TRIANGULAR NUMBERS IN THE JACOBSTHAL FAMILY

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Abstract
Using congruences, second-order diophantine equations, and linear algebra, we identify Jacobsthal and Jacobsthal-Lucas numbers that are also triangular numbers.

1. Introduction

Like the well-known Fibonacci and Lucas numbers [4], Jacobsthal numbers J_n and Jacobsthal-Lucas numbers j_n provide gratifying opportunities for both experimentation and exploration. They satisfy the same Jacobsthal recurrence and are often defined recursively [3, 8]:

$$\begin{aligned} J_1 &= 1, \;\; J_2 = 1 & \qquad j_1 &= 1, \;\; j_2 = 5 \\ J_n &= J_{n-1} + 2J_{n-2}, \;\; n \geq 3; & \qquad j_n &= j_{n-1} + 2j_{n-2}, \;\; n \geq 3. \end{aligned}$$

The two definitions differ only in the second initial conditions, as in the case of Fibonacci and Lucas numbers.

Table 1 gives the first twelve members of each family, and the first twelve triangular numbers $t_n = \frac{n(n+1)}{2}$.

n	1	2	3	4	5	6	7	8	9	10	11	12
J_n	1	1	$\boxed{3}$	5	11	21	43	85	171	341	683	1365
j_n	1	5	7	17	31	65	127	257	511	1025	2047	4097
t_n	1	$\boxed{3}$	6	10	15	21	28	36	45	55	66	78

Table 1

It follows from the above recursive definitions that J_n and j_n satisfy the *Binet-like* formulas $J_n = \dfrac{2^n - (-1)^n}{3}$ and $j_n = 2^n + (-1)^n$, respectively [3, 8]. When n is even, $J_n = \frac{M_n}{3}$, where M_n denotes the nth Mersenne number $2^n - 1$ and $n \geq 1$; and when n is odd, $j_n = M_n$ [5].

2. Quadratic Diophantine Equation $u^2 - Nv^2 = C$

Our goal is to identify all Jacobsthal and Jacobsthal-Lucas numbers that are also triangular. This task hinges on solving the quadratic diophantine equation (QDE) $u^2 - Nv^2 = C$, where N is a nonsquare positive integer and C a positive integer. The solutions of the QDE are closely related to those of Pell's equation $u^2 - Nv^2 = 1$. So we will first give a very brief introduction to solving the QDE [2, 6, 7]. In the interest of brevity, we will confine our discussion to solutions (u, v) with $u > 0$.

Let (α, β) be the fundamental solution of Pell's equation $u^2 - Nv^2 = 1$, and (u_0, v_0) a solution of the QDE. Let $u_m + v_m\sqrt{N} = (u_0 + v_0\sqrt{N})(\alpha + \beta\sqrt{N})^m$, where m is a positive integer. Then (u_m, v_m) is a solution of the QDE. We then say that the solution (u_m, v_m) is *associated* with the solution (u_0, v_0). Such solutions belong to a class of solutions of the QDE. Suppose (u_0, v_0) has the property that it has the least possible positive value of u among the solutions in the class; then (u_0, v_0) is the *fundamental solution* of the class.

The QDE can have different classes of solutions. Although each class is infinite, the number of distinct classes is finite. Two solutions (u, v) and (u', v') belong to the same class if and only if $uu' \equiv Nvv' \pmod{C}$ and $uv' \equiv u'v \pmod{C}$ [6].

The following theorem provides a mechanism for finding the solutions of the QDE, when it is solvable.

Theorem. Let (α, β) be the fundamental solution of Pell's equation $u^2 - Nv^2 = 1$, (u_0, v_0) a fundamental solution of the QDE $u^2 - Nv^2 = C$, and m a positive integer. Then:

i) $0 < u_0 \le \sqrt{\dfrac{(\alpha+1)C}{2}}$ and $0 < |v_0| \le \beta\sqrt{\dfrac{C}{2(\alpha+1)}}$.

