THE NUMBER OF REPRESENTATIONS OF A NUMBER AS SUMS OF VARIOUS POLYGONAL NUMBERS

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Abstract

In this paper, we present twenty-five analogues of Jacobi's two-square theorem which involve squares, triangular numbers, pentagonal numbers, heptagonal numbers, octagonal numbers, decagonal numbers, hendecagonal numbers, dodecagonal numbers, and octadecagonal numbers.

1. Introduction

Jacobi's celebrated two-square theorem is as follows.

Theorem 1.1. ([7]). *Let $r\{\Box+\Box\}(n)$ denote the number of representations of n as a sum of two squares and $d_{i,j}(n)$ denote the number of positive divisors of n congruent to i modulo j. Then*

$$r\{\Box+\Box\}(n)=4(d_{1,4}(n)-d_{3,4}(n)). \tag{1}$$

Simple proofs of (1) can be seen in [2] and [4]. Similar representation theorems involving squares and triangular numbers were found by Dirichlet [3], Lorenz [10], Legendre [9], and Ramanujan [1]. For example, another classical result due to Lorenz [10] is stated below.

Theorem 1.2. *Let $r\{l\Box+m\Box\}(n)$ denote the number of representations of n as a sum of l times a square and m times a square. Then*

$$r\{\Box+3\Box\}(n)=2(d_{1,3}(n)-d_{2,3}(n))+4(d_{4,12}(n)-d_{8,12}(n)). \tag{2}$$

In [5], M.D. Hirschhorn obtained sixteen identities (including those obtained by Legendre and Ramanujan) simply by dissecting the q-series representations of the identities obtained by Jacobi, Dirichlet and Lorenz. Hirschhorn [6] further extended his work and obtained twenty-nine more identities involving squares, triangular numbers, pentagonal numbers and octagonal numbers. For more work on this topic one can see [8], [11] and [12]. In [12], R. S. Melham presented an informal account of analogues of Jacobi's two-square theorem which are verified using computer algorithms.

In this paper, we find twenty-five more such identities involving squares, triangular numbers, pentagonal numbers, heptagonal numbers, octagonal numbers, decagonal numbers, hendecagonal numbers, dodecagonal numbers, and octadecagonal numbers, by employing Ramanujan's theta function identities.

For $k \geq 3$, the n^{th} k-gonal number $F_k(n)$ is given by

$$F_k := F_k(n) = \frac{(k-2)n^2 - (k-4)n}{2}.$$

By allowing the domain for $F_k(n)$ to be the set of all integers, we see that the generating function $G_k(q)$ of $F_k(n)$ is given by

$$G_k(q) = \sum_{n=-\infty}^{\infty} q^{F_k} = \sum_{n=-\infty}^{\infty} q^{\frac{(k-2)n^2 - (k-4)n}{2}}.$$

We note an exception for the case $k = 3$. We observe that $G_3(q)$ generates each triangular number twice while $G_6(q)$ generates each only once. As such, we take $G_6(q)$ as the generating function for triangular numbers instead of $G_3(q)$. We further observe that

$$G_k(q) = f(q, q^{k-3}), \tag{3}$$

where $f(a, b)$ is Ramanujan's general theta function defined by [1, p. 34, Eq. (18.1)]:

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

Two important special cases of $f(a, b)$ are

$$\varphi(q) := f(q, q),$$
$$\psi(q) := f(q, q^3).$$

In view of (3), the respective generating functions of squares, triangular numbers, pentagonal numbers, heptagonal numbers, octagonal numbers, decagonal numbers, hendecagonal numbers, dodecagonal numbers, and octadecagonal numbers are

$$\begin{aligned}
G_4(q) &= f(q,q) = \varphi(q),\\
G_6(q) &= f(q,q^3) = \psi(q),\\
G_5(q) &= f(q,q^2),\\
G_7(q) &= f(q,q^4),\\
G_8(q) &= f(q,q^5),\\
G_{10}(q) &= f(q,q^7),\\
G_{11}(q) &= f(q,q^8),\\
G_{12}(q) &= f(q,q^9),
\end{aligned}$$

and

$$G_{18}(q) = f(q,q^{15}).$$

In Section 2, we give dissections of $\varphi(q)$, $\psi(q)$, $G_5(q)$, and $G_{12}(q)$ and recall some identities established in [5] and [6]. In the remaining five sections, we successively present sets of identities involving decagonal numbers, hendecagonal numbers, dodecagonal numbers, heptagonal numbers, and octadecagonal numbers.

