2 the sequence with the index set $I_{T_1} \cup I_{T_2}$. We identify two subsequences S_1 and S_2 of S if and only if $I_{S_1} = I_{S_2}$.

Let $\sigma(S) = \sum_{i=1}^{l} g_i \in G$ denote the sum of S. We call the sequence S

- a *zero-sum* sequence if $\sigma(S) = 0$,
- a *zero-sum free* sequence if S contains no nonempty zero-sum subsequence,
- a *minimal zero-sum* sequence if S is a nonempty zero-sum sequence and S contains no proper zero-sum subsequence.

Every map of abelian groups $\phi : G \to H$ extents to a map from the sequences over G to the sequences over H by $\phi(S) = \phi(g_1) \cdot \ldots \cdot \phi(g_l)$. If ϕ is a homomorphism, then $\phi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \ker(\phi)$.

Let $D(G)$ be the Davenport constant of G which is the smallest integer d such that every sequence of d elements from G is not zero-sum free. $D(G)$ can also be defined equivalently as the maximal length of the minimal zero-sum sequences over G.

Let

$$k(S) = \sum_{i=1}^{l} \frac{1}{\operatorname{ord}(g_i)}$$

denote the *cross number* of S. Define

$$K(G) = \max\{k(S) | S \text{ is a minimal zero-sum sequence over } G\},$$

the maximum taken when S runs over all minimal zero-sum sequences over G.

The following invariant $N_1(G)$ was introduced by Narkiewicz in 1979 [13] which like $D(G)$ and $K(G)$ plays an important role in the study of non-unique factorization problems in algebraic number theory (see [7], [12], [16] and [6]). Let S be a zero-sum sequence over $G \setminus \{0\}$, i.e., S is a zero-sum sequence of non-zero elements from G. Clearly, S can be written in the form $S = S_1 \cdots S_r$ with all S_i being minimal zero-sum subsequences of S, and we call $S = S_1 \cdots S_r$ an *irreducible factorization* of S. We identify two irreducible factorizations $S = S_1 \cdots S_r$ and $S = T_1 \cdots T_m$ if and only if $m = r$, and there is a permutation τ on $\{1, \ldots, r\}$ such that $S_i = T_{\tau(i)}$ for every $i \in [1, r]$. A zero-sum sequence S over $G \setminus \{0\}$ is called *a unique factorization se-quence* if S has only one irreducible factorization. Narkiewicz constant $N_1(G)$ is the maximal length of the unique factorization sequences over $G \setminus \{0\}$. Unique factorization sequences and therefore $N_1(G)$ can also be formulated in terms of the concept of "type" just like what Geroldinger and Hater-Koch did in ([6], Chapter 9).

For $|G| > 1$, define

$$K_1(G) = \max\{k(S) | S \text{ is a unique factorization sequence over } G \setminus \{0\}\}$$

where the maximum is taken when S runs over all unique factorization sequences over $G \setminus \{0\}$, and let $K_1(G) = 0$ if $|G| = 1$.

The study of the cross number has attracted a lot of attention since it was introduced by Krause [8] in 1984 (for example, see [5], [9], [2], [6], [10] and [11]).

Every nontrivial finite abelian group G can be written uniquely in the form $G = \oplus_{i=1}^{r} \oplus_{j=1}^{t_i} C_{{p_i}^{e_{ij}}}$, where $p_1, \cdots, p_r$ are distinct primes. Set

$$K_1^*(G) = \sum_{i=1}^{r} \sum_{j=1}^{t_i} \frac{p_i^{e_{ij}} - 1}{p_i^{e_{ij}} - p_i^{e_{ij}-1}},$$

and let $K_1^*(G) = 0$ if $|G| = 1$.

It is not difficult to see that $K_1(G) \geq K_1^*(G)$ holds for all finite abelian groups G (see Proposition 3 in Section 2). We propose the following conjecture.

Conjecture 1. *$K_1(G) = K_1^*(G)$ holds for any finite abelian group G.*

In this paper we shall verify Conjecture 1 for some special groups by showing the following main result.

Theorem 2. *Let p be a prime, and let G be a finite abelian group. Then, $K_1(G) = K_1^*(G)$ if G is one of the following groups:*

(1) $G = C_{p^m}$ *with* $m \in \mathbb{N}$;

(2) $G = C_{pq}$ *with q a prime;*

(3) $G = C_2^r$ *with* $r \in \mathbb{N}$;

(4) $G = C_3^r$ *with* $r \in \mathbb{N}$;

(5) $G = C_p^2$.

