GENERALIZED BINOMIAL EXPANSIONS AND BERNOULLI POLYNOMIALS

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Abstract

We investigate generalized binomial expansions that arise from two-dimensional sequences satisfying a broad generalization of the triangular recurrence for binomial coefficients. In particular, we present a new combinatorial formula for such sequences in terms of a 'shift by rank' quasi-expansion based on ordered set partitions. As an application, we give a new proof of Dilcher's formula for expressing generalized Bernoulli polynomials in terms of classical Bernoulli polynomials.

1. Introduction

The binomial expansion formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \tag{1}$$

where $\binom{n}{k}$ denotes the binomial coefficients given by the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (0 \le k \le n), \tag{2}$$

is well known to all mathematicians. In this paper, we consider instead a generalized binomial expansion of two sequences $a(k)$ and $b(k)$ where the first few cases are given by the quasi-expansion that we shall refer to as 'shift by rank':

$$\begin{aligned}
(a(k)+b(k))^0 &:= 1 \\
(a(k)+b(k))^1 &:= a(k)+b(k) \\
(a(k)+b(k))^2 &:= a(k)^2 + a(k-1)b(k) + b(k)a(k) + b(k-1)b(k) \\
(a(k)+b(k))^3 &:= a(k)^3 + b(k)a(k)^2 + a(k-1)b(k)a(k) + a(k-1)^2 b(k) \\
&\quad + b(k-1)b(k)a(k) + b(k-1)a(k-1)b(k) \\
&\quad + a(k-2)b(k-1)b(k) + b(k-2)b(k-1)b(k).
\end{aligned} \tag{3}$$

Can the reader guess the correct expansion for the next case, $(a(k)+b(k))^4$, and more generally the formula for $(a(k)+b(k))^n$ where n is any non-negative integer? Observe that if $a(k)$ and $b(k)$ are constant sequences, then the pattern reduces to that for binomial coefficients.

As we shall explain in this paper, the 'shift-by-rank' expansion above appears in a formula for two-dimensional sequences $x(n,k)$ satisfying the generalized triangular recurrence

$$x(n,k) = a(n,k)x(n-1,k) + b(n,k)x(n-1,k-1) \tag{4}$$

where $a(n,k)$ and $b(n,k)$ are arbitrary two-dimensional sequences. If $a(n,k)$ and $b(n,k)$ are constant and equal to one, then (4) reduces to the classic triangular recurrence satisfied by the binomial coefficients:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \tag{5}$$

This recurrence is commonly illustrated by arranging the binomial coefficients into the much-celebrated figure known as Pascal's triangle:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \\ &&&&\cdots&&&& \end{array} \tag{6}$$

Moreover, the recurrence (5) uniquely defines the binomial coefficients if we initialize the values on the boundary of Pascal's triangle to be 1, i.e.

$$\binom{n}{0} = \binom{n}{n} = 1, \tag{7}$$

since all other entries can then be generated by this recurrence.

Of course, the mathematical literature contains many extensions of (5) which are special cases of (4) (see [1], [2], and [5]). For example, the unsigned Stirling numbers of the first kind are defined by the recurrence

$$\left[\begin{array}{c} n \\ k \end{array}\right] = (n-1)\left[\begin{array}{c} n-1 \\ k \end{array}\right] + \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right] \tag{8}$$

and Stirling numbers of the second kind are defined by

$$\left\{\begin{array}{c} n \\ k \end{array}\right\} = k\left\{\begin{array}{c} n-1 \\ k \end{array}\right\} + \left\{\begin{array}{c} n-1 \\ k-1 \end{array}\right\}. \tag{9}$$

An even more interesting recurrence is one satisfied by the generalized Bernoulli polynomials $B_k^{(n)}$ of order n:

$$B_k^{(n)}(z) = \left(1 - \frac{k}{n-1}\right) B_k^{(n-1)}(z) + k\left(\frac{z}{n-1} - 1\right) B_{k-1}^{(n-1)}(z). \tag{10}$$

We shall consider this recurrence further in Section 3.

