THE DIOPHANTINE EQUATION $X^4 + Y^4 = D^2 Z^4$ IN QUADRATIC FIELDS

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Received: 12/19/11, Revised: 10/30/12, Accepted: 11/28/12, Published: 11/30/12

Abstract

A. Aigner proved in 1934 that, except in $\mathbb{Q}(\sqrt{-7})$, there are no nontrivial quadratic solutions to the Diophantine equation $x^4+y^4 = z^4$. The result was later re-proven by D.K. Faddeev and the argument was simplified by L.J. Mordell. This paper extends this result and shows that nontrivial quadratic solutions exist to $X^4 + Y^4 = D^2 Z^4$ precisely when either $D = 1$ or D is a congruent number.

1. Introduction

In a letter to Huygens, Fermat proved that $x^4 + y^4 = z^4$ has no nontrivial integer solutions, where 'nontrivial' means that all variables are nonzero; see p. 75-79, of [15]. Generalizing this result, Hilbert proved in Theorem 169 of his Zahlbericht that there exist no nontrivial solutions in the Gaussian field either [7]. Then in 1934, A. Aigner [1] proved that nontrivial quadratic solutions to $x^4 + y^4 = z^4$ exist only in $\mathbb{Q}(\sqrt{-7})$. Aigner's result was re-proven by D.K. Faddeev [6] and the argument was simplified by L.J. Mordell [11]. We are interested in generalizing this result to $x^4 + y^4 = D^2 z^4$ with $D \in \mathbb{Z}$ and x, y, z algebraic integers and so consider

$$x^4 + y^4 = D^2 \tag{1}$$

for x, y in some quadratic field. Using Mordell's methods, also used in [10], we prove the following result:

Theorem. *Let D be a positive square-free integer. Nontrivial solutions to $x^4+y^4 = D^2$ exist in a quadratic number field precisely when either $D = 1$ or D is a congruent number. More specifically, there are two possible types of solutions: Type 1 and type 2 solutions, depending on whether neither x^2 nor y^2 are rational or both x^2 and y^2 are rational. The detailed presentation of the solutions is found in Theorem 7 and Theorem 9, respectively.*

A positive integer D is a *congruent number* if it is the area of a right triangle with rational sides [8]. For further reading on congruent numbers see [2] or [3]. In [9], Lucas noted that D is a congruent number precisely when there are nontrivial rational solutions to

$$u^4 - D^2v^4 = z^2. \tag{2}$$

Additionally, Tunnell [14] remarked that it is well-known that D is a congruent number precisely when there are nontrivial rational solutions to the elliptic curve

$$y^2 = x^3 - D^2x. \tag{3}$$

For further details on (2) and (3), see [4].

Remark 1. Both of the curves (2) and (3) are genus 1 with infinitely many solutions in the rationals. In contrast, $x^4+y^4 = D^2z^4$ is a nonsingular curve of genus 3. Since the genus exceeds 1, there are finitely many points with coordinates in any number field, by Faltings' Theorem.

The outline of the paper follows. We first consider when one of x^2 or y^2 are rational, and prove no solutions exist to $x^4+y^4 = D^2$. The second section considers when neither x^2 nor y^2 are rational, corresponding to type 1 solutions. We prove type 1 solutions exist to $x^4 + y^4 = D^2$ when there exist rational numbers n, s that satisfy $2n^2 - s^4 = D^2$. Furthermore, in this case there are conjugate solutions $x', y' \in \mathbb{Q}(\sqrt{4n - 3s^2})$ such that $(x')^4 + (y')^4 = D^2$. The third section considers when both x^2 and y^2 are rational, corresponding to type 2 solutions. This case only occurs when $D \neq 1$. We prove that since D is a congruent number, there are infinitely many solutions to $x_1^4 + y_1^2 = D^2$ and each solution gives a quadratic solution to $x^4 + y^4 = D^2$ in $\mathbb{Q}(\sqrt{y_1})$. Moreover, the list of such fields $\mathbb{Q}(\sqrt{y_1})$ is infinite. Admittedly, the more interesting case is when neither x^2 nor y^2 are rational. The second case is included for the sake of the reader.

