GENERALIZATION OF UNIVERSAL PARTITION AND BIPARTITION THEOREMS

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Received: 5/24/12, Revised: 5/4/13, Accepted: 8/10/13, Published: 9/26/13

Abstract

Let $A = (a_{i,j})$, $i = 1, 2, \ldots,$ $j = 0, 1, 2, \ldots,$ be an infinite matrix with elements $a_{i,j} = 0$ or 1; $p(n, k; A)$ the number of partitions of n into k parts whose number y_i of parts which are equal to i belongs to the set $Y_i = \{j : a_{i,j} = 1\}$, $i = 1, 2, \ldots.$ The universal theorem on partitions states that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} p(n, k; A)\, u^k t^n = \prod_{i=1}^{\infty}\left(\sum_{j=0}^{\infty} a_{i,j} u^j t^{ij}\right).$$

In this paper, we present a generalization of this result. We show that this generalization remains true when $a_{i,j}$ are indeterminate. We also take into account the bi-partite and multi-partite situations.

1. Introduction

Let $A = (a_{i,j})$, $i = 1, 2, \ldots,$ $j = 0, 1, 2, \ldots$ be an infinite matrix with elements $a_{i,j} = 0$ or 1; $p(n, k; A)$ the number of partitions of n into k parts whose number y_i of parts which are equal to i belongs to the set $Y_i = \{j : a_{i,j} = 1\}$, $i = 1, 2, \ldots.$ The universal theorem on partitions states that

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} p(n, k; A)\, u^k\right) t^n = \prod_{i=1}^{\infty}\left(\sum_{j=0}^{\infty} a_{i,j} u^j t^{ij}\right); \tag{1}$$

[1]The research is partially supported by the LITIS Laboratory of Rouen University and the PNR project 8/u160/664.
[2]This research is supported by the PNR project 8/u160/3172.

see for instance [2] and [3].

In Section 2, we will provide an extension of the above identity and show that it remains true when $a_{i,j}$ are indeterminate. In Section 3, we will present an equivalent version in terms of complete Bell polynomials when $a_{i,0} = 1,\ i \geq 1$.

Similarly, a partition of an ordered pair $(m, n) \neq (0, 0)$, of nonnegative integers, is a non-ordered collection of nonnegative integers $(x_i, y_i) \neq (0, 0)$, $i = 1, 2, \ldots$, whose sum equals (m, n). Given a partition of (m, n), let $k_{i,j}$ be the number of parts which are equal to (i, j), $i = 0, 1, 2, \ldots, m,\ j = 0, 1, 2, \ldots, n,\ (i, j) \neq (0, 0)$, such that

$$\sum_{i=0}^{m} i \sum_{j=0}^{n} k_{i,j} = m, \quad \sum_{j=0}^{n} j \sum_{i=0}^{m} k_{i,j} = n. \tag{2}$$

For a partition of (m, n) into k parts, we add

$$\sum_{i=0}^{m}\sum_{j=0}^{n} k_{i,j} = k. \tag{3}$$

Let $p(m, n)$ be the number of partitions of the bi-partite number (m, n) with $p(0, 0) = 1$ and $p(m, n, k)$ be the number of partitions of (m, n) into k parts with $p(0, 0, 0) = 1$. The universal bipartition theorem states that

$$F(t, u, w) = \sum_{m,n\geq 0} \left(\sum_{k=0}^{m+n} p(m, n, k)\, w^k \right) t^m u^n = \prod_{\substack{j=0 \\ (i,j)\neq(0,0)}}^{\infty} \prod_{i=0}^{\infty} \left(1 - w t^i u^j\right)^{-1}; \tag{4}$$

see [2, p. 403, pb. 24]. A generalization of identity (4) is dealt with in Section 4. Section 5 is devoted to the concept of multipartition.