Our approach to studying $x(n,k)$ relies on a different initialization of Pascal's triangle, which will appear quite natural if we realize it as an infinite rectangular array (also referred to as Pascal's matrix or square; see [3] and [6]), where infinitely many 0's are appended to both ends of each row as shown in (11):

$$\begin{array}{llccccccl} & & (k=-1) & (k=0) & (k=1) & (k=2) & (k=3) & ... & \\ (n=0) & ... & 0 & 1 & 0 & 0 & 0 & 0 & ... \\ (n=1) & ... & 0 & 1 & 1 & 0 & 0 & 0 & ... \\ (n=2) & ... & 0 & 1 & 2 & 1 & 0 & 0 & ... \\ (n=3) & ... & 0 & 1 & 3 & 3 & 1 & 0 & ... \\ ... & & & & & & & & \end{array} \tag{11}$$

In this case, it suffices to initialize only those values along the first row ($n = 0$) as follows:

$$\binom{0}{k} \equiv \delta_{0,k}, \quad k \in \mathbb{Z} \tag{12}$$

where

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{13}$$

is Kronecker's delta. All other binomial coefficients in this array can then be generated from the triangular recurrence as before.

As an example, let $x(n,k)$ be any two-dimensional sequence which satisfies the classical triangular recurrence, i.e.

$$x(n,k) = x(n-1,k) + x(n-1,k-1), \tag{14}$$

but whose values along row $n = 0$ are prescribed by an arbitrary bi-directional sequence $f(k)$, i.e.

$$x(0,k) = f(k), \quad k \in \mathbb{Z}. \tag{15}$$

It is straightforward to show that $x(n,k)$ satisfies the following convolution formula:

$$x(n,k) = \sum_{m=0}^{n} \binom{n}{m} f(k-m). \tag{16}$$

This demonstrates that binomial coefficients can be considered as fundamental building blocks for generating all two-dimensional sequences which satisfy the triangular recurrence.

In this paper we shall describe the fundamental building blocks for two-dimensional sequences $x(n,k)$ satisfying the generalized triangular recurrence (4). This will require us to introduce the notion of rank for comparing the relative size of an integer with respect to a given subset. As a result, we obtain what seems to be a new combinatorial formula for $x(n,k)$ (Theorem 1) involving ordered set partitions; an extensive search of the literature, including the standard reference [1], which mentions (4) but only discusses special cases, did not reveal any formula similar to ours. As an application, we give a different proof of Dilcher's formula ([1], Theorem 2), which expresses generalized Bernoulli polynomials in terms of classical Bernoulli polynomials. We conclude by presenting a generalization of our results to three-term recurrences.

2. Generalized Triangular Recurrences

We begin our study of generalized triangular recurrences by first considering the simple generalization

$$x(n,k) = ax(n-1,k) + bx(n-1,k-1), \tag{17}$$

where a and b are constants. Cadogan has proven in [2] that

$$x(n,k) = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m f(k-m) \tag{18}$$

which shows that the coefficients in (18) can be generated by expanding the binomial $(ax+by)^n$.

As a next step, let us assume that the more general triangular recurrence holds:

$$x(n,k) = a(n)x(n-1,k) + b(n)x(n-1,k-1), \tag{19}$$

where $a(n)$ and $b(n)$ are one-dimensional sequences. In this case, the situation becomes much more interesting. To gain some intuition, let's write out $x(n,k)$ explicitly for the first several values of n:

$$\begin{aligned}
x(0,k) &= f(k) \\
x(1,k) &= a(1)f(k) + b(1)f(k-1) \\
x(2,k) &= a(1)a(2)f(k) + [b(1)a(2) + a(1)b(2)]f(k-1) + b(1)b(2)f(k-2) \\
x(3,k) &= a(1)a(2)a(3)f(k) \\
&\quad +[a(1)a(2)b(3) + a(1)b(2)a(3) + b(1)a(2)a(3)]f(k-1) \\
&\quad +[a(1)b(2)b(3) + b(1)a(2)b(3) + b(1)b(2)a(3)]f(k-2) \\
&\quad +b(1)b(2)b(3)f(k-3).
\end{aligned} \tag{20}$$

Guided by the combinatorial definition of binomial coefficients as counting the number of subsets of a given set, we look to describe the coefficients of f in (20) by ordered partitions of $\{1,2,3\}$.

