ENUMERATION OF THE DEGREE SEQUENCES OF LINE–HAMILTONIAN MULTIGRAPHS

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Abstract

Recently, Gu, Lai and Liang proved necessary and sufficient conditions for a given sequence of positive integers $d_1, d_2, \ldots, d_n$ to be the degree sequence of a line–Hamiltonian multigraph. Our goal in this note is to utilize this result to prove a closed formula for the function $d_{lh}(2m)$, the number of degree sequences with degree sum $2m$ representable by line–Hamiltonian multigraphs. Indeed, we give a truly elementary proof that

$$d_{lh}(2m) = p(2m) - 2\left(\sum_{j=0}^{m-1} p(j)\right) + 1$$

where $p(j)$ is the number of unrestricted integer partitions of j.

1. Introduction and Statement of Results

In this note, all graphs $G = (V, E)$ under consideration will be finite, undirected, and loopless but may contain multiple edges. We denote the *degree sequence* of the vertices $v_1, v_2, \ldots, v_m$ by $d_1, d_2, \ldots, d_m$ with the convention that $d_1 \geq d_2 \geq \cdots \geq d_m$. We say that a sequence $d_1, d_2, \ldots, d_m$ with $d_1 \geq d_2 \geq \cdots \geq d_m$ is *multigraphic* if there exists a multigraph G with this degree sequence. (This multigraph G is called a *realization* of this degree sequence.) Lastly, we say that a degree sequence is *line–Hamiltonian* if it has a multigraphic realization G such that $L(G)$, the line graph of G, is Hamiltonian.

In 2008, Fan, Lai, Shao, Zhang, and Zhou [1] characterized those degree sequences for which there exists a *simple* line–Hamiltonian graph realization. More recently, Gu, Lai and Liang [2] provided a characterization of those degree sequences for which there exists a *multigraphic* line–Hamiltonian realization. Their characterization (in this multigraphic setting) is the following:

Theorem 1. *Let $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ be integers with $n \geq 2$. There exists a multigraphic line–Hamiltonian realization of $d_1, d_2, \ldots, d_n$ if and only if*

1) $d_1 + d_2 + \cdots + d_n$ *is even, and*

2) $d_1 \leq d_2 + d_3 + \cdots + d_n$, *and*

3a) $\displaystyle\sum_{d_i=1} d_i \leq \sum_{d_j>1} (d_j - 2)$ *or* 3b) $d_1 = n - 1$.

In this brief note, our goal is to enumerate all degree sequences of sum $2m$ for which there exists a multigraphic line–Hamiltonian realization. We will denote the number of degree sequences of sum $2m$ with a multigraphic line–Hamiltonian realization by $d_{lh}(2m)$. Then our ultimate goal is to prove the following:

Theorem 2. *For all $m \geq 2$,*

$$d_{lh}(2m) = p(2m) - 2\left(\sum_{j=0}^{m-1} p(j)\right) + 1$$

where $p(k)$ is the number of unrestricted integer partitions of k.

It should be noted that similar enumeration results were discovered recently by Rødseth, Sellers, and Tverberg [3] when they proved that the number of degree sequences with multigraphic connected (respectively, non–separable) realizations equal similar linear combinations of the partition function $p(k)$. In [3], the proofs utilized included generating function manipulations and bijections. In this paper, the proof techniques are even more elementary and involve simply understanding the combinatorial nature of the integer partitions in question. Even so, these techniques should not be discounted for their simplicity.

2. Proof of the Theorem

We begin our proof of Theorem 2 by considering the enumeration of those degree sequences satisfying the criteria 1, 2, and 3a of Theorem 1 above. To this end, assume $d_1 + d_2 + \cdots + d_n = 2m$ for some fixed value of m, so that the graphs in question have exactly m edges. Then we see that criterion 2 is equivalent to $2d_1 \leq d_1 + d_2 + d_3 + \cdots + d_n$ or $2d_1 \leq 2m$ or $d_1 \leq m$. From a partition–theoretic point of view, this means that the largest part in our partition is at most m. Note also that criterion 3a is equivalent then to saying that $n \le m$ because of the following:

$$\begin{aligned}
\sum_{d_i=1} d_i \le \sum_{d_j>1}(d_j-2) \quad &\text{if and only if} \quad \sum_{d_i=1} d_i \le \sum_{d_j>1} d_j - 2\sum_{d_j>1} 1\\
&\text{if and only if} \quad 2\sum_{d_i=1} d_i \le \sum_{\text{all } d_j} d_j - 2\sum_{d_j>1} 1\\
&\text{if and only if} \quad 2\sum_{d_i=1} 1 \le \sum_{\text{all } d_j} d_j - 2\sum_{d_j>1} 1\\
&\text{if and only if} \quad 2\sum_{\text{all } d_i} 1 \le \sum_{\text{all } d_j} d_j\\
&\text{if and only if} \quad 2n \le 2m\\
&\text{if and only if} \quad n \le m\,.
\end{aligned}$$