2. A Lower Bound for $K_1(G)$

Proposition 3. *Let G be a finite abelian group.* (1) *If $G = G_1 \oplus G_2$ for some finite abelian groups G_1 and G_2 then $K_1(G) \geq K_1(G_1) + K_1(G_2)$;* (2) *$K_1(G) \geq K_1^*(G)$ holds for any finite abelian group G.*

Proof. If one of G, G_1 and G_2 is trivial then the proposition holds trivially. So, we may assume that none of G, G_1 and G_2 is trivial.
(1). Let $S_1 = a_1 \cdots a_u$ be a unique factorization sequence over G_1 with $k(S_1) = K_1(G_1)$, and Let $S_2 = b_1 \cdots b_v$ be a unique factorization sequence over G_2 with

$k(S_2) = K_1(G_2)$. Let $\mathbf{0}_{G_1}$ denote the identity element of G_1, and let $\mathbf{0}_{G_2}$ denote the identity element of G_2. Let

$$S_1' = (a_1, \mathbf{0}_{G_2})(a_2, \mathbf{0}_{G_2}) \cdots (a_u, \mathbf{0}_{G_2}) \quad \text{and} \quad S_2' = (\mathbf{0}_{G_1}, b_1)(\mathbf{0}_{G_1}, b_2) \cdots (\mathbf{0}_{G_1}, b_v).$$

Then both S_1' and S_2' are sequences over $G = G_1 \oplus G_2$ with $|S_1'| = |S_1|, |S_2'| = |S_2|, k(S_1') = k(S_1)$ and $k(S_2') = k(S_2)$. Let $S = S_1'S_2'$. Clearly, S is a unique factorization sequence over G. Therefore, $K_1(G) \geq k(S) = k(S_1') + k(S_2') = k(S_1) + k(S_2) = K_1(G_1) + K_1(G_2)$.
(2). By (1), it suffices to prove $K_1(G) \geq K_1^*(G)$ for every cyclic group G of prime power order. Let $G = C_{p^m}$ with p a prime, and let g be a generating element of G. Let

$$S = g^{p-1} \cdot ((1-p)g) \cdot (pg)^{p-1} \cdot ((1-p)pg) \cdots (p^{m-2}g)^{p-1} \cdot ((1-p)p^{m-2}g) \cdot (p^{m-1}g)^p,$$

i.e., S is the sequence with $v_{p^i g}(S) = p - 1$ and $v_{(1-p)p^i g}(S) = 1$ for every $i \in [0, m-2]$, and $v_{p^{m-1}g}(S) = p$. Clearly, S is a unique factorization sequence. So, $K_1(C_{p^m}) \geq k(S) = 1 + \frac{1}{p} + \cdots + \frac{1}{p^{m-1}} = \frac{p^m - 1}{p^m - p^{m-1}} = K_1^*(G)$. □

3. Proof of Theorem 2

To prove Theorem 2 we need some preliminaries and we begin with a result of Olson [15].

Let p be a prime, and let G be a finite abelian p-group. For $g \in G$, define $\alpha(g) = p^n$ where n is the largest integer such that $g \in p^n G = \{p^n x | x \in G\}$ ($\alpha(0) = \infty$). Let $S = g_1 \cdot \ldots \cdot g_l$ be a sequence over G. Define

$$\alpha(S) = \sum_{i=1}^{l} \alpha(g_i).$$

Lemma 4. ([15]) *Let p be a prime, and let $G = C_{p^{e_1}} \oplus \cdots \oplus C_{p^{e_r}}$. Let $S = g_1 \cdots g_k$ be a sequence over G. If $\alpha(S) = \sum_{i=1}^r \alpha(g_i) \geq 1 + \sum_{i=1}^r (p^{e_i} - 1)$, then S is not zero-sum free.*

Lemma 5. ([3]) *Let S be a zero-sum sequence over $G \setminus \{0\}$. Then, the following statements are equivalent.*
(1) *S is a unique factorization sequence;*
(2) *For any two zero-sum subsequences S_1 and S_2 of S we have that the intersection $S_1 \cap S_2$ is also a zero-sum sequence.*

Let G be a finite abelian group. It is well known that either $|G| = 1$ or G can be written uniquely in the form $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 | \cdots | n_r$. Narkiewicz

[13] conjectured that $N_1(G) = n_1 + \cdots + n_r$ holds for any finite abelian group G. This conjecture has been verified only for some very special groups. Some of these groups are listed below and will be used in the proof of Theorem 2.