2. Preliminary Results

Let $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$ for each integer n. Then we have [1, p. 48, Entry 31]

$$f(a,b) = f(U_1,V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right).$$

Replacing a by q^a and b by q^b, we find that

$$\begin{aligned}
f(q^a,q^b) = \sum_{r=0}^{n-1} & q^{\left(\frac{a+b}{2}\right)r^2+\left(\frac{a-b}{2}\right)r}\\
&\times f\left(q^{\left(\frac{a+b}{2}\right)n^2+(a+b)nr+\left(\frac{a-b}{2}\right)n}, q^{\left(\frac{a+b}{2}\right)n^2-(a+b)nr-\left(\frac{a-b}{2}\right)n}\right).
\end{aligned} \tag{4}$$

Setting $a=b=1$ and then letting $n=3,5$ and 8 in (4), we obtain

$$\varphi(q) = \varphi(q^9) + 2qG_8(q^3), \tag{5}$$

$$\varphi(q) = \varphi(q^{25}) + 2qA(q^5) + 2q^4G_{12}(q^5), \tag{6}$$

and

$$\varphi(q) = \varphi(q^{64}) + 2qB(q^{16}) + 2q^4\psi(q^{32}) + 2q^9G_{10}(q^{16}) + 2q^{16}\psi(q^{128}), \tag{7}$$

respectively, where $A(q) = f(q^3, q^7)$ and $B(q) = f(q^3, q^5)$.

Setting $a = 1$, $b = 3$ and then putting $n = 2, 4$ and 6 in (4), we deduce that

$$\psi(q) = B(q^2) + qG_{10}(q^2), \tag{8}$$

$$\psi(q) = f(q^{28}, q^{36}) + qf(q^{20}, q^{44}) + q^3f(q^{12}, q^{52}) + q^6G_{18}(q^4), \tag{9}$$

and

$$\psi(q) = f(q^{66}, q^{78}) + qB(q^{18}) + q^3f(q^{42}, q^{102}) + q^6f(q^{30}, q^{114}) + q^{10}G_{10}(q^{18}) + q^{15}G_{26}(q^6), \tag{10}$$

respectively.

Setting $a = 1$, $b = 0$ and then choosing $n = 3$ and 5 in (4) and noting that $\psi(q) = \frac{1}{2}f(1, q)$, we obtain

$$\psi(q) = G_5(q^3) + q\psi(q^9) \tag{11}$$

and

$$\psi(q) = C(q^5) + qG_7(q^5) + q^3\psi(q^{25}), \tag{12}$$

respectively, where $C(q) = f(q^2, q^3)$.

Next, setting $a = 1$, $b = 2$ and $n = 3$ in (4), we find that

$$G_5(q) = f(q^{12}, q^{15}) + qf(q^6, q^{21}) + q^2G_{11}(q^3). \tag{13}$$

Again, setting $a = 1$, $b = 9$ and $n = 2$ in (4), we obtain

$$G_{12}(q) = A(q^4) + qG_7(q^8). \tag{14}$$

We also require a few identities deduced in [5] and [6]. Throughout the sequel, $r\{lF_i + mF_j\}(n)$ denotes the number of representations of n as a sum of l times a polygonal number F_i and m times a polygonal number F_j. Note that $r\{2\Box + \triangle\}(n)$ that appears in (16) is $r\{2F_4 + F_6\}(n)$. However, we have kept the former notation in those cases which involve squares and/or triangular numbers. The first seven of the following identities appeared in [5] as equations (1.1), (1.3), (1.4), (1.5), (1.11), (1.12), and (1.14), respectively, while the last six identities appeared in [6] as equations (1.2), (1.3), (1.4), (1.6), (1.13), and (1.14), respectively.

$$r\{\triangle + \triangle\}(n) = d_{1,4}(4n+1) - d_{3,4}(4n+1), \tag{15}$$
$$r\{2\square + \triangle\}(n) = d_{1,4}(8n+1) - d_{3,4}(8n+1), \tag{16}$$
$$r\{\triangle + 4\triangle\}(n) = \frac{1}{2}(d_{1,4}(8n+5) - d_{3,4}(8n+5)), \tag{17}$$
$$r\{\triangle + 2\triangle\}(n) = \frac{1}{2}(d_{1,8}(8n+3) + d_{3,8}(8n+3) - d_{5,8}(8n+3) - d_{7,8}(8n+3)), \tag{18}$$
$$r\{6\square + \triangle\}(n) = d_{1,3}(8n+1) - d_{2,3}(8n+1), \tag{19}$$
$$r\{\triangle + 12\triangle\}(n) = \frac{1}{2}(d_{1,3}(8n+13) - d_{2,3}(8n+13)), \tag{20}$$
$$r\{3\triangle + 4\triangle\}(n) = \frac{1}{2}(d_{1,3}(8n+7) - d_{2,3}(8n+7)), \tag{21}$$
$$\begin{aligned} r\{\triangle + 4F_5\}(n) = d_{1,24}(24n+7) + d_{19,24}(24n+7) - d_{5,24}(24n+7) \\ - d_{23,24}(24n+7), \end{aligned} \tag{22}$$
$$r\{3\triangle + F_5\}(n) = d_{1,12}(12n+5) - d_{11,12}(12n+5), \tag{23}$$
$$r\{3\triangle + 2F_5\}(n) = d_{1,8}(24n+11) - d_{7,8}(24n+11), \tag{24}$$
$$r\{6\triangle + F_5\}(n) = d_{1,8}(24n+19) - d_{7,8}(24n+19), \tag{25}$$
$$\begin{aligned} r\{3\square + F_5\}(n) = d_{1,8}(24n+1) + d_{3,8}(24n+1) - d_{5,8}(24n+1) \\ - d_{7,8}(24n+1), \end{aligned} \tag{26}$$
$$r\{3\square + 4F_5\}(n) = d_{1,8}(6n+1) + d_{3,8}(6n+1) - d_{5,8}(6n+1) - d_{7,8}(6n+1). \tag{27}$$