(These two inequalities provide computable upper bounds for u_0 and v_0. The number of solutions (u_0, v_0) resulting from these inequalities determines the number of different classes of solutions.)

ii) Every solution (u_m, v_m) belonging to the class of (u_0, v_0) is given by

$$u_m + v_m\sqrt{N} = (u_0 + v_0\sqrt{N})(\alpha + \beta\sqrt{N})^m. \tag{1}$$

iii) The QDE is *not* solvable if it has no solution satisfying the inequalities in (i).

3. Recurrence for (u_m, v_m)

Equation (1) can be employed to derive a recurrence for (u_m, v_m):

$$\begin{aligned} u_{m+1} + v_{m+1}\sqrt{N} &= (u_0 + v_0\sqrt{N})(\alpha + \beta\sqrt{N})^{m+1} \\ &= (u_m + v_m\sqrt{N})(\alpha + \beta\sqrt{N}) \\ &= (\alpha u_m + N\beta v_m) + (\beta u_m + \alpha v_m)\sqrt{N}. \end{aligned}$$

Thus we have the following recurrence for (u_m, v_m):

$$\begin{aligned} u_{m+1} &= \alpha u_m + N\beta v_m \\ v_{m+1} &= \beta u_m + \alpha v_m. \end{aligned} \tag{2}$$

These recurrences can be used to develop a second-order recurrence for both u_m and v_m.

4. A Second-Order Recurrence for (u_m, v_m)

These recurrences can be combined into a matrix equation:

$$\begin{bmatrix} u_{m+1} \\ v_{m+1} \end{bmatrix} = M \begin{bmatrix} u_m \\ v_m \end{bmatrix},$$

where $M = \begin{bmatrix} \alpha & N\beta \\ \beta & \alpha \end{bmatrix}$.

By the well known Cayley-Hamilton Theorem [1], M satisfies its characteristic equation $|M - \lambda I| = 0$, where I denotes the 2×2 identity matrix; that is, $\lambda^2 - 2\alpha\lambda + 1 = 0$. So $M^2 = 2\alpha M - I$.

Consequently, we have:

$$\begin{bmatrix} u_{m+2} \\ v_{m+2} \end{bmatrix} = M^2 \begin{bmatrix} u_m \\ v_m \end{bmatrix} = (2\alpha M - I) \begin{bmatrix} u_m \\ v_m \end{bmatrix} = 2\alpha \begin{bmatrix} u_{m+1} \\ v_{m+1} \end{bmatrix} - \begin{bmatrix} u_m \\ v_m \end{bmatrix}.$$

Thus both u_m and v_m satisfiy the recurrence

$$r_{m+2} = 2\alpha r_{m+1} - r_m, \tag{3}$$

where $m \geq 0$.

With these facts at our finger tips, we are now ready to identify all triangular Jacobsthal numbers.

5. Triangular Jacobsthal Numbers

Clearly, $J_1 = t_1$; $J_2 = t_1$; $J_3 = t_2, J_6 = t_6$, and $J_9 = t_{18}$; see Table 1. So there are at least five triangular Jacobsthal numbers.

Next, we will show that there are *no* other triangular Jacobsthal numbers. To this end, we will employ the fact that $8t_k + 1 = (2k + 1)^2$, discovered around 250 A.D. by Diophantus of Alexandria, Egypt. Consequently, J_n is a triangular number if and only if $8J_n + 1$ is the square of an odd integer [5].

`Case 1.` Suppose J_{2n} is a triangular number, where $n \geq 5$. Then $8J_{2n} + 1 = y^2$ for some odd positive integer y. This yields:

$$\begin{aligned} 8 \cdot \frac{2^{2n} - 1}{3} + 1 &= y^2 \\ 2w^2 - 3y^2 - 5 &= 0 \qquad (4) \\ x^2 - 6y^2 &= 10, \qquad (5) \end{aligned}$$

where $w = 2^{n+1}$ and $x = 2w \geq 128$.

Using the theorem, we can solve the QDE (5). The fundamental solution of Pell's equation $x^2 - 6y^2 = 1$ is $(\alpha, \beta) = (5, 2)$; and $(\pm 4, \pm 1)$ are solutions of the QDE (5). Since $x > 0$, we can safely ignore the solutions $(-4, \pm 1)$. This leaves just two fundamental solutions: $(x_0, y_0) = (4, 1)$ and $(x_0', y_0') = (4, -1)$.