Definition 1. Let $A_n = \{1,2,...,n\}$. We define $P = (\sigma_1, \sigma_2)$ to be a *2-block ordered partition* of A_n if σ_1 and σ_2 are disjoint sets whose union is A_n. Morever, let $\mathcal{P}_2(A_n)$ denote the set of all 2-block ordered partitions of A_n.

Observe that in the definition above we allow either σ_1 or σ_2 to be empty (but not both), which differs from the standard definition of set partitions (see [8]). Also, the correspondence $(\sigma_1, \sigma_2) \leftrightarrow \sigma_2$ gives a bijection between $\mathcal{P}_2(A_n)$ and the power set of A_n, i.e., the set of all subsets of A_n, since σ_1 and σ_2 are complements of each other in A_n. Thus, we can express any 2-block ordered partition as $(\bar{\sigma}, \sigma)$, where σ is a subset of A_n. Moreover, $\mathcal{P}_2(A_n)$ has cardinality 2^n.

Definition 2. Given $(\bar{\sigma}, \sigma) \in \mathcal{P}_2(A_n)$ where the elements of σ and $\bar{\sigma}$ are denoted explicitly by $\{i_1, i_2, ..., i_m\}$ and $\{j_1, j_2, ..., j_{n-m}\}$, respectively, we define the product

$$\pi_{a,b}(\bar{\sigma}, \sigma) \equiv a(j_1)a(j_2)\cdots a(j_{n-m})b(i_1)b(i_2)\cdots b(i_m). \tag{21}$$

It follows from this definition that $x(n, k)$ can be described by the formula

$$x(n,k) = \sum_{(\bar{\sigma},\sigma)\in\mathcal{P}_2(A_n)} \pi_{a,b}(\bar{\sigma},\sigma) f(k - |\sigma|), \tag{22}$$

or equivalently,

$$x(n,k) = \sum_{m=0}^{n} \left(\sum_{\sigma\in A_n(m)} \pi_{a,b}(\bar{\sigma},\sigma) \right) f(k-m). \tag{23}$$

where $A_n(m)$ denotes the set of m-element subsets of A_n. We shall prove a more general formula later on.

Let us now assume that $x(n, k)$ satisfies the recurrence

$$x(n,k) = a(k)x(n-1,k) + b(k)x(n-1,k-1) \tag{24}$$

where $a(k)$ and $b(k)$ are again one-dimensional sequences as in (19), but this time dependent on the second index k instead of the first index n. How does this affect our formula for $x(n, k)$? Again, we write out $x(n, k)$ explicitly for the first several values of n:

$$\begin{aligned}
x(0,k) &= f(k)\\
x(1,k) &= a(k)f(k) + b(k)f(k-1)\\
x(2,k) &= a(k)a(k)f(k) + [a(k-1)b(k) + b(k)a(k)]f(k-1)\\
&\quad +vb(k-1)b(k)f(k-2)\\
x(3,k) &= a(k)a(k)a(k)f(k) + [b(k)a(k)a(k)\\
&\quad +a(k-1)b(k)a(k) + a(k-1)a(k-1)b(k)]f(k-1)\\
&\quad +[b(k-1)b(k)a(k) + b(k-1)a(k-1)b(k)\\
&\quad +a(k-2)b(k-1)b(k)]f(k-2) + b(k-2)b(k-1)b(k)f(k-3).
\end{aligned} \tag{25}$$

The pattern in this case appears more complicated than the previous case due to shifts in certain indices; in fact, (25) is essentially the 'shift and rank' expansion (modulo f) discussed at the beginning of this paper. To shed some light on this pattern, we first fix an ordering for how the factors a and b should appear in each coefficient: indices should be written left to right in ascending order and if two factors have the same index, then they are written left to right in reverse-alphabetical order. For example, the expansion in (25) is written using this ordering.

Next, we compare the shifts of each index with the corresponding positions of the factor a and b. In particular, if we denote by s_j the amount of shift in the index of the j-th factor of a given coefficient of f, then what pattern does s_j satisfy? The answer is that the shift in the index of a given factor depends on the number of factors involving b that are higher in position, i.e., further to the right.

We formalize this pattern by introducing a rank function to indicate the relative size of an integer with respect to a given subset.