2. Generalized Universal Partition Theorem

Theorem 1. *Let* $X = (x_{i,j})$, $i = 1, 2, \ldots,\ j = 0, 1, 2, \ldots,$ *be an infinite matrix of indeterminates;* $\pi(n, k)$ *the set of all nonnegative integer solutions of*

$$k_1 + k_2 + \cdots + k_n = k \quad \text{and} \quad k_1 + 2k_2 + \cdots + nk_n = n;$$

and $\pi(n) = \bigcup_{k=1}^{n} \pi(n, k)$ *be the set of all nonnegative integer solutions of* $k_1 + 2k_2 + \cdots + nk_n = n$. *For every solution* $k_1, k_2, \ldots, k_n$, *we set*

$$p(n, k; X) := \sum_{\pi(n,k)} x_{1,k_1} x_{2,k_2} \cdots x_{n,k_n}.$$

Then

$$G(t, u; X) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} p(n, k; X)\, u^k \right) t^n = \prod_{i=1}^{\infty} \left(\sum_{j=0}^{\infty} x_{i,j} u^j t^{ij} \right).$$

Proof. We have

$$G(t,u;X)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\sum_{\pi(n,k)}x_{1,k_1}x_{2,k_2}\cdots x_{n,k_n}u^{k_1+\cdots+k_n}t^{k_1+2k_2+\cdots+nk_n}\right).$$

Since these sums apply for all $k=0,1,\ldots,n$ and $n=0,1,\ldots,$ it follows that

$$\begin{aligned}G(t,u;X)&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\sum_{\pi(n,k)}\left(x_{1,k_1}(ut)^{k_1}\right)\left(x_{2,k_2}\left(ut^2\right)^{k_2}\right)\cdots\left(x_{n,k_n}\left(ut^n\right)^{k_n}\right)\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{\pi(n)}\left(x_{1,k_1}(ut)^{k_1}\right)\left(x_{2,k_2}\left(ut^2\right)^{k_2}\right)\cdots\left(x_{n,k_n}\left(ut^n\right)^{k_n}\right)\right)\\&=\left(\sum_{k_1=0}^{\infty}x_{1,k_1}(ut)^{k_1}\right)\left(\sum_{k_2=0}^{\infty}x_{2,k_2}\left(ut^2\right)^{k_2}\right)\cdots\\&=\prod_{i=1}^{\infty}\left(\sum_{j=0}^{\infty}x_{i,j}\left(ut^i\right)^j\right),\end{aligned}$$

which is the required expression. □

For $x_{i,j}=a_{i,j}$ with $i=1,2,\ldots,\ j=0,1,2,\ldots,$ in Theorem 1, we obtain the universal theorem on partitions. For $x_{i,j}=\frac{a_{i,j}}{j!i^j}$ with $i=1,2,\ldots,\ j=0,1,2,\ldots,$ in Theorem 1, we obtain:

Corollary 2. *Let $A=(a_{i,j}),\ i=1,2,\ldots,\ j=0,1,2,\ldots,$ be an infinite matrix with elements $a_{i,j}=0$ or 1 and $c(n,k;A)$ the number of permutations of a finite set W_n, of n elements, that are decomposed into k cycles such that the number of cycles of length i belongs to the set $Y_i=\{j:a_{i,j}=1\}$. Then*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}c(n,k;A)\,u^k\frac{t^n}{n!}=\prod_{i=1}^{\infty}\left(\sum_{j=0}^{\infty}a_{i,j}\frac{u^j}{j!}\left(\frac{t^i}{i}\right)^j\right).$$

For $x_{i,j}=\frac{1}{j!}\left(\frac{z_i}{i!}\right)^j a_{i,j},\ z_i\in\mathbb{C},\ i=1,2,\ldots,\ j=0,1,2,\ldots,$ in Theorem 1, we obtain a remarkable identity according to the partial Bell polynomials:

Corollary 3. *Let $A=(a_{i,j}),\ i=1,2,\ldots,\ j=0,1,2,\ldots,$ be an infinite matrix with elements $a_{i,j}=0$ or 1 and*

$$B_{n,k;A}(z_1,z_2,\ldots,z_n):=\sum_{\pi(n,k)}\frac{n!}{k_1!\cdots k_n!}\left(\frac{z_1}{1!}\right)^{k_1}\cdots\left(\frac{z_n}{n!}\right)^{k_n}a_{1,k_1}\cdots a_{n,k_n}.$$