3. Identities Involving Decagonal Numbers

Theorem 3.1. *We have*

$$r\{\square + 3F_{10}\}(n) = d_{1,3}(16n+27) - d_{2,3}(16n+27), \tag{28}$$
$$r\{2\triangle + 3F_{10}\}(n) = \frac{1}{2}(d_{1,3}(16n+31) - d_{2,3}(16n+31)), \tag{29}$$
$$r\{2\triangle + F_{10}\}(n) = \frac{1}{2}(d_{1,4}(16n+13) - d_{3,4}(16n+13)), \tag{30}$$
$$r\{\square + F_{10}\}(n) = d_{1,4}(16n+9) - d_{3,4}(16n+9), \tag{31}$$
$$r\{6\triangle + F_{10}\}(n) = \frac{1}{2}(d_{1,3}(16n+21) - d_{2,3}(16n+21)), \tag{32}$$

$$r\{3\square + F_{10}\}(n) = d_{1,3}(16n+9) - d_{2,3}(16n+9), \tag{33}$$

$$r\{F_8 + F_{10}\}(n) = \frac{1}{2}(d_{1,3}(48n+43) - d_{2,3}(48n+43)), \tag{34}$$

$$r\{F_5 + 3F_{10}\}(n) = d_{1,8}(48n+83) - d_{7,8}(48n+83), \tag{35}$$

$$\begin{aligned} r\{2F_5 + F_{10}\}(n) &= d_{1,24}(48n+31) + d_{19,24}(48n+31) \\ &\quad - d_{5,24}(48n+31) - d_{23,24}(48n+31), \end{aligned} \tag{36}$$

$$\begin{aligned} r\{\triangle + F_{10}\}(n) &= \frac{1}{2}(d_{1,8}(16n+11) + d_{3,8}(16n+11) \\ &\quad - d_{5,8}(16n+11) - d_{7,8}(16n+11)). \end{aligned} \tag{37}$$

Proof. Identity (19) is equivalent to

$$\varphi(q^6)\psi(q) = \sum_{n\geq 0}(d_{1,3}(8n+1) - d_{2,3}(8n+1))q^n. \tag{38}$$

Employing (10) in (38), we have

$$\begin{aligned} &\varphi(q^6)(f(q^{66},q^{78}) + qB(q^{18}) + q^3 f(q^{42},q^{102}) + q^6 f(q^{30},q^{114}) \\ &\quad + q^{10}G_{10}(q^{18}) + q^{15}G_{26}(q^6)) = \sum_{n\geq 0}(d_{1,3}(8n+1) - d_{2,3}(8n+1))q^n. \end{aligned} \tag{39}$$

Extracting the terms involving q^{6n+4} in (39) and then dividing the resulting identity by q^4 and replacing q^6 by q, we find that

$$q\varphi(q)G_{10}(q^3) = \sum_{n\geq 0}(d_{1,3}(48n+33) - d_{2,3}(48n+33))q^n. \tag{40}$$

Equating the coefficients of q^{n+1} on both sides of (40) and noting that $d_{1,3}(48n+33) = d_{1,3}(16n+11)$ and $d_{2,3}(48n+33) = d_{2,3}(16n+11)$, we arrive at (28).

Next, (20) is equivalent to

$$\psi(q)\psi(q^{12}) = \frac{1}{2}\sum_{n\geq 0}(d_{1,3}(8n+13) - d_{2,3}(8n+13))q^n,$$

which, with the aid of (10), can be rewritten as

$$\begin{aligned} &\psi(q^{12})\big(f(q^{66},q^{78}) + qB(q^{18}) + q^3 f(q^{42},q^{102}) + q^6 f(q^{30},q^{114}) \\ &\qquad + q^{10}G_{10}(q^{18}) + q^{15}G_{26}(q^6)\big) \\ &= \frac{1}{2}\sum_{n\geq 0}(d_{1,3}(8n+13) - d_{2,3}(8n+13))q^n. \end{aligned} \tag{41}$$

Collecting the terms in (41) in which the power of q is congruent to 4 modulo 6, we find that

$$q\psi(q^2)G_{10}(q^3) = \frac{1}{2}\sum_{n\geq 0}(d_{1,3}(48n+45) - d_{2,3}(48n+45))q^n. \tag{42}$$

Equating the coefficients of q^{n+1} on both sides of (42) and noting that $d_{1,3}(48n+45) = d_{1,3}(16n+15)$ and $d_{2,3}(48n+45) = d_{2,3}(16n+15)$, we arrive at (29).