Example 2. (Aigner [1]) $D = 1$ is not a congruent number, as proven by Fermat, see pg. 615 of [5]. Thus, there are no type 2 solutions. Observe, $D^2 = 2n^2 - s^4$ where $n = -1$ and $s = 1$. Therefore, there are type 1 solutions in $\mathbb{Q}(\sqrt{4n - 3s^2})$. For example,

$$\left(\frac{1+\sqrt{-7}}{2}\right)^4 + \left(\frac{1-\sqrt{-7}}{2}\right)^4 = 1.$$

Example 3. $D = 7$ is a congruent number, as proven by Fibonacci; see pg. 462 of [5]. Observe, $D^2 = 2n^2 - s^4$ where $n = \pm 5$ and $s = 1$. Therefore, type 1 solutions to $x^4 + y^4 = 49$ exist in $\mathbb{Q}(\sqrt{4n - 3s^2})$. For example,

$$\left(\frac{1+\sqrt{-23}}{2}\right)^4 + \left(\frac{1-\sqrt{-23}}{2}\right)^4 = 49 \text{ and } \left(\frac{1+\sqrt{17}}{2}\right)^4 + \left(\frac{1-\sqrt{17}}{2}\right)^4 = 49.$$

Since 7 is a congruent number, there are also type 2 solutions. For example, $(7/5)^4 + (168/25)^2 = 49$. Thus,

$$\left(\frac{7}{5}\right)^4 + \left(\frac{2\sqrt{42}}{5}\right)^4 = 49.$$

Note that $2n^2 - s^4 = 49$ is an elliptic curve of rank 3. Thus, there are infinitely many rational solutions to $2n^2 - s^4 = 49$. By a similar argument as in the end of Theorem 9, the list of such fields $\mathbb{Q}(\sqrt{4n - 3s^2})$ is infinite.

Also, when $s = 1$, $D^2 = 2n^2 - s^4$ is a Pell equation. For the history of Pell equations, see [15]. To calculate the specific D which satisfy $D^2 = 2n^2 - 1$, see chapter 6 of Nagell [12] or look up sequence ID Number A002315 in Neil Sloane's On-Line Encyclopedia of Integer Sequences.

Example 4. $D = 6$ is a congruent number, since it is equal to the area of the right triangle with sides 3,4,5. Therefore, solutions to $x^4 + y^4 = 36$ in a quadratic extension of $\mathbb{Q}$ exist. For example,

$$\left(\frac{12}{5}\right)^4 + \left(\frac{\sqrt{42}}{5}\right)^4 = 36$$

is a type 2 solution. There are no type 1 solutions, as can be seen by Theorem 7 and Lemma 8.

Our proof of the theorem begins by following Mordell. Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field containing x, y. Note, we assume d is a square-free integer. Let $x_1 = x^2$ and $y_1 = y^2$ so $x_1^2 + y_1^2 = D^2$. Geometrically, this equation represents a circle, centered at $(0, 0)$ of radius D with a rational point at $(D, 0)$. The slope, call it t, of the line perpendicular to the line through $(D, 0)$ and (x_1, y_1) is $t = (D - x_1)/y_1 = (D - x^2)/y^2$.

Solving for x^2, we obtain $x^2 = D - ty^2$. Substituting into $x^4 + y^4 = D^2$ and solving for y^2 yields

$$y^2 = 2Dt/(t^2 + 1). \tag{4}$$

Therefore,

$$x^2 = D(1 - t^2)/(t^2 + 1). \tag{5}$$

We have three cases to consider, when neither x^2 nor y^2 are rational, one is rational and the other is non-rational, or both are rational. Let $x = p_1 + q_1\sqrt{d}$ and $y = p_2 + q_2\sqrt{d}$, where p_1, p_2, q_1, q_2 are rationals. Thus,