Then we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B_{n,k;A}\left(z_1,z_2,\ldots,z_n\right)u^k\frac{t^n}{n!}=\prod_{i=1}^{\infty}\left(\sum_{j=0}^{\infty}\frac{a_{i,j}}{j!}\left(uz_i\frac{t^i}{i!}\right)^j\right).$$

Remark 4. For $a_{i,j}=1,\ i=1,2,\ldots$ and $j=0,1,\ldots$, Corollary 3 gives

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B_{n,k}\left(z_1,z_2,\ldots,z_n\right)u^k\frac{t^n}{n!}=\exp\left(u\sum_{i=1}^{\infty} z_i\frac{t^i}{i!}\right),$$

which is the definition of the partial Bell polynomials $B_{n,k}\left(z_1,\ldots,z_n\right)$. See [1, 3, 4].

3. Connection With the Complete Bell Polynomials

Recall that the complete Bell polynomials $A_n\left(x_1,x_2,\ldots\right)$ are defined by

$$\sum_{n=k}^{\infty} A_n\left(x_1,x_2,\ldots\right)\frac{t^n}{n!}=\exp\left(\sum_{m=1}^{\infty} x_m\frac{t^m}{m!}\right).$$

See [1, 3, 4].

In this section, we provide another formulation of Theorem 1 according to the complete Bell polynomials. We determine the generating functions of the sequences $\left(p\left(n,k;X\right)\right)_n$ and $\left(p\left(n,k;X\right)\right)_k$, where $X=\left(x_{i,j}\right),\ i=1,2,\ldots,\ j=0,1,2,\ldots,$ is an infinite matrix with indeterminates $x_{i,j}$ such that $x_{i,0}=1$ for every $i\geq 1$.

Theorem 5. *Let q,u be indeterminate. Then, for $n\geq 1$, we have*

$$\sum_{j=n}^{\infty} p\left(j,n;X\right)q^j = \frac{1}{n!}A_n\left(\rho_1\left(q;X\right),\rho_2\left(q;X\right),\ldots,\rho_n\left(q;X\right)\right), \tag{5}$$

$$\sum_{j=0}^{n} p\left(n,j;X\right)u^j = \frac{1}{n!}A_n\left(\sigma_1\left(u;X\right),\sigma_2\left(u;X\right),\ldots,\sigma_n\left(u;X\right)\right), \tag{6}$$

where

$$\rho_n\left(q;X\right):=\sum_{i=1}^{\infty} b_n\left(i\right)q^{ni} \quad \text{and} \quad \sigma_n\left(u;X\right):=n!\sum_{k|n} b_k\left(\frac{n}{k}\right)\frac{u^k}{k!},$$

with

$$b_n\left(i\right)=\sum_{k=1}^{n}\left(-1\right)^{k-1}\left(k-1\right)!B_{n,k}\left(1!x_{i,1},2!x_{i,2},\ldots,j!x_{i,j},\ldots\right).$$

Proof. From Theorem 1, we get

$$G(q,u;X) := \sum_{n=0}^{\infty}\sum_{k=0}^{n} p(n,k;X)\,u^k q^n = \exp\left(\sum_{i=1}^{\infty} \ln\left(1+\sum_{j=1}^{\infty}(j!x_{i,j})\frac{(uq^i)^j}{j!}\right)\right).$$