Identity (1) is equivalent to

$$\varphi^2(q) = 1 + 4\sum_{n\geq 1}(d_{1,4}(n) - d_{3,4}(n))q^n, \tag{43}$$

which can be rewritten, with the aid of (7), as

$$\begin{aligned}&(\varphi(q^{64}) + 2qB(q^{16}) + 2q^4\psi(q^{32}) + 2q^9G_{10}(q^{16}) + 2q^{16}\psi(q^{128}))^2\\ &= 1 + 4\sum_{n\geq 1}(d_{1,4}(n) - d_{3,4}(n))q^n.\end{aligned} \tag{44}$$

Now, we extract those terms in (44) where the power of q is congruent to 13 modulo 16, divide the resulting identity by q^{13} and replace q^{16} by q, to obtain

$$\psi(q^2)G_{10}(q) = \frac{1}{2}\sum_{n\geq 0}(d_{1,4}(16n+13) - d_{3,4}(16n+13))q^n,$$

which readily yields (30).

Next, extracting those terms in (44) where the power of q is congruent to 9 modulo 16, then dividing the resulting identity by q^9 and replacing q^{16} by q, we have

$$G_{10}(q)(\varphi(q^4) + 2q\psi(q^8)) = \sum_{n\geq 0}(d_{1,4}(16n+9) - d_{3,4}(16n+9))q^n. \tag{45}$$

But, setting $a = b = 1$ and $n = 2$ in (4), or from [1, p. 40, Entries 25(i) and 25(ii)], we have

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \tag{46}$$

Employing (46) in (45), we find that

$$\varphi(q)G_{10}(q) = \sum_{n\geq 0}(d_{1,4}(16n+9) - d_{3,4}(16n+9))q^n,$$

which implies (31).

Now, (2) is equivalent to

$$\begin{aligned}\varphi(q)\varphi(q^3) &= 1 + 2\sum_{n\geq 1}(d_{1,3}(n) - d_{2,3}(n))q^n + 4\sum_{n\geq 1}(d_{4,12}(n) - d_{8,12}(n))q^n\\ &= 1 + 2\sum_{n\geq 1}(d_{1,3}(n) - d_{2,3}(n))q^n + 4\sum_{n\geq 1}(d_{1,3}(n) - d_{2,3}(n))q^{4n}.\end{aligned} \tag{47}$$

Employing (7) in (47), we have

$$\begin{aligned}&(\varphi(q^{64})+2qB(q^{16})+2q^4\psi(q^{32})+2q^9G_{10}(q^{16})+2q^{16}\psi(q^{128}))\\ &\times(\varphi(q^{192})+2q^3B(q^{48})+2q^{12}\psi(q^{96})+2q^{27}G_{10}(q^{48})+2q^{48}\psi(q^{384}))\\ &=1+2\sum_{n\geq 1}(d_{1,3}(n)-d_{2,3}(n))q^n+4\sum_{n\geq 1}(d_{1,3}(n)-d_{2,3}(n))q^{4n}. \qquad (48)\end{aligned}$$

Extracting the terms in (48) involving q^{16n+5}, then dividing the resulting identity by q^5 and replacing q^{16} by q, we find that

$$q\psi(q^6)G_{10}(q)=\frac{1}{2}\sum_{n\geq 0}(d_{1,3}(16n+5)-d_{2,3}(16n+5))q^n,$$

from which (32) can be easily deduced.

Again, using (5) in (40), we have

$$q(\varphi(q^9)+2qG_8(q^3))G_{10}(q^3)=\sum_{n\geq 0}(d_{1,3}(16n+11)-d_{2,3}(16n+11))q^n. \qquad (49)$$

Separating the terms involving q^{3n+1} and q^{3n+2} in (49), we obtain

$$\varphi(q^3)G_{10}(q)=\sum_{n\geq 0}(d_{1,3}(48n+27)-d_{2,3}(48n+27))q^n \qquad (50)$$

and

$$2G_8(q)G_{10}(q)=\sum_{n\geq 0}(d_{1,3}(48n+43)-d_{2,3}(48n+43))q^n, \qquad (51)$$

respectively. Now the identities (33) and (34) follow easily from (50) and (51), respectively.

