ON q-ANALOG OF WOLSTENHOLME TYPE CONGRUENCES FOR MULTIPLE HARMONIC SUMS

Jianqiang Zhao
Department of Mathematics, Eckerd College, St. Petersburg, Florida
zhaoj8888@gmail.com

Received: 9/25/12 , Accepted: 4/20/13, Published: 4/25/13

Abstract

Multiple harmonic sums are iterated generalizations of harmonic sums. Recently Dilcher has considered congruences involving q-analogs of these sums in depth one. In this paper we shall study the homogeneous case for arbitrary depth by using generating functions and shuffle relations of the q-analog of multiple harmonic sums. At the end, we also consider some non-homogeneous cases.

1. Introduction

In [8] Shi and Pan extended Andrews' result [1] on the q-analog of Wolstenholme Theorem to the following two cases: for all prime $p \geq 5$

$$\sum_{j=1}^{p-1} \frac{1}{[j]_q} \equiv \frac{p-1}{2}(1-q) + \frac{p^2-1}{24}(1-q)^2[p]_q \pmod{[p]_q^2}, \tag{1}$$

$$\sum_{j=1}^{p-1} \frac{1}{[j]_q^2} \equiv -\frac{(p-1)(p-5)}{12}(1-q)^2 \pmod{[p]_q}, \tag{2}$$

where $[n]_q = (1-q^n)/(1-q)$ for any $n \in \mathbb{N}$ and $q \neq 1$. This type of congruences is considered in the polynomial ring $\mathbb{Z}[q]$ throughout this paper. Notice that the modulus $[p]_q$ is an irreducible polynomial in q when p is a prime. In [3] Dilcher generalized the above two congruences further to sums of the form $\sum_{j=1}^{p-1} \frac{1}{[j]_q^n}$ and $\sum_{j=1}^{p-1} \frac{q^n}{[j]_q^n}$ for all positive integers n in terms of certain determinants of binomial coefficients. However, his modulus is always $[p]_q$. He also expressed these congruences using Bernoulli numbers, Bernoulli numbers of the second kind, and Stirling numbers of the first kind, which we briefly recall now.

The well-known Bernoulli numbers are defined by the following generating series:

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = 1 - \frac{1}{2}\frac{x}{1!} + \frac{1}{6}\frac{x^2}{2!} - \frac{1}{30}\frac{x^4}{4!} + \cdots .$$

On the other hand, the Bernoulli numbers of the second kind are defined by the power series (cf. [7, p. 114]).

$$\frac{x}{\log(1+x)} = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = 1 + \frac{1}{2}\frac{x}{1!} - \frac{1}{6}\frac{x^2}{2!} + \frac{1}{4}\frac{x^3}{3!} - \frac{19}{24}\frac{x^4}{4!} + \cdots.$$

This is a little different from the definition of $\tilde{b}_n$ in [3], which is changed to b_n later in the same paper. Finally, the Stirling numbers of the first kind $s(n,j)$ are defined by

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{j=0}^{n} s(n,j)x^j.$$

Define

$$K_n(p) := (-1)^{n-1}\frac{b_n}{n!} - \frac{(-1)^n}{(n-1)!}\sum_{j=1}^{[n/2]} \frac{B_{2j}}{2j} s(n-1, 2j-1)p^{2j}. \tag{3}$$

By [3, Thm. 1, (6.5) and Thm. 4] and [4, Thm. 3.1] one gets:

Theorem 1.1. *If $p > 3$ is a prime, then for all integers $n > 1$ we have*

$$\sum_{j=1}^{p-1} \frac{q^j}{[j]_q^n} \equiv K_n(p)(1-q)^n \pmod{[p]_q}.$$

We will need the following easy generalization of this theorem.