Definition 3. Let $\sigma = \{i_1, i_2, ..., i_m\} \in A_n(m)$. We define the *rank* of a positive integer j relative to σ, and denote it by $R_\sigma(j)$, to be the number of elements in σ that are greater than j, i.e.

$$R_\sigma(j) = |\{i \in \sigma : i > j\}|$$

.

For example, if $\sigma = \{1, 2, 4\} \in A_5(3)$, then $R_\sigma(3) = 1$ since there is only one element in σ, namely 4, that is greater than 3. We also have $R_\sigma(5) = 0$. In general, $R_\sigma(n) = 0$ for every subset $\sigma \in A_n$. Then given $(\bar{\sigma}, \sigma) \in \mathcal{P}_2(A_n)$, we define the product (analogous to (21))

$$\pi_{a,b}(\bar{\sigma}, \sigma; k) = \prod_{s=1}^{n-m} a(k - R_\sigma(j_s)) \prod_{r=1}^{m} b(k - R_\sigma(i_r)). \tag{26}$$

As a result, we are now able to express $x(n, k)$ by the *shift-by-rank* expansion formula

$$x(n, k) = \sum_{m=0}^{n} \left(\sum_{\sigma \in A_n(m)} \pi_{a,b}(\bar{\sigma}, \sigma; k) \right) f(k - m). \tag{27}$$

This brings us to the last case where we assume that $x(n, k)$ satisfies the most general recurrence given by (17) in terms of arbitrary two-dimensional sequences $a(n, k)$ and $b(n, k)$. Again, given $(\bar{\sigma}, \sigma) \in \mathcal{P}_2(A_n)$, we define the product

$$\pi_{a,b}(\bar{\sigma}, \sigma; k) = \prod_{s=1}^{n-m} a(j_s, k - R_\sigma(j_s)) \prod_{r=1}^{m} b(i_r, k - R_\sigma(i_r)). \tag{28}$$

By combining the results of the two previous cases, we obtain the following main theorem, which we prove rigorously.

Theorem 4. *Suppose a two-dimensional sequence $x(n,k)$ satisfies the generalized triangular recurrence*

$$\begin{aligned} &x(n,k) = a(n,k)x(n-1,k) + b(n,k)x(n-1,k-1) \\ &x(0,k) = f(k) \end{aligned}$$

where n is a non-negative integer and $k \in \mathbb{Z}$. Then

$$x(n,k) = \sum_{m=0}^{n} \left(\sum_{\sigma \in A_n(m)} \pi_{a,b}(\bar{\sigma}, \sigma; k) \right) f(k-m). \tag{29}$$

Proof. For $n = 0$, formula (29) reduces to $x(0,k) = f(k)$. By induction on n, we have

$$\begin{aligned} x(n+1,k) &= a(n+1,k)x(n,k) + b(n+1,k)x(n,k-1) \\ &= a(n+1,k)\textstyle\sum_{m=0}^{n} \left(\sum_{\sigma \in A_n(m)} \pi_{a,b}(\bar{\sigma}, \sigma; k) \right) f(k-m) \\ &\quad + b(n+1,k)\textstyle\sum_{m=0}^{n} \left(\sum_{\sigma \in A_n(m)} \pi_{a,b}(\bar{\sigma}, \sigma; k-1) \right) f(k-1-m) \\ &= \textstyle\sum_{m=0}^{n} \left(\sum_{\sigma \in A_n(m)} a(n+1,k)\pi_{a,b}(\bar{\sigma}, \sigma; k) \right) f(k-m) \\ &\quad + \textstyle\sum_{m=0}^{n} \left(\sum_{\sigma \in A_n(m)} b(n+1,k)\pi_{a,b}(\bar{\sigma}, \sigma; k-1) \right) f(k-1-m). \end{aligned}$$