Next, (24) is equivalent to

$$\psi(q^3)G_5(q^2)=\sum_{n\geq 0}(d_{1,8}(24n+11)-d_{7,8}(24n+11))q^n. \qquad (52)$$

Invoking (8) in (52), we have

$$(B(q^6)+q^3G_{10}(q^6))G_5(q^2)=\sum_{n\geq 0}(d_{1,8}(24n+11)-d_{7,8}(24n+11))q^n. \qquad (53)$$

Extracting the terms involving q^{2n+1} in (53), we obtain

$$qG_{10}(q^3)G_5(q)=\sum_{n\geq 0}(d_{1,8}(48n+35)-d_{7,8}(48n+35))q^n. \qquad (54)$$

Comparing the coefficients of q^{n+1} on both sides of (54), we arrive at (35).

Identity (22) is equivalent to

$$\psi(q)G_5(q^4) = \sum_{n\geq 0}(d_{1,24}(24n+7)+d_{19,24}(24n+7)-d_{5,24}(24n+7)-d_{23,24}(24n+7))q^n. \tag{55}$$

Using (8) in (55), we have

$$(B(q^2)+qG_{10}(q^2))G_5(q^4) = \sum_{n\geq 0}(d_{1,24}(24n+7)+d_{19,24}(24n+7) - d_{5,24}(24n+7)-d_{23,24}(24n+7))q^n. \tag{56}$$

Extracting the terms involving odd powers of q in (56), we obtain

$$G_{10}(q)G_5(q^2) = \sum_{n\geq 0}(d_{1,24}(48n+31)+d_{19,24}(48n+31)-d_{5,24}(48n+31)-d_{23,24}(48n+31))q^n,$$

which readily yields (36).

Identity (18) is equivalent to

$$\psi(q)\psi(q^2) = \frac{1}{2}\sum_{n\geq 0}(d_{1,8}(8n+3)+d_{3,8}(8n+3)-d_{5,8}(8n+3)-d_{7,8}(8n+3))q^n,$$

which, with the aid of (8), can be written as

$$(B(q^2)+qG_{10}(q^2))\psi(q^2) = \frac{1}{2}\sum_{n\geq 0}(d_{1,8}(8n+3)+d_{3,8}(8n+3)-d_{5,8}(8n+3)-d_{7,8}(8n+3))q^n. \tag{57}$$

Extracting the terms involving q^{2n+1} in (57), we obtain

$$G_{10}(q)\psi(q) = \frac{1}{2}\sum_{n\geq 0}(d_{1,8}(16n+11)+d_{3,8}(16n+11)-d_{5,8}(16n+11)-d_{7,8}(16n+11))q^n. \tag{58}$$

Equating the coefficients of q^n on both sides of (58), we arrive at (37). □

4. Identities Involving Hendecagonal Numbers

Theorem 4.1. *We have*

$$r\{\triangle + F_{11}\}(n) = d_{1,12}(36n+29) - d_{11,12}(36n+29), \tag{59}$$

$$r\{\triangle + 2F_{11}\}(n) = d_{1,8}(72n+107) - d_{7,8}(72n+107), \tag{60}$$

$$r\{2\triangle + F_{11}\}(n) = d_{1,8}(72n+67) - d_{7,8}(72n+67), \tag{61}$$

$$r\{\square + F_{11}\}(n) = d_{1,8}(72n+49) + d_{3,8}(72n+49) - d_{5,8}(72n+49) - d_{7,8}(72n+49), \tag{62}$$

$$r\{\square + 4F_{11}\}(n) = d_{1,8}(18n+49) + d_{3,8}(18n+49) - d_{5,8}(18n+49) - d_{7,8}(18n+49), \tag{63}$$

$$r\{F_{10} + F_{11}\}(n) = d_{1,8}(144n+179) - d_{7,8}(144n+179). \tag{64}$$

Proof. Identity (23) is equivalent to

$$\psi(q^3)G_5(q) = \sum_{n\geq 0}(d_{1,12}(12n+5) - d_{11,12}(12n+5))q^n,$$

which we rewrite, by (13), as

$$\psi(q^3)(f(q^{12},q^{15}) + qf(q^6,q^{21}) + q^2G_{11}(q^3)) = \sum_{n\geq 0}(d_{1,12}(12n+5) - d_{11,12}(12n+5))q^n. \tag{65}$$

Extracting the terms involving q^{3n+2} in (65), we obtain

$$\psi(q)G_{11}(q) = \sum_{n\geq 0}(d_{1,12}(36n+29) - d_{11,12}(36n+29))q^n,$$

which readily yields (59).

Next, (24) is equivalent to

$$\psi(q^3)G_5(q^2) = \sum_{n\geq 0}(d_{1,8}(24n+11) - d_{7,8}(24n+11))q^n. \tag{66}$$

Invoking (13) in (66), we find that

$$\psi(q^3)(f(q^{24},q^{30}) + q^2f(q^{12},q^{42}) + q^4G_{11}(q^6)) = \sum_{n\geq 0}(d_{1,8}(24n+11) - d_{7,8}(24n+11))q^n. \tag{67}$$

Extracting the terms involving q^{3n+1} in (67), we obtain

$$q\psi(q)G_{11}(q^2) = \sum_{n\geq 0}(d_{1,8}(72n+35) - d_{7,8}(72n+35))q^n, \tag{68}$$

from which (60) follows.