Lemma 6. ([14], [1], [4]) *Let p be a prime. Then $N_1(G) = n_1 + \cdots + n_r$ if G is one of the following groups*

1. $G = C_n$ *with* $n \in \mathbb{N}$;
2. $G = C_2^r$;
3. $G = C_3^r$;
4. $G = C_p^2$.

Lemma 7. *Let p be a prime, and let r be a positive integer. Then, $N_1(C_p^r) = rp$ if and only if $K_1(C_p^r) = r$.*

Proof. Let $G = C_p^r$. Since every nonzero element of G has order p, the result follows from the definitions of $N_1(G)$ and $K_1(G)$. □

Proof of Theorem 2. We start with the proof of (1). By Proposition 3, it suffices to prove the upper bound.

We proceed by induction on m. If $m = 1$, let $S = g_1 \cdots g_k$ be a zero-sum sequence over $G \setminus \{0\}$ with $k(S) = \frac{k}{p} > 1$. Since $N_1(C_p) = p$ we know that S is not a unique factorization sequence. It follows that $K_1(C_p) = 1$.

Now let $m \geq 2$. Let S be a unique factorization zero-sum sequence over $G^* = C_{p^m} \setminus \{0\}$. We need to show that $k(S) \leq 1 + \frac{1}{p} + \cdots + \frac{1}{p^{m-1}}$.

Assume to the contrary that $k(S) > 1 + \frac{1}{p} + \cdots + \frac{1}{p^{m-1}}$. We shall derive a contradiction. Write S in the form

$$S = g_{11} \cdots g_{1r_1} g_{21} \cdots g_{2r_2} \cdots g_{m1} \cdots g_{mr_m} = \prod_{i=1}^{m} \prod_{j=1}^{r_i} g_{ij}$$

with $g_{ij} \in C_{p^m}$ and $\mathrm{ord}(g_{ij}) = p^i$ for all $i \in [1, m]$ and $j \in [1, r_i]$. Then

$$k(S) = \sum_{i=1}^{m} \sum_{j=1}^{r_i} \frac{1}{\mathrm{ord}(g_{ij})} = \frac{r_1}{p} + \cdots + \frac{r_m}{p^m}.$$

Therefore, $\frac{r_1}{p} + \cdots + \frac{r_m}{p^m} > 1 + \frac{1}{p} + \cdots + \frac{1}{p^{m-1}}$. Multiplying the two sides of the above inequality with p we obtain

$$r_1 + \frac{r_2}{p} + \cdots + \frac{r_m}{p^{m-1}} > p + 1 + \frac{1}{p} + \cdots + \frac{1}{p^{m-2}}.$$

Let ϕ be the canonical epimorphism from C_{p^m} to C_{p^m}/C_p. Let $T = g_{11} \cdots g_{1r_1}$ and let $S' = ST^{-1}$. Then $\phi(S') = \phi(ST^{-1}) = \prod_{i=2}^{m} \prod_{j=1}^{r_i} \phi(g_{ij})$ and

$$k(\phi(S')) = \frac{r_2}{p} + \cdots + \frac{r_m}{p^{m-1}} > p - r_1 + 1 + \frac{1}{p} + \cdots + \frac{1}{p^{m-2}}.$$

By multipling the two sides of the above inequality with p^{m-1} we obtain that $r_2p^{m-2} + r_3p^{m-3} + \cdots + r_m \geq p^{m-1}(p - r_1 + 1) + p^{m-2} + \cdots + p + 1$. Therefore,

$$\alpha(\phi(S')) = r_2p^{m-2} + r_3p^{m-3} + \cdots + r_m \geq p^{m-1}(p - r_1 + 1) + p^{m-2} + \cdots + p + 1.$$