Next, we simplify the inner summation in each of double summations above as follows. First, assume $\sigma = \{i_1, i_2, ..., i_m\} \in A_n(m)$ so that $\bar{\sigma} = \{j_1, j_2, ..., j_{n-m}\} \in A_n(n-m)$. Then observe that we can view σ as a subset of $A_{n+1}(m)$; moreover, the complement of σ in $A_{n+1}(m)$ equals $\bar{\sigma} \cup \{n+1\}$. Set $j_{n-m+1} = n+1$ and recall that $R_\sigma(n+1) = 0$. It follows that the inner summation of the first term can be rewritten as

$$\begin{aligned} &\sum_{\sigma \in A_n(m)} a(n+1,k)\pi_{a,b}(\bar{\sigma}, \sigma; k) \\ &\qquad = \textstyle\sum_{\sigma \in A_n(m)} a(n+1,k) \prod_{s=1}^{n-m} a(j_s, k - R_\sigma(j_s)) \prod_{r=1}^{m} b(i_r, k - R_\sigma(i_r)) \\ &\qquad = \textstyle\sum_{\substack{\sigma \in A_{n+1}(m) \\ n+1 \notin \sigma}} \prod_{s=1}^{n+1-m} a(j_s, k - R_\sigma(j_s)) \prod_{r=1}^{m} b(i_r, k - R_\sigma(i_r)) \\ &\qquad = \textstyle\sum_{\substack{\sigma \in A_{n+1}(m) \\ n+1 \notin \sigma}} \pi_{a,b}(\bar{\sigma}, \sigma; k). \end{aligned}$$

As for the second term, since $R_{\sigma'}(j) = R_\sigma(j) + 1$ for $\sigma \in A_n(m)$, where $\sigma' = \sigma \cup \{n+1\}$ and $1 \le j < n+1$, we can simplify it as

$$\sum_{\sigma\in A_n(m)} b(n+1,k)\pi_{a,b}(\bar{\sigma},\sigma;k-1)$$

$$= \sum_{\sigma\in A_n(m)} b(n+1,k)\prod_{s=1}^{n-m} a(j_s,k-1-R_\sigma(j_s))\prod_{r=1}^{m} b(i_r,k-1-R_\sigma(i_r))$$

$$= \sum_{\substack{\sigma'=\sigma\cup\{n+1\}\\ \sigma\in A_n(m)}} \prod_{s=1}^{n-m} a(j_s,k-R_{\sigma'}(j_s))\prod_{r=1}^{m+1} b(i_r,k-R_{\sigma'}(i_r))$$

$$= \sum_{\substack{\sigma'\in A_{n+1}(m+1)\\ n+1\in\sigma'}} \pi_{a,b}(\bar{\sigma}',\sigma';k).$$

It follows that

$$x(n+1,k) = \sum_{m=0}^{n}\left(\sum_{\substack{\sigma\in A_{n+1}(m)\\ n+1\notin\sigma}} \pi_{a,b}(\bar{\sigma},\sigma;k)\right) f(k-m)$$

$$+ \sum_{m=0}^{n}\left(\sum_{\substack{\sigma'\in A_{n+1}(m+1)\\ n+1\in\sigma'}} \pi_{a,b}(\bar{\sigma}',\sigma';k)\right) f(k-(m+1)).$$

But the two summations above can be viewed as a partition of A_{n+1} into those subsets σ' that contain the integer $n+1$ and those subsets σ that do not. Then by re-indexing the second summation on the right hand side above and making the limits of summation the same for both, which involve inserting empty cases, we can combine both summations into one as follows:

$$x(n+1,k) = \sum_{m=0}^{n+1}\left(\sum_{\substack{\sigma\in A_{n+1}(m)\\ n+1\notin\sigma}} \pi_{a,b}(\bar{\sigma},\sigma;k)\right) f(k-m)$$

$$+ \sum_{m=0}^{n+1}\left(\sum_{\substack{\sigma'\in A_{n+1}(m)\\ n+1\in\sigma'}} \pi_{a,b}(\bar{\sigma}',\sigma';k)\right) f(k-m)$$

$$= \sum_{m=0}^{n+1}\left(\sum_{\sigma\in A_{n+1}(m)} \pi_{a,b}(\bar{\sigma},\sigma;k)\right) f(k-m).$$

This completes the proof. □

3. Application to Bernoulli Polynomials

As an application of Theorem 1, we prove Dilcher's formula ([1]) for expressing generalized Bernoulli polynomials in terms of classical Bernoulli polynomials $B_n(z)$. The latter arises in many important applications involving special functions such as the Riemann zeta function.