Using the following known expansion (see [2, Theorem 11.17])

$$\ln\left(1+\sum_{k=1}^{\infty} g_k \frac{q^k}{k!}\right) = \sum_{n=1}^{\infty} c_n \frac{q^n}{n!},$$

with $c_n = \sum_{k=1}^{n} (-1)^{k-1}(k-1)!B_{n,k}(g_1, g_2, \ldots)$, we obtain

$$\begin{aligned} G(q,u;X) = \exp\left(\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} b_k(i)\frac{u^k}{k!} q^{ki}\right) &= \exp\left(\sum_{k=1}^{\infty}\frac{u^k}{k!}\sum_{i=1}^{\infty} b_k(i)\, q^{ki}\right) \\ &= \exp\left(\sum_{k=1}^{\infty}\rho_k(q;X)\frac{u^k}{k!}\right) \\ &= 1+\sum_{k=1}^{\infty} A_k(\rho_1(q;X), \ldots, \rho_k(q;X))\frac{u^k}{k!}. \end{aligned}$$

On the other hand, we have

$$G(q,u;X) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} p(n,k;X)\,u^k q^n = \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} p(n,k;X)\,q^n\right)u^k.$$

The first identity follows from identification, where as the second identity follows from the expansion

$$\begin{aligned} G(q,u;X) &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} p(n,k;X)\,u^k\right)q^n \\ &= \exp\left(\sum_{i=1}^{\infty}\ln\left(1+\sum_{j=1}^{\infty}(j!x_{i,j})\frac{(uq^i)^j}{j!}\right)\right) \\ &= \exp\left(\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} b_k(i)\,u^k\frac{q^{ik}}{k!}\right) \\ &= \exp\left(\sum_{n=1}^{\infty} q^n \sum_{k|n} b_k\left(\frac{n}{k}\right)\frac{u^k}{k!}\right) \\ &= \exp\left(\sum_{n=1}^{\infty}\sigma_n(u;X)\frac{q^n}{n!}\right) \\ &= 1+\sum_{n=1}^{\infty} A_n(\sigma_1(u;X), \sigma_2(u;X), \ldots, \sigma_n(u;X))\frac{q^n}{n!}. \end{aligned}$$

□

Corollary 6. *Let α and q be two indeterminates. We then have*

$$A_n\left(\frac{1}{1-q}-\alpha,\ldots,(n-1)!\left(\frac{1}{1-q^n}-\alpha^n\right)\right)=n!\left(\frac{1}{1-q^n}-\alpha\right)\prod_{i=1}^{n-1}\left(1-q^i\right)^{-1}$$

and

$$A_n\left((1-\alpha)\,u,\ldots,(n-1)!\left(\sum_{k|n}ku^{n/k}-(\alpha u)^n\right)\right)=n!\left(p_n\left(u\right)-\alpha u p_{n-1}\left(u\right)\right),$$

where $p_n\left(u\right):=\sum_{j=0}^{n}p\left(n,j\right)u^j$.

Proof. We put in identity (5) $x_{1,0}=1,\ x_{1,j}=q^{-j}\left(1-\alpha\right)$ for $j\geq 1$ and $x_{i,j}=q^{-j}$ for $i\geq 2,\ j\geq 0$, and use identity $B_{n,k}\left(1!,2!,3!,\ldots\right)=\frac{(n-1)!}{(k-1)!}\binom{n}{k}$ (Lah numbers). We obtain

$$\begin{aligned}
b_n\left(1\right)&=(n-1)!\left(1-\alpha^n\right)q^{-n},\ b_n\left(i\right)=(n-1)!q^{-n},\quad i\geq 2,\\
\rho_n\left(q;X\right)&=\sum_{i=1}^{\infty}b_n\left(i\right)q^{ni}=(n-1)!\left(\frac{1}{1-q^n}-\alpha^n\right),\\
p\left(n,k;X\right)&=q^{-k}\sum_{\pi(n,k),\ k_1=0}1+q^{-k}\sum_{\pi(n,k),\ k_1\geq 1}\left(1-\alpha\right)\\
&=q^{-k}\sum_{\pi(n-k,k)}1+\left(1-\alpha\right)q^{-k}\sum_{\pi(n-1,k-1)}1\\
&=q^{-k}\left[p\left(n-k,k\right)+\left(1-\alpha\right)p\left(n-1,k-1\right)\right],
\end{aligned}$$