Theorem 1.2. *If $p > 3$ is a prime, then for every integer $n > t \geq 1$ we have*

$$\sum_{j=1}^{p-1} \frac{q^{tj}}{[j]_q^n} \equiv (1-q)^n \sum_{i=0}^{t-1} \binom{t-1}{i} (-1)^i K_{n-i}(p) \pmod{[p]_q}. \tag{4}$$

Moreover,

$$\sum_{j=1}^{p-1} \frac{1}{[j]_q^n} \equiv (1-q)^n \left(\frac{p-1}{2} + \sum_{j=2}^{n} K_j(p) \right) \pmod{[p]_q}. \tag{5}$$

Proof. If $t > 1$ it is clear that

$$q^{tj} = q^j \left(1 - (1-q^j)\right)^{t-1} = q^j \sum_{i=0}^{t-1} \binom{t-1}{i} (-1)^i (1-q^j)^i.$$

So (4) follows from Theorem 1.1 immediately. Congruence (5) is a variation of [3, (5.11)]. □

All the sums in Theorem 1.1 and 1.2 are special cases of the q-analog of multiple harmonic sums. The congruence properties of the classical multiple harmonic sums (MHS for short) are systematically investigated in [10]. In this paper we shall study their q-analogs which are natural generalizations of the congruences obtained by Shi and Pan [8] and Dilcher [3].

Similar to its classical case (cf. [10]) a q-analog of multiple harmonic sum (q-MHS for short) is defined as follows. For $\mathbf{s} := (s_1, \dots, s_\ell) \in \mathbb{N}^\ell$, $\mathbf{t} := (t_1, \dots, t_\ell) \in \mathbb{N}^\ell$ and $n \in \mathbb{Z}_{\geq 0}$ set

$$H_q^{(\mathbf{t})}(\mathbf{s}; n) := \sum_{1 \le k_1 < \cdots < k_\ell \le n} \frac{q^{k_1 t_1 + \cdots + k_\ell t_\ell}}{[k_1]_q^{s_1} \cdots [k_\ell]_q^{s_\ell}}, \qquad H_q^{*(\mathbf{t})}(\mathbf{s}; n) = H_q^{(\mathbf{t})}(\mathbf{s}; n)/(1-q)^{w(\mathbf{s})}, \tag{6}$$

where $w(\mathbf{s}) := s_1 + \cdots + s_\ell$ is the *weight*, $\ell(\mathbf{s}) := \ell$ the *depth* and $\mathbf{t}$ the *modifier*. For trivial modifier we set

$$H_q(\mathbf{s}; n) := H_q^{(0,\dots,0)}(\mathbf{s}; n), \qquad H_q^*(\mathbf{s}; n) = H_q(\mathbf{s}; n)/(1-q)^{w(\mathbf{s})}.$$

Note that in [3] $\tilde{H}_q(s; p-1) := H_q^{(1)}(s; p-1)$ are studied in some detail and are related to $H_q(s; p-1)$. Also note that $H_q^{(s_1-1,\dots,s_\ell-1)}(\mathbf{s}; n)$ are the partial sums of the most convenient form of q-multiple zeta functions (see [9]).

In this paper we mainly consider q-MHS with the trivial modifier. By convention we set $H_q^{(\mathbf{t})}(\mathbf{s}; r) = 0$ for $r = 0, \dots, \ell(\mathbf{s}) - 1$, and $H_q^{(\mathbf{t})}(\emptyset; n) = 1$. To save space, for an ordered set $(e_1, \dots, e_t)$ we denote by $\{e_1, \dots, e_t\}^d$ the ordered set formed by repeating $(e_1, \dots, e_t)$ d times. For example, $H_q(\{s\}^\ell; n)$ will be called a *homogeneous* sum.

Throughout the paper, we use short-hand $H_q(\mathbf{s})$ to denote $H_q(\mathbf{s}; p-1)$ for some fixed prime p.

2. Homogeneous q-MHS

It is extremely beneficial to study the so-called stuffle (or quasi-shuffle) relations among MHS (see, for e.g., [10]). The same mechanism works equally well for q-MHS.

Recall that for any two ordered sets $(r_1, \dots, r_t)$ and $(r_{t+1}, \dots, r_n)$ the shuffle operation is defined by

$$\mathrm{Shfl}\big((r_1, \dots, r_t), (r_{t+1}, \dots, r_n)\big) := \bigcup_{\substack{\sigma \text{ permutes } \{1,\dots,n\}, \\ \sigma^{-1}(1) < \cdots < \sigma^{-1}(t), \\ \sigma^{-1}(t+1) < \cdots < \sigma^{-1}(n)}} \big(r_{\sigma(1)}, \dots, r_{\sigma(n)}\big).$$