Let p be a positive integer. Then the generalized Bernoulli polynomials $B_n^{(p)}(z)$ of order p are defined by the generating function

$$\frac{t^p e^{zt}}{(e^t-1)^p} = \sum_{n=0}^{\infty} B_n^{(p)}(z)\frac{t^n}{n!}. \tag{30}$$

If $p=1$, then $B_n^p(z)$ reduces to the classical Bernoulli polynomials $B_n(z)$. Since

$$\frac{t^p e^{zt}}{(e^t-1)^p} = \prod_{q=1}^{p}\left(\frac{te^{z_q t}}{e^t-1}\right) = \prod_{q=1}^{p}\left(\sum_{n=0}^{\infty} B_n(z_q)\frac{t^n}{n!}\right) \tag{31}$$

where $z = z_1 + ... + z_p$, it follows from equating (30) and (31) that

$$S_{n,p}(z) \equiv \sum_{\substack{i_1\geq 0, i_2\geq 0,...,.i_p\geq 0\\ i_1+i_2+...+i_p=n}} \frac{n!}{i_1!\cdots i_p!} B_{i_1}(z_1)\cdots B_{i_p}(z_p) = B_n^{(p)}(z). \tag{32}$$

In [3], Nörlund proved that the generalized Bernoulli polynomials satisfy the generalized triangular recurrence

$$B_n^{(p)}(z) = \left(1-\frac{n}{p-1}\right)B_n^{(p-1)}(z) + n\left(\frac{z}{p-1}-1\right)B_{n-1}^{(p-1)}(z) \tag{33}$$

for $n > p \geq 1$. By applying Theorem 1 to this recurrence, we obtain the following corollary.

Corollary 5. *The generalized Bernoulli polynomials are given in terms of the classical Bernoulli polynomials by*

$$B_n^{(p)}(z) = \sum_{k=0}^{p-1}\left(\sum_{\sigma\in A_{p-1}(k)} \pi_{a,b}(\bar{\sigma},\sigma;n)\right)B_{n-k}(z). \tag{34}$$

Proof. Formula (34) follows immediately from (29) by setting $x(p,n) = B_n^{(p+1)}(z)$, $a(p,n) = 1-\frac{n}{p}$ and $b(p,n) = n\left(\frac{z}{p}-1\right)$. □

The following lemma describing certain properties of the rank function will be useful in demonstrating that formula (34) reduces to Dilcher's formula, which we show afterwards.

Lemma 6. *Let* $\sigma = \{i_1, i_2, ..., i_m\} \in A_n(m)$ *with* $i_1 < i_2 < ... < i_m$ *and* $\bar{\sigma} = \{j_1, j_2, ..., j_{n-m}\} \in A_n(n-m)$ *with* $j_1 < j_2 < ... < j_{n-m}$*. Then*

$$\{R_\sigma(i_r)\}_{r=1}^m = \{0, ..., m-1\} \tag{35}$$

and

$$\{R_\sigma(j_s) + j_s\}_{s=1}^{n-m} = \{m+1, ..., n\}. \tag{36}$$

Proof. Since $i_1 < i_2 < ... < i_k$ we have $R_\sigma(i_r) = m - r$ for $r = 1, ..., m$ and so

$$\{R_\sigma(i_r)\}_{r=1}^m = \{0, ..., m-1\}.$$

This proves (35). To prove (36), we first show that $R_\sigma(j_s) + j_s$ is bounded between $m+1$ and n for $s = 1, ..., n-m$. This is because $R_\sigma(j_s) \geq m - j_s + 1$ and so

$$R_\sigma(j_s) + j_s \geq (m - j_s + 1) + j_s = m + 1.$$

On the other hand, we have $R_\sigma(j_s) \leq n - j_s$ and so

$$R_\sigma(j_s) + j_s \leq n - j_s + j_s = n.$$

Next, we claim that the values $R_\sigma(j_s) + j_s$ are all distinct for $1 \leq s \leq n-m$ and in particular increasing in s, i.e., $R_\sigma(j_r) + j_r < R_\sigma(j_s) + j_s$ for $r < s$. To prove this, we set $k = j_s - j_r > 0$ (recall that $j_r < j_s$). Since $R_\sigma(j_r) < R_\sigma(j_s) + k$, it follows that

$$R_\sigma(j_r) + j_r < R_\sigma(j_s) + j_r + k = R_\sigma(j_s) + j_s$$

which proves our claim. Thus, (36) holds. □

Let $s(n,k)$ denote the Stirling numbers of the first kind, which are defined by the generating function

$$z(z-1)(z-2)\cdots(z-n+1) = \sum_{m=0}^{n} s(n,m)z^m. \tag{37}$$

In particular, we have

$$s(n,m) = \sum_{\{i_1,i_2,\ldots,i_m\}\in A_n(m)} (-1)^m i_1 i_2 \cdots i_m. \tag{38}$$

We are now ready to prove Dilcher's formula.