This gives us an upper bound on the *number* of parts in each partition in question. Taking all of this information into account, we see that the number of partitions satisfying criteria 1, 2, and 3a is given by the coefficient of q^{2m} in the q–binomial coefficient $\begin{bmatrix} 2m \\ m \end{bmatrix}_q$ which equals

$$\frac{(1-q^{m+1})(1-q^{m+2})(1-q^{m+3})\dots(1-q^{2m})}{(1-q)(1-q^2)(1-q^3)\dots(1-q^m)}.$$

This generating function is equivalent to

$$(1-q^{m+1})(1-q^{m+2})(1-q^{m+3})\dots(1-q^{2m})\sum_{N=0}^{\infty} p(N;\le m)q^N$$

where $p(N;\le m)$ is the number of partitions of N where the largest part is at most m. A closer consideration reveals that the coefficient of q^{2m} in this generating function is given by

$$p(2m;\le m)-p(2m-(m+1);\le m)-p(2m-(m+2);\le m)-\dots-p(2m-2m;\le m)$$

or

$$p(2m;\le m)-p(m-1;\le m)-p(m-2;\le m)-\dots-p(0;\le m)$$

upon simplification. Next, note that whenever $a \le m$, $p(a;\le m) = p(a)$, where $p(a)$ is simply the number of (unrestricted) partitions of a. Thus, we now know that the coefficient of q^{2m} in the generating function in question is actually equal to

$$p(2m;\le m)-p(m-1)-p(m-2)-\dots-p(0).$$

Lastly, note that

$$p(2m;\leq m) = p(2m) - p(2m;m+1) - p(2m;m+2) - \cdots - p(2m;2m)$$

where $p(a;b)$ equals the number of partitions of a with largest part **exactly equal** to b. But, of course, $p(a;b) = p(a-b;\leq b)$ by simply removing this largest part b from every partition counted by $p(a;b)$. Therefore,

$$\begin{aligned} p(2m;\leq m) &= p(2m) - p(m-1;\leq m+1) - p(m-2;\leq m+2) - \cdots - p(0;\leq 2m) \\ &= p(2m) - p(m-1) - p(m-2) - \cdots - p(0). \end{aligned}$$

Combining all the results above, we know that the number of partitions satisfying criteria 1, 2, and 3a equals

$$p(2m) - 2\sum_{j=0}^{m-1} p(j). \tag{1}$$

Notice that (1) gives almost all of the formula for $d_{lh}(2m)$ as given in Theorem 2. As an aside, it should be noted that the values generated by (1), namely $1, 3, 8, 18, 39, \ldots,$ appear as the elements of sequence A128552 in Sloane's Online Encyclopedia of Integer Sequences [4].

To close the proof, we must now consider those partitions of $2m$ which satisfy criteria 1, 2, 3b and the negation of 3a of Theorem 1 above. Combining criteria 2 and 3b we know that $n-1 \leq m$. We also know from the negation of criterion 3a, and the work completed above, that $n > m$. Clearly, this implies $m = n-1$. This means $d_1 = m$ since criterion 3b states that $d_1 = n-1$. So we know that the partitions in question must have largest part equal to m. But we also know that $n = m+1$, meaning that the number of parts in our partition must be exactly $m+1$. Lastly, from criterion 1, we know that the sum of all the parts in the partition must be $2m$. Combining all these facts implies that there can be only one partition satisfying these requirements, the partition

$$2m = m + \underbrace{1 + 1 + \cdots + 1}_{m \text{ ones}}.$$

(Notice that the graph realization of this degree sequence is a star graph. The line graph of such a graph is complete, and this is clearly Hamiltonian.) This means that the contribution to the formula for $d_{lh}(2m)$ which arises from this set of criteria is simply 1 (accounting for this one partition above). Combining this with (1) yields

$$d_{lh}(2m) = p(2m) - 2\left(\sum_{j=0}^{m-1} p(j)\right) + 1.$$

This completes the proof of Theorem 2. $\square$

3. Closing Thoughts

Two sets of closing thoughts come to mind. First, it would be intriguing to know of other degree sequence characterization results which might provide such enumerative corollaries. Secondly, it would be truly satisfying to obtain a similar enumeration result for *simple* graphs. Unfortunately, this would mean altering the second criterion in the statement of Theorem 1 to a statement which is much more complicated. This makes the enumeration problem much more difficult.

References

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