Let $t \geq 0$ be maximal such that there are disjoint subsequences $S_1, \ldots, S_t$ of S' with $\sigma(S_i) \in \ker\phi \setminus \{0\}$ for every $i \in [1, t]$. By the maximality of t we infer that $\phi(S_i)$ is a minimal zero-sum sequence for each $i \in [1, t]$. It follows from Lemma 4 that $\alpha(\phi(S_i)) \leq p^{m-1}$ for each $i \in [1, t]$. We assert that $t+r_1 \geq p+1$. Assume to the contrary that $t+r_1 \leq p$. Then, $\alpha(\phi(S'(S_1 \cdots S_t)^{-1})) = \alpha(\phi(S')) - \sum_{i=1}^{t} \alpha(\phi(S_i)) \geq p^{m-1}(p-r_1+1) + p^{m-2} + \cdots + p + 1 - (p-r_1)p^{m-1} \geq p^{m-1} + p^{m-2} + \cdots + p + 1$. Let $S'' = S'(S_1 \cdots S_t)^{-1}$. We just proved that $\alpha(\phi(S'')) \geq p^{m-1} + p^{m-2} + \cdots + p + 1$. Let r''_j be the number of elements x(counted with multiple) of $\phi(S'')$ with $\mathrm{ord}(x) = p^j$ for every $j \in [1, m-1]$. It follows that $r''_1p^{m-2} + \cdots + r''_{m-2}p + r''_{m-1} = \alpha(\phi(S'')) \geq p^{m-1} + p^{m-2} + \cdots + p + 1$. Therefore,

$$K(\phi(S'')) = \frac{r''_1}{p} + \cdots + \frac{r''_{m-2}}{p^{m-2}} + \frac{r''_{m-1}}{p^{m-1}} \geq 1 + \frac{1}{p} + \cdots + \frac{1}{p^{m-2}} + \frac{1}{p^{m-1}}.$$

By the induction hypothesis, we have $K_1(\phi(C_{p^m})) = K_1(C_{p^{m-1}}) = 1 + \frac{1}{p} + \cdots + \frac{1}{p^{m-2}}$. Therefore, $\phi(S'')$ is not a unique factorization sequence. By Lemma 5 there exist two subsequences T_1, T_2 of S'' such that both $\phi(T_1)$ and $\phi(T_2)$ are minimal zero-sum sequences but $\phi(T_1 \cap T_2)$ is not a zero-sum sequence over $\phi(G) = C_{p^{m-1}}$. Hence, $T_1 \cap T_2$ is not a zero-sum sequence over C_{p^m}. Since S is a unique factorization sequence, again by Lemma 5 we obtain that either $\sigma(T_1) \in \ker\phi \setminus \{0\}$, or $\sigma(T_2) \in \ker\phi \setminus \{0\}$, a contradiction to the maximality of t. This proves that $t + r_1 \geq p + 1$.

Since $\sigma(\phi(S(TS_1 \cdots S_t)^{-1})) = 0$, $S(TS_1 \cdots S_t)^{-1} = R_1 \cdots R_\ell$ with $\phi(R_i)$ being minimal zero-sum for each $i \in [1, \ell]$. By the maximality of t, $\sigma(R_i) = 0$ for each $i \in [1, \ell]$. It follows that both $S(TS_1 \cdots S_t)^{-1}$ and $TS_1 \cdots S_t$ are zero-sum sequences. Now $T\sigma(S_1) \cdots \sigma(S_t)$ is a zero-sum sequence over $C_p \setminus \{0\}$ and $|T\sigma(S_1) \cdots \sigma(S_t)| = r_1 + t \geq p + 1$. By $N_1(C_p) = p$ we obtain that $T\sigma(S_1) \cdots \sigma(S_t)$ is not a unique factorization sequence, and so neither is S, a contradiction.

We now prove (2). From Part (1) we may assume that $p \neq q$. It suffices to prove the upper bound. Let S be a unique factorization sequence over $C_{pq} \setminus \{0\}$. We need to show that $k(S) \leq 2$. Assume to the contrary that $k(S) > 2$. Write S in the form

$$S = g_{11} \cdots g_{1m} g_{21} \cdots g_{2n} g_{31} \cdots g_{3k}$$

with

$$\mathrm{ord}(g_{ij}) = \begin{cases} p & \text{if } i = 1 \\ q & \text{if } i = 2 \\ pq & \text{if } i = 3. \end{cases}$$

Then $k(S) = \frac{m}{p} + \frac{n}{q} + \frac{k}{pq} > 2$. Therefore,

$$mq + np + k \geq 2pq + 1. \tag{1}$$

Let $T = g_{11} \cdots g_{1m}$, and let ϕ be the canonical epimorphism from C_{pq} to C_{pq}/C_p. Then

$$\phi(ST^{-1}) = \phi(g_{21}) \cdots \phi(g_{2n})\phi(g_{31}) \cdots \phi(g_{3k})$$

and $k(\phi(ST^{-1})) = \frac{n+k}{q}$. Since $\sigma(S) = 0$, we infer that $\sigma(\phi(ST^{-1})) = 0$.