Fix a positive integer s. For any $k = 1, \dots, \ell-1$, by stuffle relation we have

$$H_q^*\big((\ell-k)s\big) \cdot H_q^*\big(\{s\}^k\big) = \sum_{\mathbf{s} \in \mathrm{Shfl}\left(\{(\ell-k)s\}, \{s\}^k\right)} H_q^*(\mathbf{s}) + \sum_{\mathbf{s} \in \mathrm{Shfl}\left(\{(\ell-k+1)s\}, \{s\}^{k-1}\right)} H_q^*(\mathbf{s}).$$

Applying $\sum_{k=1}^{\ell-1}(-1)^{\ell-k-1}$ on both sides we get

$$H_q^*(\{s\}^\ell) = \frac{1}{\ell}\sum_{k=0}^{\ell-1}(-1)^{\ell-k-1}H_q^*((\ell-k)s)\cdot H_q^*(\{s\}^k). \tag{7}$$

Theorem 2.1. *Let s be a positive integer and let $\eta_s = \exp(2\pi\sqrt{-1}/s)$ be the sth primitive root of unity. Then*

$$\sum_{\ell=0}^{\infty} H_q^*(\{s\}^\ell)x^\ell \equiv \frac{(-1)^s}{p^s x}\prod_{n=0}^{s-1}\Big(1-(1-\eta_s^n(-x)^{1/s})^p\Big) \pmod{[p]_q}.$$

Proof. Let $\zeta = \exp(2\pi\sqrt{-1}/p)$ be the primitive pth root of unity and set

$$P_n = \sum_{j=1}^{p-1}\frac{1}{(1-\zeta^j)^n}. \tag{8}$$

It is easy to see that $H_q^*(n) \equiv P_n \pmod{[p]_q}$. By using partial fractions Dilcher [4, (4.2)] obtained essentially the following generating function of P_n:

$$g(x) := \sum_{n=0}^{\infty} P_n x^n = -\frac{px(x-1)^{p-1}}{1-(1-x)^p}. \tag{9}$$

Let $a_\ell = H_q^*(\{s\}^\ell)$ for all $\ell \geq 0$. Let $w(x) = \sum_{\ell=0}^{\infty} a_\ell x^\ell$ be its the generating function. By (7) we get

$$w(x) = \sum_{\ell=0}^{\infty} a_\ell x^\ell \equiv 1 + \sum_{\ell=1}^{\infty}\frac{1}{\ell}\sum_{k=0}^{\ell-1}(-1)^{\ell-k-1}P_{(\ell-k)s}a_k x^\ell \pmod{[p]_q}.$$

Differentiating both sides and changing index $\ell \to \ell+1$ we get, modulo $[p]_q$,

$$w'(x) \equiv \sum_{\ell=0}^{\infty}\sum_{k=0}^{\ell}(-1)^{\ell-k}P_{(\ell-k+1)s}a_k x^\ell \equiv \sum_{k=0}^{\infty}\sum_{\ell=k}^{\infty}(-1)^{\ell-k}P_{(\ell-k+1)s}a_k x^\ell.$$

Changing index $\ell \to \ell+k$ and then exchanging the order of summation we get

$$\begin{aligned} w'(x) \equiv & w(x)\sum_{\ell=0}^{\infty}P_{(\ell+1)s}(-x)^\ell \equiv \frac{w(x)}{-x}\left(\sum_{\ell=0}^{\infty}P_{\ell s}(-x)^\ell + 1\right) \\ \equiv & \frac{w(x)}{-sx}\left(s + \sum_{n=0}^{s-1}\sum_{\ell=0}^{\infty}P_\ell(\eta_s^n(-x)^{1/s})^\ell\right) \\ \equiv & \frac{w(x)}{-sx}\left(s + \sum_{n=0}^{s-1}g\big(\eta_s^n(-x)^{1/s}\big)\right) \\ \equiv & \frac{w(x)}{-sx}\left(s - \sum_{n=0}^{s-1}\frac{p\eta^n(-x)^{1/s}(\eta^n(-x)^{1/s}-1)^{p-1}}{1-(1-\eta_s^n(-x)^{1/s})^p}\right). \end{aligned}$$

Here $\eta_s = \exp(2\pi\sqrt{-1}/s)$ is the sth primitive root of unity. Thus

$$(\ln w(x))' = \left(-(\ln x)' + \sum_{n=0}^{s-1} \frac{(1-(1-\eta^n(-x)^{1/s})^p)'}{1-(1-\eta_s^n(-x)^{1/s})^p}\right).$$