Again, (25) is equivalent to

$$\psi(q^6)G_5(q) = \sum_{n\geq 0}(d_{1,8}(24n+19) - d_{7,8}(24n+19))q^n. \tag{69}$$

Using (13) in (69), we have

$$\psi(q^6)(f(q^{12},q^{15}) + qf(q^6,q^{21}) + q^2G_{11}(q^3)) = \sum_{n\geq 0}(d_{1,8}(24n+19) - d_{7,8}(24n+19))q^n. \tag{70}$$

Extracting the terms involving q^{3n+2} in (70), we obtain

$$\psi(q^2)G_{11}(q) = \sum_{n\geq 0}(d_{1,8}(72n+67) - d_{7,8}(72n+67))q^n,$$

which gives (61).

Identity (26) is equivalent to

$$\varphi(q^3)G_5(q) = \sum_{n\geq 0}(d_{1,8}(24n+1) + d_{3,8}(24n+1) - d_{5,8}(24n+1) - d_{7,8}(24n+1))q^n,$$

and by (13), we have

$$\varphi(q^3)(f(q^{12},q^{15}) + qf(q^6,q^{21}) + q^2G_{11}(q^3)) = \sum_{n\geq 0}(d_{1,8}(24n+1) + d_{3,8}(24n+1) - d_{5,8}(24n+1) - d_{7,8}(24n+1))q^n. \tag{71}$$

Extracting the terms involving q^{3n+2} in (71), we obtain

$$\varphi(q)G_{11}(q) = \sum_{n\geq 0}(d_{1,8}(72n+49) + d_{3,8}(72n+49) - d_{5,8}(72n+49) - d_{7,8}(72n+49))q^n,$$

which readily yields (62).

Identity (27) is equivalent to

$$\varphi(q^3)G_5(q^4) = \sum_{n\geq 0}(d_{1,8}(6n+1) + d_{3,8}(6n+1) - d_{5,8}(6n+1) - d_{7,8}(6n+1))q^n. \tag{72}$$

Using (13) in (72), we have

$$\begin{aligned}&\varphi(q^3)(f(q^{48},q^{60})+q^4f(q^{24},q^{84})+q^8G_{11}(q^{12}))\\&\quad=\sum_{n\geq 0}(d_{1,8}(6n+1)+d_{3,8}(6n+1)-d_{5,8}(6n+1)-d_{7,8}(6n+1))q^n.\end{aligned} \tag{73}$$

Extracting the terms involving q^{3n+2} in (73), we find that

$$\begin{aligned}q^2\varphi(q)G_{11}(q^4)=&\sum_{n\geq 0}(d_{1,8}(18n+13)+d_{3,8}(18n+13)\\&-d_{5,8}(18n+13)-d_{7,8}(18n+13))q^n,\end{aligned}$$

which readily yields (63).

Again, employing (8) in (68), we obtain

$$q(B(q^2)+qG_{10}(q^2))G_{11}(q^2)=\sum_{n\geq 0}(d_{1,8}(72n+35)-d_{7,8}(72n+35))q^n. \tag{74}$$

Comparing the terms in (74) where the powers of q are even, we find that

$$qG_{10}(q)G_{11}(q)=\sum_{n\geq 0}(d_{1,8}(144n+35)-d_{7,8}(144n+35))q^n. \tag{75}$$

Equating the coefficients of q^{n+1} in (75), we arrive at (64). □

5. Identities Involving Dodecagonal Numbers

Theorem 5.1. *We have*

$$r\{5\square+F_{12}\}(n)=d_{1,4}(5n+4)-d_{3,4}(5n+4), \tag{76}$$
$$r\{F_{12}+F_{12}\}(n)=d_{1,4}(5n+8)-d_{3,4}(5n+8), \tag{77}$$
$$r\{5\triangle+F_{12}\}(n)=\frac{1}{2}(d_{1,4}(20n+17)-d_{3,4}(20n+17)). \tag{78}$$

Proof. Employing (6) in (43), we find that

$$(\varphi(q^{25})+2qA(q^5)+2q^4G_{12}(q^5))^2=1+4\sum_{n\geq 1}(d_{1,4}(n)-d_{3,4}(n))q^n. \tag{79}$$

Extracting those terms in (79) in which the power of q is congruent to 4 modulo 5, we obtain

$$\varphi(q^5)G_{12}(q)=\sum_{n\geq 0}(d_{1,4}(5n+4)-d_{3,4}(5n+4))q^n,$$

from which (76) follows.