where $p\left(n,k\right)$ is the number of partitions of n into k parts, which satisfy

$$p\left(n,k\right)=p\left(n-k,k\right)+p\left(n-1,k-1\right).$$

Thus, we obtain

$$\begin{aligned}
\sum_{j=n}^{\infty}p\left(j,n;X\right)q^j&=q^{-n}\left(\sum_{j=n}^{\infty}p\left(j,n\right)q^j-\alpha\sum_{j=n}^{\infty}p\left(j-1,n-1\right)q^j\right)\\
&=\left(\frac{1}{1-q^n}-\alpha\right)\prod_{i=1}^{n-1}\left(1-q^i\right)^{-1},
\end{aligned}$$

which gives the first identity.
For the second identity, we take $x_{1,0}=1,\ x_{1,j}=1-\alpha$ for $j\geq 1$ and $x_{i,j}=1$ for

$i \geq 2$, $j \geq 0$, in relation (6) to get

$$b_n(1) = -(n-1)!(\alpha^n - 1), \ b_n(i) = (n-1)!, \quad i \geq 2,$$
$$\sigma_n(u;X) = n!\sum_{k|n} b_k \binom{n}{k}\frac{u^k}{k!} = -(n-1)!(\alpha u)^n + (n-1)!\sum_{k|n} k u^{n/k},$$
$$p(n,k;X) = p(n,k) - \alpha p(n-1,k-1),$$

thus

$$\sum_{j=0}^{n} p(n,j;X) u^j = p_n(u) - \alpha u p_{n-1}(u),$$

which provides the second identity. □

4. Generalized Universal Bipartition Theorem

In this section, we provide a generalization of identity (4) and deduce some known identities. Let us start with the following example: how do we partition $(2,3)$ into different parts? Let $p(2,3,k)$ be the number of partitions of the bi-partite number $(2,3)$ into k parts, $k = 1,\ldots,5$ and $p(2,3)$ be the total number of partitions of $(2,3)$. We have

$(2,3)$
$- -$ $p(2,3,1) = 1$
$(0,1)+(2,2) = (0,2)+(2,1) = (0,3)+(2,0) = (1,0)+(1,3) = (1,1)+(1,2)$
$- -$ $p(2,3,2) = 5$
$(0,1)+(0,1)+(2,1) = (0,1)+(0,2)+(2,0) = (0,1)+(1,1)+(1,1) =$
$(0,1)+(1,0)+(1,2) = (0,3)+(1,0)+(1,0)$
$- -$ $p(2,3,3) = 5$
$(0,1)+(0,1)+(0,1)+(2,0) = (0,1)+(0,1)+(1,0)+(1,1) =$
$(0,1)+(1,0)+(1,0)+(0,2)$
$- -$ $p(2,3,4) = 3$
$(0,1)+(0,1)+(0,1)+(1,0)+(1,0)$
$- -$ $p(2,3,5) = 1$

$$p(2,3) = \sum_{k=1}^{5} p(2,3,k) = 1+5+5+3+1 = 15.$$

Theorem 7. *Let* $X = (x_{i,j,s})$, $i,j,s = 0,1,2,\ldots,$ *be a sequence of indeterminates with* $x_{0,0,s} = 0$, $\Pi(m,n,k)$ *the set of all nonnegative integers* $k_{i,j}$ *satisfying (2) and (3) and* $\Pi(m,n) := \bigcup_{k=1}^{n+m} \Pi(m,n,k)$ *the set of all nonnegative integers satisfying (2). For every partition of the bi-partite number* (m,n) *into* k *parts, we set*

$$\mathbf{p}(m,n,k;X) := \sum_{\Pi(m,n,k)} \prod_{i=0}^{m}\prod_{j=0}^{n} x_{i,j,k_{i,j}}.$$