Since $x_0x_0' - 6y_0y_0' = 4 \cdot 4 - 6 \cdot 1 \cdot (-1) \not\equiv 0 \pmod{10}$ and $x_0y_0' - x_0'y_0 = 4 \cdot (-1) - 4 \cdot 1 \not\equiv 0 \pmod{10}$, it follows that the solutions (4,1) and $(4, -1)$ belong to two different classes of solutions of the QDE (5) [6]; each is the fundamental solution of the corresponding class.

`Subcase 1.1.` Consider the case $(x_0, y_0) = (4, 1)$. Since $(\alpha, \beta) = (5, 2)$ and $N = 6$, it follows by equation (2) that every solution (x_m, y_m) of (5) in the class of (4,1) is given by:

$$\begin{aligned} x_{m+1} &= 5x_m + 12y_m \\ y_{m+1} &= 2x_m + 5y_m, \end{aligned}$$

where $(x_0, y_0) = (4, 1)$.

Consequently, $(x_1, y_1) = (32, 13)$ and $(x_2, y_2) = (316, 129)$ are solutions of QDE (5): $32^2 - 6 \cdot 13^2 = 10 = 316^2 - 6 \cdot 129^2$.

Since y_0 and y_1 are odd, and $(\alpha, \beta) = (5, 2)$, it follows by induction that every y_m is odd. Since x_0 is even and $x_{m+1} \equiv x_m \pmod 2$, it also follows that every x_m is even.

6. Recurrence for (w_m, y_m)

Since $x_m = 2w_m$, the above equations yield the following recurrences for w_m and y_m:

$$\begin{aligned} w_{m+1} &= 5w_m + 6y_m \\ y_{m+1} &= 4w_m + 5y_m, \end{aligned}$$

where $(w_0, y_0) = (2, 1)$ and $m \geq 0$.

Since $w_{m+1} \equiv w_m \pmod 2$ and w_0 is even, it follows that every w_m is even; see Table 2.

With $\alpha = 5$, recurrence (3), satisfied by both w_m and y_m, comes in handy when computing the solutions (w_m, y_m) of (4), belonging to the class with the fundamental solution $(2, 1)$. Table 2 shows the first ten such solutions. Since the y-values do not directly impact the problem at hand, we will ignore them.

m	0	1	2	3	4	5	6	7	8	9
w_m	2	16	158	1564	15482	153256	1517078	15017524	148658162	1471564096
y_m	1	13	129	1277	12641	125133	1238689	12261757	121378881	1201527053

Table 2

We will now show that *no* w_m can be a power of 2. To see this, using the recurrence $w_{m+2} = 10w_{m+1} - w_m$, we now compute the values of $\{w_m \pmod{31}\}_{m\geq 0}$ and $\{w_m \pmod{32}\}_{m\geq 0}$. Both sequences are periodic with periods 32 and 16, respectively; see Table 3. But $2^k \pmod{31} = 1, 2, 4, 8$, or 16; and $2^k \pmod{32} = 0$ for $k \geq 5$. It now follows from the table that no w_m satisfies both conditions satisfied by 2^k, where $k \geq 5$; that is, *no* w_m is congruent to 2^k modulo 31 and 32, when $k \geq 5$.

m	0	1	2	3	4	5	6	7	8	9	10	11
$w_m \pmod{31}$	2	16	3	14	13	23	0	8	18	17	28	15
$w_m \pmod{32}$	2	16	30	28	26	8	22	20	18	0	14	12
m	12	13	14	15	16	17	18	19	20	21	22	23
$w_m \pmod{31}$	29	27	24	27	29	15	28	17	18	8	0	23
$w_m \pmod{32}$	10	24	6	4	2	16	30	28	26	8	22	20
m	24	25	26	27	28	29	30	31				
$w_m \pmod{31}$	13	14	3	16	2	4	7	4				
$w_m \pmod{32}$	18	0	14	12	10	24	6	4				

Table 3

Subcase 1.2. Suppose $(x'_0, y'_0) = (4, -1)$. This solution, coupled with $(\alpha, \beta) = (5, 2)$, can be used to generate a different family of solutions (x_m, y_m) of (5) and hence (w_m, y_m). In particular, it follows by recurrence (2) that $(x_1, y_1) = (8, 3)$ and $(x_2, y_2) = (76, 31)$ are also solutions of the QDE (5): $8^2 - 6 \cdot 3^2 = 10 = 76^2 - 6 \cdot 31$.