Therefore by comparing the constant term we get

$$w(x) \equiv \frac{(-1)^s}{p^s x} \prod_{n=0}^{s-1} \left(1-(1-\eta_s^n(-x)^{1/s})^p\right) \pmod{[p]_q}$$

as desired. □

Corollary 2.2. *For every positive integer $\ell < p$ we have*

$$H_q\big(\{1\}^\ell\big) \equiv \frac{1}{\ell+1}\binom{p-1}{\ell} \cdot (1-q)^\ell \qquad \pmod{[p]_q}.$$

Proof. By the theorem we get

$$\sum_{\ell=0}^{\infty} H_q^*\big(\{1\}^\ell\big)x^\ell \equiv \frac{(1+x)^p-1}{px}$$
$$\equiv \frac{1}{px}\sum_{\ell=0}^{\infty}\binom{p}{\ell+1}x^{\ell+1} \equiv \sum_{\ell=0}^{\infty}\frac{1}{\ell+1}\binom{p-1}{\ell}x^\ell \pmod{[p]_q}.$$

The corollary follows immediately. □

Corollary 2.3. *For every positive integer $\ell < p$ we have*

$$H_q\big(\{2\}^\ell\big) \equiv (-1)^\ell \frac{2\cdot \ell!}{(2\ell+2)!}\binom{p-1}{\ell} \cdot F_{2,\ell}(p) \cdot (1-q)^{2\ell} \pmod{[p]_q},$$

where $F_{2,\ell}(p)$ is a monic polynomial in p of degree ℓ.

Proof. By Theorem 2.1 we have modulo $[p]_q$

$$\sum_{\ell=0}^{\infty} H_q^*\big(\{2\}^\ell\big)x^\ell \equiv \frac{1}{p^2x}\Big(1-(1-\sqrt{-1}\sqrt{x})^p\Big)\Big(1-(1+\sqrt{-1}\sqrt{x})^p\Big)$$
$$\equiv \frac{1}{p^2x}\left|\sum_{j=1}^{(p-1)/2}\binom{p}{2j}(-1)^j x^j + \sqrt{-1}\sqrt{x}\sum_{j=0}^{(p-1)/2}\binom{p}{2j+1}(-1)^j x^j\right|^2,$$

which easily yields

$$H_q^*\big(\{2\}^\ell\big) \equiv \frac{(-1)^\ell}{p^2}\left\{\sum_{\substack{j+k=\ell\\0\le j,k<p/2}}\binom{p}{2j+1}\binom{p}{2k+1} - \sum_{\substack{j+k=\ell+1\\1\le j,k<p/2}}\binom{p}{2j}\binom{p}{2k}\right\}.$$

In the first sum above if $j+k=\ell+1$ and $1\le j,k<p/2$ then we may assume $j>\ell/2$. Then $(\ell+1)!\binom{p}{\ell+1}$ is a factor of $(2j+1)!\binom{p}{2j+1}$ as a polynomial of p, and so is $\ell!\binom{p-1}{\ell}$. Similarly we can see that $\ell!\binom{p-1}{\ell}$ is a factor of the second sum.

In order to determine the leading coefficient we set

$$C_1(x)=\sum_{j=0}^{\ell}\frac{(2\ell+2)!\,x^{2j+1}}{(2j+1)!(2l-2j+1)!}=\frac{(x+1)^{2\ell+2}-(x-1)^{2\ell+2}}{2},$$
$$C_2(x)=\sum_{j=0}^{\ell+1}\frac{(2\ell+2)!\,x^{2j}}{(2j)!(2l-2j+2)!}=\frac{(x+1)^{2\ell+2}+(x-1)^{2\ell+2}}{2}.$$

Hence

$$\sum_{\substack{j+k=\ell\\ 0\le j,k<p/2}}\frac{1}{(2j+1)!(2k+1)!}-\sum_{\substack{j+k=\ell+1\\ 1\le j,k<p/2}}\frac{1}{(2j)!(2k)!}$$
$$=\frac{C_1(1)-(C_2(1)-2)}{(2\ell+2)!}=\frac{2}{(2\ell+2)!}.$$