$$x^2 = (p_1 + q_1\sqrt{d})^2 = p_1^2 + 2p_1q_1\sqrt{d} + q_1^2 d \tag{6}$$

$$y^2 = (p_2 + q_2\sqrt{d})^2 = p_2^2 + 2p_2q_2\sqrt{d} + q_2^2 d. \tag{7}$$

If one of x^2 and y^2 is rational, and the other non-rational, assume without loss of generality that $x^2 \in \mathbb{Q}$ and y^2 is non-rational. By $x^4 + y^4 = D^2$ and $x^2 \in \mathbb{Q}$, it follows that $y^4 \in \mathbb{Q}$. Hence, squaring (7), $4p_2q_2\sqrt{d}(p_2^2 + dq_2^2) = 0$. Again by (7), and since $y^2 \notin \mathbb{Q}$, we have $p_2q_2 \neq 0$, hence $p_2^2 + dq_2^2 = 0$ which can hold only if the square-free integer $d = -1$ which contradicts Hilbert's Theorem 169 [7].

2. Type 1 Solutions: Neither x^2 Nor y^2 Are Rational

Since x^2 is non-rational, it is clear from (5) that the slope t is non-rational. Thus $K = \mathbb{Q}(t)$ where t is a root of the monic irreducible quadratic equation $F(t) = t^2 + Bt + C$ with $B, C \in \mathbb{Q}$. Let $X = (1+t^2)xy$ and $Y = (1+t^2)y$. Squaring X and Y, replacing $D(1-t^2)/(t^2+1)$ for x^2 and $2Dt/(t^2+1)$ for y^2, we obtain

$$X^2 = 2D^2t(1-t^2), Y^2 = 2Dt(1+t^2). \tag{8}$$

Because $\{1, t\}$ is a basis for K over $\mathbb{Q}$, there are $a, b, a_1, b_1 \in \mathbb{Q}$ such that

$$X = a + bt, Y = a_1 + b_1t. \tag{9}$$

Substitute (9) into (8). Observe that t is therefore a root of the cubic polynomials $(a+bz)^2 - 2D^2z(1-z^2)$ and $(a_1+b_1z)^2 - 2Dz(1+z^2)$ which must be divisible by $F(z)$. Therefore,

$$(a+bz)^2 - 2D^2z(1-z^2) = F(z)(P+Qz) \tag{10}$$

and

$$(a_1+b_1z)^2 - 2Dz(1+z^2) = F(z)(P_1+Q_1z). \tag{11}$$

for some $P, Q, P_1, Q_1 \in \mathbb{Q}$. Notice that $(-P/Q)$ and $(-P_1/Q_1)$ are rational roots of the right hand sides of (10) and (11), and thus must be roots of the left hand sides as well.

Lemma 5. *The only rational values of z that satisfy $(a+bz)^2 = 2D^2z(1-z^2)$, are $z = 0, 1, -1$.*

Proof. The equation $(a+bz)^2 = 2D^2z(1-z^2)$ is of the form $Y^2 = 2X(X^2-1)$ where $X = -z$ and $Y = (a+bz)/D$. This equation defines an elliptic curve of zero rank and torsion points $(\pm1, 0), (0, 0)$ and the point at infinity. Hence, $z = 0, \pm1$ are the only possibilities. Alternatively, putting $Y^2 = 2X(X^2-1)$ into standard form we obtain $y^2 = x^3 - 4x$, which is birationally equivalent to the Fermat curve $u^4 + v^4 = w^2$. For the explicit mapping; see p. 55, of [8]. □

Before proving Theorem 7, we need the following result.

Proposition 6. *Let D be a square -free integer > 1, for which there exist $n, s \in \mathbb{Q}$ satisfying $2n^2 - s^4 = D^2$. Then,*

(i) *D is a congruent number.*

(ii) *The polynomial $X^2 - sX + s^2 - n$ is irreducible over $\mathbb{Q}$ and its roots $x', y' \in \mathbb{Q}(\sqrt{4n - 3s^2})$ furnish a solution to (1) with neither x'^2 nor y'^2 rational.*

Proof. (i) Note that if D is a square-free integer greater than 1, and n, s are rational and satisfy $D^2 = 2n^2 - s^4$ then $(x, y) = (\frac{n^2}{s^2}, \frac{n(n-s^2)(n+s^2)}{s^3})$ is a nontrivial point on (3), which is equivalent to D being a congruent number.