Theorem 7. (Dilcher [1]) *For positive integers n and p with $n > p \geq 1$,*

$$S_{n,p}(z) = (-1)^{p-1} p \binom{n}{p} \sum_{k=0}^{p-1} \frac{(-1)^k}{n-k} \left(\sum_{m=0}^{k} \binom{q-1}{m} s(p,q) z^m \right) B_{n-k}(z) \tag{39}$$

where $q = p - k + m$.

Proof. Because of (32) it suffices to show that $B_n^{(p)}(x)$, given by (34), equals the right-hand side of (39). Towards this end, suppose $\sigma \in A_{p-1}(k)$ with $\sigma = \{i_1, i_2, ..., i_k\}$ and $\bar{\sigma} = \{j_1, j_2, ..., j_{p-k-1}\}$. Then since $a(p,n) = 1 - \frac{n}{p}$ and $b(p,n) = n\left(\frac{z}{p} - 1\right)$, we have

$$\begin{aligned}\pi_{a,b}(\bar{\sigma},\sigma;n) &= \prod_{s=1}^{p-k-1} a(j_s, n-R_\sigma(j_s)) \prod_{r=1}^{k} b(i_r, n-R_\sigma(i_r)) \\ &= \prod_{s=1}^{p-k-1}\left(1-\frac{n-R_\sigma(j_s)}{j_s}\right)\prod_{r=1}^{k}[(n-R_\sigma(i_r))\left(\frac{z}{i_r}-1\right)] \\ &= \frac{(-1)^{p-k-1}}{(p-1)!}\prod_{s=1}^{p-k-1}(n-R_\sigma(j_s)-j_s)\prod_{r=1}^{k}(n-R_\sigma(i_r))\prod_{r=1}^{k}(z-i_r)\end{aligned}$$

where we have used the fact that $i_1 \cdots i_k j_1 \cdots j_{p-k-1} = (p-1)!$. By the previous lemma, the products involving R_σ can be combined into

$$\prod_{s=1}^{p-k-1}(n-R_{\bar{\sigma}}(j_s)-j_s)\prod_{r=1}^{k}(n-R_\sigma(i_r)) = \prod_{\substack{r=0 \\ r\neq k}}^{p-1}(n-r) = \frac{(n)_p}{n-k}$$

which yields

$$\pi_{a,b}(\bar{\sigma},\sigma;n) = \frac{(-1)^{p-k-1}}{(p-1)!}\cdot\frac{(n)_p}{n-k}\cdot\prod_{r=1}^{k}(z-i_r).$$

It follows that

$$\sum_{\sigma\in A_{p-1}(k)}\pi_{a,b}(\bar{\sigma},\sigma;n) = (-1)^{p-k-1}\frac{p}{n-k}\binom{n}{p}\sum_{\sigma\in A_{p-1}(k)}\prod_{r=1}^{k}(z-i_r).$$

Now, write the polynomial $P_\sigma(z) \equiv \prod_{r=1}^{k}(z-i_r)$, which has degree k, in the form

$$P_\sigma(z) = c_k(\sigma)z^k + c_{k-1}(\sigma)z^{k-1} + ... + c_0(\sigma).$$

Here each coefficient $c_m(\sigma)$ corresponding to z^m is a sum of terms of the form $(-1)^{k-m} i_{r_1} i_{r_2} ... i_{r_{k-m}}$, where $\{i_{r_1}, ..., i_{r_{k-m}}\}$ is an $(k-m)$-element subset of σ. Since σ is a k-element subset of $\{1, ..., p-1\}$, we claim that there are exactly $\binom{p-1-k+m}{m}$ subsets $\sigma \in A_{p-1}(k)$ that contain $\{i_{r_1}, ..., i_{r_{k-m}}\}$. This is because, assuming the elements $\{i_{r_1}, ..., i_{r_{k-m}}\}$ have already been chosen, we can fill in the remaining m elements of σ by choosing them from the $(p-1-k+m)$ un-chosen elements. It follows that