This finishes the proof of the corollary. □

Corollary 2.4. *Let ℓ be a positive integer. Set $\delta_\ell=(1+(-1)^\ell)$ and $L=3\ell+3$. Then for every prime $p\ge L$ we have modulo $[p]_q$*

$$H_q\big(\{3\}^\ell\big)\equiv\begin{cases}\dfrac{-3\cdot\ell!}{(3\ell+1)!}\dbinom{p-1}{\ell}\cdot F_{3,\ell}(p)\cdot(1-q)^{3\ell} & ,\quad \text{if } \ell \text{ is odd},\\ \dfrac{6\cdot\ell!}{(3\ell+3)!}\dbinom{p-1}{\ell}\cdot F_{3,\ell}(p)\cdot(1-q)^{3\ell} & ,\quad \text{if } \ell \text{ is even},\end{cases}\tag{10}$$

where $F_{3,\ell}(p)$ is a monic polynomial in p of degree $2\ell-1$ if ℓ is odd and of degree 2ℓ if ℓ is even.

Proof. Let $\eta=\exp(2\pi i/3)$. Then $\eta^2+\eta+1=0$. By Theorem 2.1 we have

$$\sum_{\ell=0}^{\infty}H_q^*\big(\{3\}^\ell\big)x^\ell\equiv\frac{-1}{p^3x}\prod_{a=0}^{2}\Big(1-(1-\eta^a\sqrt[3]{-x})^p\Big).\tag{11}$$

We now use two ways to expand this. Set $y=\sqrt[3]{-x}$. First, the product on the right hand side of (11) can be expressed as

$$
\begin{aligned}
&1-\sum_{a=0}^{2}(1-\eta^a y)^p+\sum_{a=0}^{2}(1-\eta^a y)^p(1-\eta^{a+1}y)^p-\prod_{a=0}^{2}(1-\eta^a y)^p\\
=&1-\sum_{j=0}^{p}\binom{p}{j}\sum_{a=0}^{2}\eta^{aj}y^j+\sum_{a=0}^{2}(1+\eta^a y+\eta^{a+1}y^2)^p-(1+x)^p\\
=&1-3\sum_{j=0}^{[p/3]}\binom{p}{3j}x^j+3\sum_{\substack{j,k\geq 0,j+k<p\\ 2j+k\equiv 0(3)}}\frac{p!\,(-x)^{(j+2k)/3}}{j!k!(p-j-k)!}-(1+x)^p.
\end{aligned}
$$

Thus for $\ell>0$ we get

$$
H_q^*(\{3\}^\ell)\equiv\frac{1}{p^3}\left\{3\delta_\ell\binom{p}{L}+(-1)^\ell\cdot 3\sum_{k\geq 1}\binom{p}{L-k}\binom{L-k}{k}+\binom{p}{\ell+1}\right\}
$$

Note that if ℓ is odd then the degree of the polynomial is reduced to $3\ell-1$ with leading coefficient given by

$$
(-1)^\ell\cdot 3\frac{1}{(L-1)!}\binom{L-1}{1}=\frac{-3}{(L-2)!}=\frac{-3}{(3\ell+1)!}
$$

as we wanted.

Now to prove $\ell!\binom{p}{\ell}$ is a factor we use the following expansion of (11):

$$
\sum_{\ell=0}^{\infty}\frac{1}{p^3x}\sum_{j,k,n\geq 1}(-1)^{j+k+n}\binom{p}{j}\binom{p}{k}\binom{p}{n}x^{(j+k+n)/3}\eta^{k+2n}.
$$

Thus

$$
H_q^*(\{3\}^\ell)\equiv\frac{1}{p^3}\sum_{\substack{1\leq j,k,n\leq p\\ j+k+n=3\ell+3}}(-1)^{j+k+n}\binom{p}{j}\binom{p}{k}\binom{p}{n}\eta^{k+2n}\qquad\pmod{[p]_q}.
$$

Notice that $j+k+n=3\ell+3$ implies one of the indices, say j, is at least $\ell+1$. Then clearly $\binom{p}{j}$ contains $\ell!\binom{p}{\ell}$ as a factor, therefore so does $H_q^*(\{3\}^\ell)\pmod{[p]_q}$. This completes the proof of the corollary. □

3. Some Non-Homogeneous q-MHS Congruences

In this section we consider some non-homogeneous q-MHS of depth two with modifiers of special type.