(ii) Since $n, s \in \mathbb{Q}$, if x', y' satisfies $x' + y' = s, (x')^2 + x'y' + (y')^2 = n$, then x', y' satisfies $X^2 - sX + s^2 - n \in \mathbb{Q}[X]$ and lie in $\mathbb{Q}(\sqrt{4n - 3s^2})$. Replacing s, n in $D^2 = 2n^2 - s^4$ yields $D^2 = (x')^4 + (y')^4$. If $n < 0$, clearly $\sqrt{4n - 3s^2} \notin \mathbb{Q}$. If $n > 0$, to see that $\sqrt{4n - 3s^2} \notin \mathbb{Q}$, assume to the contrary that $4n - 3s^2 = t^2$ for $t \in \mathbb{Q}$. Solving for n, and replacing in $2n^2 - s^4 = D^2$ we obtain $s^4 + 6s^2t^2 + t^4 = 8D^2$. Thus, $(z/t, D/t^2)$ is a point on the elliptic curve $X^4 + 6X^2 + 1 = 8Y^2$ which has rank 0. This means the only rational solutions of the last equation comes from torsion points. Hence,$(|X|, |Y|) = (1, 1)$ gives all rational solutions implying $t^2 = s^2 = D$, but D is a square-free integer greater than 1. □

Theorem 7. (Solutions of type 1) *Let D be a positive square-free integer and let (x, y) be a non-trivial solution to (1) with both x^2 and y^2 not rational but in a quadratic number field. Then, either $D = 1$ and $x, y \in \mathbb{Q}(\sqrt{-7})$, or $D > 1$ and there exist $n, s \in \mathbb{Q}$ satisfying $2n^2 - s^4 = D^2$, so that all conclusions of Proposition 1 are valid.*

Proof. Each rational value from Lemma 5 determines a possible expression for $F(z)$. For example, for $z = 0$, we find from (10) that $(bz)^2 - 2D^2z(1-z^2) = F(z)Qz$. Since $F(z)$ is monic, $Q = 2D^2$ and thus, $F(z) = z^2 + (b^2/2D^2)z - 1$. This polynomial also divides $(a_1 + b_1z)^2 - 2Dz(1 + z^2)$. Long division yields a remainder with constant term $a_1^2 + b_1^2 + b^2/D$. Since this remainder must be zero, $a_1 = b_1 = b = 0$. Thus $F(z) = z^2 - 1$ which is not irreducible. Similarly, for $z = 1$ we find from (10) that $a^2(1 - z)^2 - 2D^2z(1 - z^2) = F(z)P(1 - z)$. Since $F(z)$ is monic, $P = -2D^2$. Thus $F(z) = z^2 + (1 + a^2/2D^2)z - a^2/2D^2$. Once again, long division yields a remainder with constant terms. This time the constant term is $(2a_2^2D^3 + a^2b_1^2D + 2a^2D^2 + a^4)/2D^3$, which must be zero. Thus $a = 0$, and $F(z) = z^2 - z$, which is not irreducible. The only rational value that does not lead to an obvious contradiction is $z = -1$, as we now show.

For $z = -1$, we find for (10) that $a^2(1 + z)^2 - 2D^2z(1 - z^2) = F(z)P(1 + z)$. Thus,

$$F(z) = z^2 + \left(\frac{a^2}{2D^2} - 1\right) z + \frac{a^2}{2D^2}. \tag{12}$$

In this case, long division of $F(z)$ into $(a_1+b_1z)^2-2Dz(1+z^2)$ yields the quotient and remainder, respectively, of

$$-2Dz+\frac{b_1^2D+a^2-2D^2}{D} \tag{13}$$

$$-\frac{1}{2}\left(\frac{-4D^3a_1b_1+8D^4-6a^2D^2+a^2b_1^2D-2b_1^2D^3+a^4}{D^3}\right)z \\ +\frac{1}{2}\left(\frac{2a_1^2D^3-a^2b_1^2D-a^4+2a^2D^2}{D^3}\right) \tag{14}$$

which does not lead to an obvious contradiction.