m	0	1	2	3	4	5	6	7
w_m	2	4	38	376	3722	36844	364718	3610336
y_m	-1	3	31	307	3039	30083	297791	2947827

m	8	9
w_m	35738642	353776084
y_m	29180479	288856963

Table 4

Correspondingly, $w_0 = 2, w_1 = 4$, and $w_2 = 38$. As in Subcase 1.1, here also, w_m satisfies exactly the same recurrence. Table 4 shows the first ten solutions (w_m, y_m). Again, we can safely ignore the y-values.

We will now show that *no* w_m can be a power of 2. To see this, notice that the sequences $\{w_m \pmod{31}\}_{m \ge 0}$ and $\{w_m \pmod{32}\}_{m \ge 0}$ are both periodic with period 32; see Table 5. It follows from the table that *no* w_m is congruent to 2^k modulo 31 and 32 for any integer $k \ge 5$.

m	0	1	2	3	4	5	6	7	8	9	10	11
$w_m \pmod{31}$	2	4	7	4	2	16	3	14	13	23	0	8
$w_m \pmod{32}$	1	2	19	28	5	22	23	16	9	10	27	4

m	12	13	14	15	16	17	18	19	20	21	22	23
$w_m \pmod{31}$	18	17	28	15	29	27	24	27	29	15	28	17
$w_m \pmod{32}$	13	30	31	24	17	18	3	12	21	6	7	0

m	24	25	26	27	28	29	30	31
$w_m \pmod{31}$	18	8	0	23	13	14	3	16
$w_m \pmod{32}$	25	26	11	20	29	14	15	8

Table 5

It follows by Subcases 1 and 2 that no w_m can be a power of 2, when $m \ge 5$. Consequently, *no* J_{2n} is a triangular number when $m \ge 5$.

Case 2. Suppose J_{2n+1} is a triangular number, where $n \ge 5$. Then $8J_{2n+1} + 1 = y^2$ for some positive odd integer y. As before, this yields:

$$\begin{aligned} x^2 - 3y^2 + 11 &= 0 \\ z^2 - 3x^2 &= 33, \end{aligned} \tag{6}$$

where $x = 2^{n+2} \geq 128$, and $z = 3y$ is odd.

The fundamental solution of Pell's equation $z^2 - 3x^2 = 1$ is $(\alpha, \beta) = (2, 1)$. There are two fundamental solutions $(z_0, x_0) = (6, 1)$ and $(z_0', x_0') = (6, -1)$ of the QDE (6) with the least positive value 6 for z. Since $z_0 z_0' - 3x_0 x_0' = 6 \cdot 6 - 3 \cdot 1 \cdot (-1) \not\equiv 0 \pmod{33}$ and $z_0 x_0' - z_0' x_0 = 6 \cdot (-1) - 1 \cdot 6 \not\equiv 0 \pmod{33}$, it follows that the solutions (6,1) and $(6, -1)$ belong to two different classes of solutions of the QDE (6) [6]. Using the theorem, we can now find all solutions of (6).

`Subcase 2.1.` With the fundamental solution $(z_0, x_0) = (6, 1)$, every solution (z_m, x_m) in its class is given by the recurrence:

$$\begin{aligned} z_{m+1} &= 2z_m + 3x_m \\ x_{m+1} &= z_m + 2x_m. \end{aligned}$$

It now follows that $(z_1, x_1) = (15, 8)$ is also a solution of (6). Table 6 shows the first ten solutions (z_m, x_m) of (6), where $m \geq 0$.

m	0	1	2	3	4	5	6	7	8	9
z_m	6	15	54	201	750	2799	10446	38985	145494	542991
x_m	1	8	31	116	433	1616	6031	22508	84001	313496

Table 6

7. Recurrence for (y_m, x_m)

Since $z_m = 3y_m$, the above recurrences yield:

$$\begin{aligned} y_{m+1} &= 2y_m + x_m \\ x_{m+1} &= 3y_m + 2x_m. \end{aligned}$$

8. A Second-Order Recurrence for x_m

As before, it follows by (3) that x_m satisfies the recurrence

$$x_{m+2} = 4x_{m+1} - x_m, \tag{7}$$

where $m \geq 0$.