$$\begin{aligned}\sum_{\sigma\in A_{p-1}(k)} P_\sigma(z) &= \sum_{m=0}^{k}\binom{q-1}{m}\left(\sum_{\{r_1,\ldots,r_{k-m}\}\in A_{p-1}(k-m)}(-1)^{k-m} i_{r_1} i_{r_2} ... i_{r_{k-m}}\right) z^m \\ &= \sum_{m=0}^{k}\binom{q-1}{m} s(p,q) z^m\end{aligned}$$

where $q = p - k + m$. Thus,

$$\begin{aligned} S_{n,p}(x) &= B_n^{(p)}(x) \\ &= \sum_{k=0}^{p-1} \left(\sum_{\sigma \in A_{p-1}(k)} \pi_{a,b}(\bar{\sigma}, \sigma; n) \right) B_{n-k}(z) \\ &= \sum_{k=0}^{p-1} \left((-1)^{p-k-1} \frac{p}{n-k} \binom{n}{p} \sum_{\sigma \in A_{p-1}(k)} P_\sigma(z) \right) B_{n-k}(z) \\ &= (-1)^{p-1} p \binom{n}{p} \sum_{k=0}^{p-1} \frac{(-1)^k}{n-k} \left(\sum_{m=0}^{k} \binom{q-1}{m} s(p,q) z^m \right) B_{n-k}(z) \end{aligned}$$

as desired. □

4. Generalization to Three-Term Recurrences

Our results for two-term triangular recurrences can be generalized to three-term recurrences of the form

$$x(n,k) = a(n,k)x(n-1,k) + b(n,k)x(n-1,k-1) + c(n,k)x(n-1,k-2) \quad (40)$$

where $a(n,k)$, $b(n,k)$ and $c(n,k)$ are given arbitrary two-dimensional sequences. Towards this end, let $\sigma = \{i_1, i_2, ..., i_{m_1}\} \in A_n(m_1)$ and $\tau = \{j_1, j_2, ..., j_{m_2}\} \in A_n(m_2)$ be two disjoint subsets of $\{1, 2, ..., n\}$. We define the *rank* of a positive integer j with respect to (σ, τ) to be

$$R_{\sigma,\tau}(j) = |\{i \in \sigma : j < i\}| + 2\,|\{i \in \tau : j < i\}|. \quad (41)$$

Let $\mathcal{P}_3(A_n)$ denote the set of 3-block ordered partitions of the form $(\sigma_1, \sigma_2, \sigma_3)$, where σ_1, σ_2, and σ_3 are mutually disjoint and whose union equals A_n. Write out the elements of these three sets explicitly as

$$\begin{aligned} \sigma_1 &= \{j_1, j_2, ..., j_{n-m_1-m_2}\} \\ \sigma_2 &= \{i_1, i_2, ..., i_{m_2}\} \\ \sigma_3 &= \{h_1, h_2, ..., h_{m_1}\} \end{aligned}$$

and define the product

$$\pi_{a,b,c}(\sigma_1, \sigma_2, \sigma_3; k) =$$

$$\prod_{s=1}^{m_3} a(j_s, k - R_{\sigma_2,\sigma_3}(j_s)) \prod_{r=1}^{m_2} b(i_r, k - R_{\sigma_2,\sigma_3}(i_r)) \prod_{q=1}^{m_1} c(h_q, k - R_{\sigma_2,\sigma_3}(h_q)) \quad (42)$$

where $m_3 = n - m_1 - m_2$.

We conjecture the following formula for $x(n,k)$, which generalizes (29) to three-term recurrences:

$$x(n,k) = \sum_{(\sigma_1,\sigma_2,\sigma_3)\in\mathcal{P}_3(A_n)} \pi_{a,b,c}(\sigma_1,\sigma_2,\sigma_3;k)f(k-|\sigma_2|-2|\sigma_3|). \qquad (43)$$

Formulas (29) and (43) can of course be naturally extended to higher order n-term recurrences for $n \geq 4$.

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