Note that both coefficients in (14) must be zero. Moreover, $a\neq 0$, since if $a=0$ then $F(z)=z^2-z$. Also, if $a_1=0$, then using (11), $(b_1z)^2-2Dz(1+z^2)=F(z)Q_1z$. Since $F(z)$ is monic, $F(z)=z^2-(b_1^2/2D)z+1$. However, from (12), since $1=a^2/2D^2$ implies $\sqrt{2}$ is rational, we have a contradiction. Because of the second term in (14), we have $2a_1^2D^3=a^2b_1^2D+a^4-2a^2D^2$. Dividing through by a^2, we have $b_1^2D+a^2-2D^2=2a_1^2D^3/a^2$. Substituting this into (13), $-2Dz+(2a_1^2D^2)/(a^2)$. This means that $z=-P/Q=(a_1/a)^2D$ solves $(a_1+b_1z)^2-2Dz(1+z^2)=0$. Replacing $z=(a_1/a)^2D$ into $(a_1+b_1z)^2=2Dz(1+z^2)$ we have $D^2=2n^2-s^4$ for $s=a_1D/a$ and $n=(a^2+a_1b_1D)/(2a)$ where $n,s\in\mathbb{Q}$. □

The existence of $n,s\in\mathbb{Q}$ satisfying $2n^2-s^4=D^2$, though a sufficient condition for square-free $D>1$ to be a congruent number, is not necessary.

Lemma 8. *If D is a square-free even integer, $D^2=2n^2-s^4$ for $n,s\in\mathbb{Q}$ is impossible.*

Proof. Assume that $D^2=2n^2-s^4$ for some rational numbers n and s, and multiply through by the least common multiple z of the denominators of n and s to obtain an equation $D^2z^4+s_1^4=2m^2$, where m and s_1 are integers. Write $D=2D_1, z=2^bz_1, s_1=2^cs_2$ and $m=2^dm_1$, where D_1,z_1,s_2,m_1 are odd with $c,d\geq 1$. Then $2^{4b+2}D_1^2z_1^4+2^{4c}s_2^4=2^{2d+1}m_1^2$. The highest power of 2 dividing the right-hand side is 2^{2d+1}. We look now at the highest power of 2 dividing the left-hand side. Since $4b+2\neq 4c$, this highest power is exactly equal to $\min\{2^{4b+2},2^{4c}\}$. Clearly, in any case, this is not equal to 2^{2d+1}, and this contradiction completes the proof. □

3. Type 2 Solutions: Both x^2 and y^2 Are Rational

Theorem 9. *(Solutions of type 2.) If nontrivial quadratic solutions exist to $x^4+y^4=D^2$ and $x^2,y^2\in\mathbb{Q}$, then D is a congruent number. Conversely, if D is a* *congruent number, then there are infinitely many rational solutions* (x_1, y_1) *to the equation* $x_1^4 + y_1^2 = D^2$*. Therefore, for each such solution* (x_1, y_1) *in positive rationals,* $(x, y) = (x_1, \sqrt{y_1})$ *is a solution to* (1) *in* $\mathbb{Q}(\sqrt{y_1})$ *with* $x^2, y^2 \in \mathbb{Q}$*. Furthermore, the list of such fields* $\mathbb{Q}(\sqrt{y_1})$ *is infinite.*

Proof. If $x^2, y^2 \in \mathbb{Q}$, clearly $t = (D - x^2)/y^2$ is rational. Using (6) and (7) $x^2 = (p_1 + q_1\sqrt{d})^2 = p_1^2 + 2p_1q_1\sqrt{d} + q_1^2 d$ and $y^2 = (p_2 + q_2\sqrt{d})^2 = p_2^2 + 2p_2q_2\sqrt{d} + q_2^2 d$, $2p_1q_1\sqrt{d} = 0$ and $2p_2q_2\sqrt{d} = 0$. Thus, we have four cases to consider, when $p_1 = p_2 = 0$, $q_1 = q_2 = 0$, $p_1 = q_2 = 0$, and $q_1 = p_2 = 0$.