It follows from this recurrence that $x_{m+2} \equiv x_m \pmod 2$. Since x_0 is odd and x_1 is even, it follows by induction that $x_m \equiv m+1 \pmod 2$; so x_m and $m+1$ have the same parity. Likewise, z_m is always even; see Table 6. Since $y_{m+2} \equiv y_m \pmod 2$, it follows that y_m and m have the same parity.

Using recurrence (7), we now compute $x_m \pmod 7$ and $x_m \pmod{16}$; see Table 7.

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$x_m \pmod 7$	1	1	3	4	6	6	4	3	1	1	3	4	6
$x_m \pmod{16}$	1	8	15	4	1	0	15	12	1	8	15	4	1
m	13	14	15										
$x_m \pmod 7$	6	4	3										
$x_m \pmod{16}$	0	15	12										

Table 7

The sequence $\{x_m \pmod 7\}$ is periodic with period 8:

$$\underbrace{1\ 1\ 3\ 4\ 6\ 6\ 4\ 3}\ \underbrace{1\ 1\ 3\ 4\ 6\ 6\ 4\ 3}\cdots\ ;$$

and so is the sequence $\{x_m \pmod{16}\}$: $\underbrace{1\ 8\ 15\ 4\ 1\ 0\ 15\ 12}\ \underbrace{1\ 8\ 15\ 4\ 1\ 0\ 15\ 12}\cdots$. But $2^k \pmod 7 = 1, 2,$ or 4; and $2^k \pmod{16} = 0$ when $k \geq 5$. Consequently, *no* x_m satisfies both conditions when $k \geq 5$.

`Subcase 2.2.` Consider the solution $(z_0', x_0') = (6, -1)$. Then $(z_1, x_1) = (9, 4)$ and $(z_2, x_2) = (30, 17)$. Since we want $x_m > 0$, we will ignore the solution $(6, -1)$.

Using (7), we now compute the sequences $\{x_m \pmod{127}\}$ and $\{x_m \pmod{128}\}$; see Table 8. Again, no x_m satisfies the conditions satisfied by $\{2^k \pmod{127}\}_{k\geq 5}$ and $\{2^k \pmod{128}\}_{k\geq 5}$. So *no* x_m in this class can be be a power of 2 when $k \geq 5$.

m	1	2	3	4	5	6	7	8
$x_m \pmod{127}$	4	17	64	112	3	27	105	12
$x_m \pmod{128}$	4	17	64	111	124	1	8	31
m	9	10	11	12	13	14	15	16
$x_m \pmod{127}$	70	14	113	57	115	22	100	124
$x_m \pmod{128}$	116	49	80	15	108	33	24	63

m	17	18	19	20	21	22	23	24
$x_m \pmod{127}$	15	63	110	123	1	8	31	116
$x_m \pmod{128}$	100	81	96	47	92	65	40	95
m	25	26	27	28	29	30	31	32
$x_m \pmod{127}$	52	92	62	29	54	60	59	49
$x_m \pmod{128}$	84	113	112	79	76	97	56	127

m	33	34	35	36	37	38	39	40
$x_m \pmod{127}$	10	118	81	79	108	99	34	37
$x_m \pmod{128}$	68	17	0	111	60	1	72	31

m	41	42	43	44	45	46	47	48
$x_m \pmod{127}$	114	38	38	114	37	34	99	108
$x_m \pmod{128}$	52	49	16	15	44	33	88	63

m	49	50	51	52	53	54	55	56
$x_m \pmod{127}$	79	81	118	10	49	59	60	54
$x_m \pmod{128}$	36	81	32	47	28	65	104	95

m	57	58	59	60	61	62	63	64
$x_m \pmod{127}$	29	62	92	52	116	31	8	1
$x_m \pmod{128}$	20	113	48	79	12	97	120	127

m	65	66	67	68	69	70	71	72
$x_m \pmod{127}$	123	110	63	15	124	100	22	115
$x_m \pmod{128}$	4	17	64	111	124	1	8	31