Again, extracting the terms involving q^{5n+3} in (79), we have

$$qG_{12}^2(q) = \sum_{n\geq 0}(d_{1,4}(5n+3) - d_{3,4}(5n+3))q^n, \tag{80}$$

which immediately gives (77).

Furthermore, extracting the terms involving q^{5n+2} in (79), we find that

$$A^2(q) = \sum_{n\geq 0}(d_{1,4}(5n+2) - d_{3,4}(5n+2))q^n. \tag{81}$$

But, from [1, p. 46, Entries 30(v) and 30(vi)], we have

$$A^2(q) = f^2(q^3, q^7) = A(q^2)\varphi(q^{10}) + 2q^3G_{12}(q^4)\psi(q^{20}). \tag{82}$$

From (81) and (82), we obtain

$$A(q^2)\varphi(q^{10}) + 2q^3G_{12}(q^4)\psi(q^{20}) = \sum_{n\geq 0}(d_{1,4}(5n+2) - d_{3,4}(5n+2))q^n. \tag{83}$$

Collecting the terms involving q^{4n+3} in (83), we find that

$$2G_{12}(q)\psi(q^5) = \sum_{n\geq 0}(d_{1,4}(20n+17) - d_{3,4}(20n+17))q^n,$$

which readily yields (78). □

6. Identities Involving Heptagonal Numbers

Theorem 6.1. *We have*

$$r\{F_7 + F_7\}(n) = d_{1,4}(20n+9) - d_{3,4}(20n+9), \tag{84}$$

$$r\{5\triangle + F_7\}(n) = \frac{1}{2}(d_{1,4}(20n+17) - d_{3,4}(20n+17)), \tag{85}$$

$$r\{2F_{12} + F_7\}(n) = \frac{1}{2}(d_{1,4}(40n+73) - d_{3,4}(40n+73)). \tag{86}$$

Proof. With the aid of (14), we rewrite (80) as

$$q(A(q^4) + qG_7(q^8))^2 = \sum_{n\geq 0}(d_{1,4}(5n+3) - d_{3,4}(5n+3))q^n. \tag{87}$$

Extracting the terms involving q^{8n+3} in (87), we find that

$$G_7^2(q) = \sum_{n\geq 0}(d_{1,4}(40n+18) - d_{3,4}(40n+18))q^n. \tag{88}$$

Equating the coefficients of q^n in (88) and noting the fact that $d_{1,4}(40n+18) = d_{1,4}(20n+9)$ and $d_{3,4}(40n+18) = d_{3,4}(20n+9)$, we arrive at (84).

Next, (15) is equivalent to

$$\psi^2(q) = \sum_{n\geq 0}(d_{1,4}(4n+1) - d_{3,4}(4n+1))q^n. \tag{89}$$

Invoking (12) in (89), we obtain

$$(C(q^5) + qG_7(q^5) + q^3\psi(q^{25}))^2 = \sum_{n\geq 0}(d_{1,4}(4n+1) - d_{3,4}(4n+1))q^n. \tag{90}$$

Extracting the terms involving q^{5n+4} in (90), we get

$$2G_7(q)\psi(q^5) = \sum_{n\geq 0}(d_{1,4}(20n+17) - d_{3,4}(20n+17))q^n. \tag{91}$$

Equating the coefficients of q^n in (91), we easily arrive at (85).

Next, (16) is equivalent to

$$\varphi(q^2)\psi(q) = \sum_{n\geq 0}(d_{1,4}(8n+1) - d_{3,4}(8n+1))q^n. \tag{92}$$

Using (6) and (12) in (92), we find that

$$\begin{aligned}&(\varphi(q^{50}) + 2q^2A(q^{10}) + 2q^8G_{12}(q^{10}))(C(q^5) + qG_7(q^5) + q^3\psi(q^{25}))\\ &= \sum_{n\geq 0}(d_{1,4}(8n+1) - d_{3,4}(8n+1))q^n.\end{aligned} \tag{93}$$

Extracting the terms involving q^{5n+4} in (93), we obtain

$$2qG_{12}(q^2)G_7(q) = \sum_{n\geq 0}(d_{1,4}(40n+33) - d_{3,4}(40n+33))q^n,$$

from which (86) can be deduced by equating the coefficients of q^{n+1}. □

7. Identities Involving Octadecagonal Numbers

Theorem 7.1. *We have*

$$\begin{aligned}r\{F_5 + F_{18}\}(n) = d_{1,24}(96n+151) + d_{19,24}(96n+151)&\\ - d_{5,24}(96n+151) - d_{23,24}(96n+151),&\end{aligned} \tag{94}$$

$$r\{\triangle + F_{18}\}(n) = \frac{1}{2}(d_{1,4}(32n+53) - d_{3,4}(32n+53)), \tag{95}$$

$$r\{3\triangle + F_{18}\}(n) = \frac{1}{2}(d_{1,3}(32n+61) - d_{2,3}(32n+61)). \tag{96}$$