`Case A`. If either $p_1 = p_2 = 0$ or $q_1 = q_2 = 0$, then either $x/y = (q_1/q_2)^2$ or $(p_1/p_2)^2$. In either case, $(x/y)^2 \in \mathbb{Q}^2$ so $x/y = m$. Since $(x/y)^2 = (1 - t^2)/2t$, $t^2 + 2m^2t - 1 = 0$. Because t is rational, the discriminate must be in $\mathbb{Q}^2$. Then there exists $n^2 \in \mathbb{Q}$ such that $m^4 + 1 = n^2$ which Fermat proved has no nontrivial solutions.

`Case B`. If either $q_1 = p_2 = 0$ or $p_1 = q_2 = 0$, then $x^2 = p_1^2$, $y^2 = dq_2^2$ or $x^2 = dq_1^2$, $y^2 = p_2^2$. Without loss of generality, assume $x^2 = p_1^2, y^2 = dq_2^2$, replace into (1) to obtain $p_1^4 + d^2q_2^4 = D^2$ for non-zero rational numbers p_1, q_2, and d, a non-zero integer.

Lucas' equation (2), $u^4 - D^2v^4 = z^2$, and $p_1^4 + (dq_2^2)^2 = D^2$ are birationally equivalent with the following mutual inverse correspondences:

$$(u, v, z) \to \left(1, \frac{p_1}{D}, \frac{dq_2^2}{D}\right), \quad (p_1, dq_2^2) \to \left(\frac{Dm}{v}, \frac{Dn}{v^2}\right).$$

Since (2) and (3) are birationally equivalent, there are infinitely many rational solutions to $x_1^4 + y_1^2 = D^2$ [13] . Thus, each such solution x_1, y_1 in positive rationals, $(x, y) = (x_1, \sqrt{y_1})$ is a solution to (1) which lie in $\mathbb{Q}(\sqrt{y_1})$. To see that there are infinitely many $\mathbb{Q}(\sqrt{y_1})$, assume to the contrary that there are a finite number of fields $\mathbb{Q}(\sqrt{y_1})$. Since there are an infinite number of points on the curve $x^4 + y^4 = D^2z^4$, due to these type 2 solutions, there must be at least one field with infinitely many solutions. However, $x^4 + y^4 = D^2z^4$ is a curve of genus 3, which by Falting's Theorem, has finitely many points in the number field. Thus, the list of such fields $\mathbb{Q}(\sqrt{y_1})$ is infinite. □

4. Conclusions

We have proven that nontrivial solutions to $x^4 + y^4 = D^2$ exist in a quadratic number field precisely when either D is a congruent number or $D = 1$. If there are solutions of type 1, when neither x^2 nor y^2 are rational, we have proven that there are conjugate solutions of type 1 in the quadratic field $\mathbb{Q}(\sqrt{4n - 3s^2})$ where $n, s \in \mathbb{Q}$ satisfies $D^2 = 2n^2 - s^4$. We have not proven that all solutions of type 1 occur as conjugate pairs. We have proven that $\{D : D \neq 1, \exists n, s \in \mathbb{Q}, D^2 = 2n^2 - s^4\}$ is a proper subset of the congruent numbers, but we have not characterized this proper subset any further. We have proven solutions of type 2, when $x^2, y^2 \in \mathbb{Q}$, to $x^4 + y^4 = D^2$ occur if and only if D is a congruent number and there are infinitely many quadratic fields with solutions.

Acknowledgments. This work was partially supported by a NASA Nebraska Space Grant Fellowship. I would like to thank my advisor, Griff Elder, for providing this research opportunity, for his patience, guidance, and support. We wish to thank the anonymous referee and Jeff Lagarias for the helpful comments that improved the exposition. A warm thank you to Ari Shnidman for the insight into the proof that there are infinitely many quadratic fields with solutions.

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