Let $t \geq 0$ be maximal such that there are disjoint subsequences $S_1, \ldots, S_t$ of ST^{-1} with $\sigma(S_i) \in \ker \phi \setminus \{0\}$ for every $i \in [1, t]$. By the maximality of t we infer that $\phi(S_i)$ is a minimal zero-sum sequence over $\phi(C_{pq}) \cong C_q$ for each $i \in [1, t]$. It follows from $D(C_q) = q$ that $|S_i| = |\phi(S_i)| \leq q$ for each $i \in [1, t]$. As in Part (1) we obtain that $T\sigma(S_1) \cdots \sigma(S_t)$ is a zero-sum sequence over $C_p \setminus \{0\}$. If $m+t \geq p+1 > p = N_1(C_p)$ then $T\sigma(S_1) \cdots \sigma(S_t)$ is not a unique factorization sequence, and so neither is S, a contradiction. Therefore, $m + t \leq p$.

If $n \geq q+1$, then by switching p for q and repeating the procedure above we can derive a contradiction. Therefore, $n \leq q$.

From Equation (1) we obtain that $np + k - (p - m)q \geq pq + 1$. This together with $n \leq q$ gives that $k - (p - m)q > 0$. Therefore, $np + (k - (p - m)q)p > np + k - (p - m)q \geq pq + 1$. Hence, $n + k - (p - m)q \geq q + 1$.

Now we have that $|S(TS_1 \cdots S_t)^{-1}| \geq |S| - m - tq = n + k - tq \geq n + k - (p - m)q \geq q + 1 > q = N_1(C_q)$. So, $\phi(S(TS_1 \cdots S_t)^{-1})$ is not a unique factorization sequence. As in Part (1) we can derive a contradiction.

The proofs of (3)–(5) result follow from Lemma 6 and Lemma 7. □

4. Concluding Remarks

For the general case we have the following result.

Proposition 8. *Let G be a nontrivial finite abelian group, and p be the smallest prime divisor of $|G|$. Then $K_1(G) < \ln |G| + \frac{1}{p} \log_2 |G|$.*

Proof. Let S be a unique factorization sequence over $G \setminus \{0\}$. Let $S = S_1 \cdots S_t$ be an irreducible factorization of S, where $t \in \mathbb{N}$, and all $S_1, \ldots, S_t$ are minimal zero-sum subsequences of S. Then we have $|S_i| \geq 2$ for every $i \in [1, t]$. By a result of Narkiewicz (see [14], Proposition 6; or [1], Lemma 2), $\Pi_{i=1}^{t} |S_i| \leq |G|$. Therefore, $t \leq \log_2 |G|$.

For every $i \in [1, t]$ we choose an element $g_i \in \operatorname{supp}(S_i)$. Since S is a unique factorization sequence, we infer that the sequence $T = g_1^{-1} S_1 \cdots g_t^{-1} S_t$ is zero-sum free. Now by a result of Geroldinger and Schneider [9], $k(T) \leq \ln |G|$. Therefore,

$$k(S) = k(T) + \sum_{i=1}^{t} \frac{1}{\operatorname{ord}(g_i)} \leq \ln |G| + t\frac{1}{p} \leq \ln |G| + \frac{\log_2 |G|}{p}.$$ □

Let G be a finite abelian group. It is easy to see that $K(G) \leq K_1(G)$ holds for all nontrivial finite abelian groups. Unlike the Davenport constant $D(G)$, the exact values of $K(G)$ for most of cyclic groups are not known. Also, very little is known about the Narkiewicz constant $N_1(G)$. So, at the moment we can't expect much results in the determining of $K_1(G)$ since this is essentially involved in the determining of $K(G)$ and $N_1(G)$.

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