m	73	74	75	76	77	78	79	80
$x_m \pmod{127}$	57	113	14	70	12	105	27	3
$x_m \pmod{128}$	116	49	80	15	108	33	24	63

m	81	82	83	84	85	86	87	88
$x_m \pmod{127}$	112	64	17	4	126	119	96	11
$x_m \pmod{128}$	100	81	96	47	92	65	40	95

m	89	90	91	92	93	94	95	96
$x_m \pmod{127}$	75	35	65	98	73	67	68	78
$x_m \pmod{128}$	84	113	112	79	76	97	56	127

m	97	98	99	100	101	102	103	104
$x_m \pmod{127}$	117	9	46	48	19	28	93	90
$x_m \pmod{128}$	68	17	0	111	60	1	72	31

m	105	106	107	108	109	110	111	112
$x_m \pmod{127}$	13	89	89	13	90	93	28	19
$x_m \pmod{128}$	52	49	16	15	44	33	88	63

m	113	114	115	116	117	118	119	120
$x_m \pmod{127}$	48	46	9	117	78	68	67	73
$x_m \pmod{128}$	36	81	32	47	28	65	104	95
m	121	122	123	124	125	126	127	128
$x_m \pmod{127}$	98	65	35	75	11	96	119	126
$x_m \pmod{128}$	20	113	48	79	12	97	120	127

Table 8

Combining these two subcases, it follows that *no* J_{2n+1} is a triangular number when $n \geq 5$.

Thus, by Cases 1 and 2, *no* J_n is a triangular number when $n \geq 5$. Consequently, the only triangular Jacobsthal numbers are J_1, J_2, J_3, J_6, and J_9; see Table 1.

Remark. Since $y_{m+2} \equiv y_m \pmod 2$ and $y_0 = 2$ in both subcases under Case 2, it follows by induction that y_{2m} is even for $m \geq 0$. But every y-value must be odd. Consequently, we could drop the columns with even values of m from Tables 7 and 8.

Next we investigate Jacobsthal-Lucas numbers j_n that are also triangular. To this end, first notice that the sequence $\{t_n \pmod 9\}$ follows an interesting pattern: $\underbrace{1\ 3\ 6\ 1\ 6\ 3\ 1\ 9\ 9}\ \underbrace{1\ 3\ \cdots\ 9\ 9}\cdots$. Consequently, $t_n \pmod 9$ equals 1, 3, 6, or 9.

9. Triangular Jacobsthal-Lucas Numbers

It follows from the Binet-like formulas that $9J_n^2 = \left(2^{2n}+1\right) - 2(-2)^n = j_{2n} - 2(-2)^n$, so $j_{2n} \equiv 2(-2)^n \pmod 9$. But the sequence $\{2(-2)^n \pmod 9\}$ follows the pattern $\underbrace{5\ 8\ 2}\ \underbrace{5\ 8\ 2}\cdots$; so $j_{2n} \pmod 9$ equals 2, 5, or 8. Consequently, *no* Jacobsthal-Lucas number j_{2n} is triangular.

Now, consider the Jacobsthal-Lucas numbers j_{2n+1}, where $n \geq 0$. Since $j_1 = t_1$, we let $n \geq 1$. Then $8j_{2n+1} + 1 = 2^{2(n+2)} - 8$. Since

$$\left(2^{n+2}-1\right)^2 < 8j_{2n+1} + 1 < 2^{2(n+2)},$$

and $\left(2^{n+2}-1\right)^2$ and $2^{2(n+2)}$ are consecutive squares, it follows that $8j_{2n+1}+1$ cannot be a square. Consequently, j_{2n+1} *cannot* be triangular when $n \geq 1$.

Thus $j_1 = 1$ is the only triangular Jacobsthal-Lucas number; see Table 1.

10. Conclusion

The Jacobsthal family is a delightful source for experimentation and exploration for both amateurs and professionals alike. Using congruences, quadratic diophantine equations, and linear algebra, we established that J_1, J_2, J_3, J_6, and J_9 are the only triangular Jacobsthal numbers; and that j_1 is the only triangular Jacobsthal-Lucas number.

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