Proof. Identity (22) is equivalent to

$$\psi(q)G_5(q^4) = \sum_{n\geq 0}(d_{1,24}(24n+7)+d_{19,24}(24n+7)-d_{5,24}(24n+7)-d_{23,24}(24n+7))q^n. \tag{97}$$

Employing (9) in (97), we have

$$(f(q^{28},q^{36})+qf(q^{20},q^{44})+q^3f(q^{12},q^{52})+q^6G_{18}(q^4))G_5(q^4) = \sum_{n\geq 0}(d_{1,24}(24n+7)+d_{19,24}(24n+7)-d_{5,24}(24n+7)-d_{23,24}(24n+7))q^n. \tag{98}$$

Extracting those terms in (98) in which the power of q is congruent to 2 modulo 4, we obtain

$$qG_{18}(q))G_5(q) = \sum_{n\geq 0}(d_{1,24}(96n+55)+d_{19,24}(96n+55)-d_{5,24}(96n+55)-d_{23,24}(96n+55))q^n,$$

which readily implies (94).

Again, (17) is equivalent to

$$\psi(q)\psi(q^4) = \frac{1}{2}\sum_{n\geq 0}(d_{1,4}(8n+5)-d_{3,4}(8n+5))q^n. \tag{99}$$

Using (9) in (99), we have

$$(f(q^{28},q^{36})+qf(q^{20},q^{44})+q^3f(q^{12},q^{52})+q^6G_{18}(q^4))\psi(q^4) = \frac{1}{2}\sum_{n\geq 0}(d_{1,4}(8n+5)-d_{3,4}(8n+5))q^n. \tag{100}$$

Extracting the terms involving q^{4n+2} from both sides of the above, we obtain

$$qG_{18}(q)\psi(q) = \frac{1}{2}\sum_{n\geq 0}(d_{1,4}(32n+21)-d_{3,4}(32n+21))q^n,$$

which readily implies (95).

Next, (21) is equivalent to

$$\psi(q^3)\psi(q^4) = \frac{1}{2}\sum_{n\geq 0}(d_{1,3}(8n+7)-d_{2,3}(8n+7))q^n. \tag{101}$$

With the help of (9) and (11), we rewrite (101) as

$$(f(q^{84},q^{108})+q^3f(q^{60},q^{132})+q^9f(q^{36},q^{156})+q^{18}G_{18}(q^{12}))(G_5(q^{12})+q^4\psi(q^{36}))$$

$$=\frac{1}{2}\sum_{n\geq 0}(d_{1,3}(8n+7)-d_{2,3}(8n+7))q^n. \qquad (102)$$

Extracting the terms involving q^{12n+10} in (102), we obtain

$$qG_{18}(q)\psi(q^2)=\frac{1}{2}\sum_{n\geq 0}(d_{1,3}(96n+87)-d_{2,3}(96n+87))q^n.$$

Equating the coefficients of q^{n+1} and noting that $d_{1,3}(96n+87)=d_{1,3}(32n+29)$ and $d_{2,3}(96n+87)=d_{2,3}(32n+29)$, we deduce (96) to finish the proof. □

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References

[1] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[2] S. Bhargava, C. Adiga, *Simple proofs of Jacobi's two and four square theorems*, Int. J. Math. Educ. Sci. Technol. 19(1988), 779–782.

[3] P. G. L. Dirichlet, J. Math. 21 (1840) 3, 6; Werke 463, 466.

[4] M. D. Hirschhorn, *A simple proof of Jacobi's two-square theorem*, Amer. Math. Monthly, 92(1985), 579–580.

[5] M. D. Hirschhorn, *The number of representation of a number by various forms*, Discrete Math. 298 (2005) 205–211.

[6] M. D. Hirschhorn, *The number of representation of a number by various forms involving triangles, squares, pentagons and octagons,* in: Ramanujan Rediscovered, N.D. Baruah, B.C. Berndt, S. Cooper, T. Huber, M. Schlosser (eds.), RMS Lecture Note Series, No. 14, Ramanujan Mathematical Society, 2010, 113–124.

[7] C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum* (1829) 107; Werke I 162–163; Letter to Legendre 9/9/1828, Werke I 424.

[8] H. Y. Lam, *The number of representations by sums of squares and triangular numbers*, Integers 7(2007), A28.

[9] A. M. Legendre, *Traité des fonctions elliptiques et des intégrales Euleriennes*, t. III, Huzard-Courcier, Paris, 1828, 133–134.

[10] L. Lorenz, Tidsskrift for Mathematik 3 (1) (1871), 106–108.

[11] R. S. Melham, *Analogues of two classical theorems on the representation of a number*, Integers 8(2008), A51.

[12] R. S. Melham, *Analogues of Jacobi's two-square theorem: An informal account*, Integers 10(2010), 83–100, A8.