Theorem 3.1. *Let m, n be two positive integers. For every prime p we have*

$$H_q^{(m,n)}(2m, 2n) \equiv \frac{1}{2}\left\{f(m;p)f(n;p) - f(m+n;p)\right\} \pmod{[p]_q}.$$

where

$$f(N;p) = (1-q)^{2N}\sum_{i=0}^{N-1}\binom{N-1}{i}(-1)^i K_{2N-i}(p)$$

Proof. By definition and substitution $i \to p-i$ and $j \to p-j$ we have

$$\begin{aligned}
H_q^{*(m,n)}(2m,2n) &= \sum_{1\le i<j<p}\frac{q^{mi+nj}}{(1-q^i)^{2m}(1-q^j)^{2n}} \\
&= \sum_{1\le j<i<p}\frac{q^{pm+pn-mi-nj}}{(1-q^{p-i})^{2m}(1-q^{p-j})^{2n}} \\
&\equiv \sum_{1\le j<i<p}\frac{q^{mi+nj}}{(q^i-q^p)^{2m}(q^j-q^p)^{2n}} && \pmod{[p]_q} \\
&\equiv \sum_{1\le j<i<p}\frac{q^{mi+nj}}{(1-p^i)^{2m}(1-p^j)^{2n}} && \pmod{[p]_q} \\
&\equiv H_q^{*(n,m)}(2n,2m) && \pmod{[p]_q} \qquad (12)
\end{aligned}$$

By shuffle relation we have

$$H_q^{*(m)}(2m)H_q^{*(n)}(2n) = H_q^{*(m,n)}(2m,2n) + H_q^{*(n,m)}(2n,2m) + H_q^{*(m+n)}(2m+2n).$$

Together with (12) this yields

$$2H_q^{*(m,n)}(2m,2n) \equiv H_q^{*(m)}(2m)H_q^{*(n)}(2n) - H_q^{*(m+n)}(2m+2n) \pmod{[p]_q}.$$

Our theorem follows from (4) quickly. □

In the study of q-multiple zeta functions the following function appears naturally (see [9, (47)] or [2, Theorem 1]):

$$\varphi_q(n) = \sum_{k=1}^{\infty}(k-1)\frac{q^{(n-1)k}}{[k]_q^n} = \sum_{k=1}^{\infty}\frac{kq^{(n-1)k}}{[k]_q^n} - \zeta_q(n),$$

where $\zeta_q(n) = \sum_{k=1}^{\infty}\frac{q^{(n-1)k}}{[k]_q^n}$ is the q-Riemann zeta value defined by Kaneko et al. in [5]. Using the results we have obtained so far in this paper we discover a congruence related to the partial sums of $\varphi_q(2)$.

Proposition 3.2. *For every prime p we have*

$$\sum_{k=1}^{p-1}\frac{kq^k}{[k]_q^2} \equiv -\frac{p(p-1)(p+1)}{24}(1-q)^2 \pmod{[p]_q}. \qquad (13)$$

Proof. To save space, all congruences are modulo $[p]_q$ throughout this proof.

We can check the congruence (13) for $p = 2$ and $p = 3$ easily by hand. Now we assume $p \geq 5$. By definition we have

$$H_q^*(2,1) = \sum_{1 \leq i < j < p} \frac{1}{(1-q^i)^2(1-q^j)}.$$

With the substitutions $i \to p - i$ and $j \to p - j$ we get

$$\begin{aligned}
-H_q^*(2,1) &= -\sum_{1 \leq j < i < p} \frac{q^{2i} \cdot q^j}{(q^i - q^p)^2(q^j - q^p)} \\
&\equiv -\sum_{1 \leq j < i < p} \frac{q^{2i} \cdot q^j}{(q^i - 1)^2(q^j - 1)} \\
&\equiv -\sum_{1 \leq j < i < p} \frac{(q^i-1)^2 + 2(q^i-1) + 1}{(q^i-1)^2} \cdot \frac{1 - q^j - 1}{1 - q^j} \\
&\equiv H_q^*(1,2) - 2H_q^*(1,1) + \sum_{k=1}^{p-1} \frac{p-3+k}{1-q^k} - \sum_{k=1}^{p-1} \frac{k-1}{(1-q^k)^2} - \binom{p-1}{2} \\
&\equiv H_q^*(1,2) - 2H_q^*(1,1) + (p-3)H_q^*(1) + H_q^*(2) \\
&\qquad\qquad - \binom{p-1}{2} - \sum_{k=1}^{p-1} \frac{kq^k}{(1-q^k)^2}.
\end{aligned}$$

Notice that we have the stuffle relations

$$\begin{aligned}
H_q^*(2,1) + H_q^*(1,2) &= H_q^*(1)H_q^*(2) - H_q^*(3), \\
2H_q^*(1,1) &= H_q^*(1)^2 - H_q^*(2).
\end{aligned}$$

Hence

$$\sum_{k=1}^{p-1} \frac{kq^k}{(1-q^k)^2} \equiv (H_q^*(1) + 2)H_q^*(2) - H_q^*(3) - H_q^*(1)^2 + (p-3)H_q^*(1) - \binom{p-1}{2}.$$

Notice that by [3, Theorem 2]

$$H_q^*(3) \equiv -\frac{(p-1)(p-3)}{8}. \tag{14}$$

The proposition now follows from (1) and (2) immediately. □

4. A Congruence of Lehmer Type

Instead of the harmonic sums up to $(p-1)$-st term Lehmer also studied the following type of congruence (see [6]): for every odd prime p

$$\sum_{j=1}^{(p-1)/2} \frac{1}{j} \equiv -2q_p(2) + q_p(2)^2 p \pmod{p^2},$$

where $q_p(2) = (2^{p-1}-1)/p$ is the Fermat quotient. It is also easy to see that for every positive integer n and prime $p > 2n+1$

$$\sum_{j=1}^{(p-1)/2} \frac{1}{j^{2n}} \equiv 0 \pmod{p}.$$

As a q-analog of the above we have

Theorem 4.1. *Let n be a positive integer. For every odd prime p we have*

$$H_q^{(n)}(2n; (p-1)/2) \equiv \frac{1}{2}(1-q)^{2n} \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j K_{2n-j}(p) \pmod{[p]_q}.$$

Proof. By definition and substitution $i \to p-i$ we have

$$\begin{aligned} H_q^{*(n)}(2n) =& H_q^{*(n)}(2n; (p-1)/2) + \sum_{1\le i\le (p-1)/2} \frac{q^{n(p-i)}}{(1-q^{p-i})^{2n}} \\ \equiv& 2H_q^{*(n)}(2n; (p-1)/2) \pmod{[p]_q} \end{aligned}$$

which yields the theorem by (4) easily. □

To conclude the paper we remark that the congruence for general q-MHS should involve some type of q-analog of Bernoulli numbers and Euler numbers similar to the classical cases treated in [10]. We hope to return to this theme in the future.

Acknowledgement. This work is partly supported by the Faculty Development Fund of Eckerd College and the NSF grant DMS1162116.

References

[1] G.E. Andrews, *q-analogs of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher*, Discrete Math., **204**(1) (1999), 15-25.

[2] D.M. Bradley, *A q-analog of Euler's decomposition formula for the double zeta function*, Int. J. Math. Math. Sci., Vol. 2005, Issue 21, 2005, pp. 3453–3458.

[3] K. Dilcher, *Determinant expressions for q-harmonic congruences and degenerate Bernoulli numbers*, Electron. J. Combin., **15**(1) (2008), R63.

[4] F. T. Howard, *Explicit formulas for degenerate Bernoulli numbers*, Discrete Math. **162** (1996), 175–185.

[5] M. Kaneko, N. Kurokawa, and M. Wakayama, *A variation of Euler's approach to values of the Riemann zeta function*, Kyushu J. Math. **57** (2003), 175–192.

[6] E. Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Ann. Math. **39** (1938), 350–360.

[7] S. Roman, The Umbral Calculus, New York: Academic Press, 1984.

[8] L.-L. Shi and H. Pan, *A q-analogue of Wolstenholme's harmonic series congruence*, Amer. Math. Monthly, **114**(6) (2007), 529-531.

[9] J. Zhao, *Multiple q-zeta functions and multiple q-polylogarithms*, Ramanujan J., **14** (2)(2007), 189–221. arXiv: math/0304448

[10] J. Zhao, *Wolstenholme type theorem for multiple harmonic sums*, Int. J. Number Theory, **4**(1) (2008), 73–106.