Then we have

$$F\left(t,u,\omega;X\right):=\sum_{m,n\geq 0}\left(\sum_{k=0}^{m+n}\mathbf{p}\left(m,n,k;X\right)\omega^{k}\right)t^{m}u^{n}=\prod_{i+j\geq 1}\left(\sum_{s=0}^{\infty}x_{i,j,s}\left(\omega t^{i}u^{j}\right)^{s}\right).$$

Proof. We have

$$\begin{aligned}F\left(t,u,\omega;X\right)&=\sum_{m,n\geq 0}\left(\sum_{k=0}^{m+n}\mathbf{p}\left(m,n,k;X\right)\omega^{k}\right)t^{m}u^{n}\\&=\sum_{m,n\geq 0}\sum_{k=0}^{m+n}\left(\sum_{\Pi(m,n,k)}t^{\sum_{i=0}^{m}i\sum_{j=0}^{n}k_{i,j}}u^{\sum_{j=0}^{n}j\sum_{i=0}^{m}k_{i,j}}\omega^{\sum_{i=0}^{m}\sum_{j=0}^{n}k_{i,j}}\prod_{i=0}^{m}\prod_{j=0}^{n}x_{i,j,k_{i,j}}\right)\\&=\sum_{m,n\geq 0}\sum_{\Pi(m,n)}\left(\prod_{i=0}^{m}\prod_{j=0}^{n}x_{i,j,k_{i,j}}\left(\omega t^{i}u^{j}\right)^{k_{i,j}}\right)\\&=\prod_{i+j\geq 1}\left(\sum_{k_{i,j}=0}^{\infty}x_{i,j,k_{i,j}}\left(\omega t^{i}u^{j}\right)^{k_{i,j}}\right).\end{aligned}$$

□

Corollary 8. *Let* $A=\left(a_{i,j,s}\right),\ i,j,s=0,1,2,\ldots,$ *with* $a_{i,j,s}=0$ *or* 1 *for* $(i,j)\neq(0,0)$ *and let* $\mathbf{p}\left(m,n,k;A\right)$ *be the number of partitions of* (m,n) *into* k *parts whose number* $y_{i,j}$ *of parts which are equal to* (i,j) *belongs to the set* $Y_{i,j}=\{s:a_{i,j,s}=1\}$, $i,j=0,1,2,\ldots,\ (i,j)\neq(0,0)$. *Then*

$$F\left(t,u,\omega;A\right):=\sum_{m,n\geq 0}\left(\sum_{k=0}^{m+n}\mathbf{p}\left(m,n,k;A\right)\omega^{k}\right)t^{m}u^{n}=\prod_{i+j\geq 1}\left(\sum_{s=0}^{\infty}a_{i,j,s}\left(\omega t^{i}u^{j}\right)^{s}\right).$$

For $x_{i,j,s}=1$ for all $i,j,s=0,1,2,\ldots,\ (i,j)\neq(0,0)$, Theorem 7 becomes:

Corollary 9. *Let* $\mathbf{p}\left(m,n\right)$ *be the number of partitions of the bi-partite number* (m,n) *with* $\mathbf{p}\left(0,0\right)=1$ *and* $\mathbf{p}\left(m,n,k\right)$ *the number of partitions of* (m,n) *into* k *parts, with* $\mathbf{p}\left(0,0,0\right)=1$. *Then*

$$\sum_{m,n\geq 0}\left(\sum_{k=0}^{m+n}\mathbf{p}\left(m,n,k\right)\omega^{k}\right)t^{m}u^{n}=\prod_{i+j\geq 1}\left(1-\omega t^{i}u^{j}\right)^{-1}.$$

Remark 10. Let $\left(y_{i,j}\right),\ i,j=0,1,\ldots,$ be a sequence of indeterminates and let $x_{i,j,s}=\frac{1}{s!}\left(\frac{y_{i,j}}{i!j!}\right)^{s}$, $i,j=0,1,2,\ldots,$ we have

$$\mathbf{p}\left(m,n,k;X\right)=\sum_{\Pi(m,n,k)}\prod_{i=0}^{m}\prod_{j=0}^{n}\frac{1}{k_{i,j}!}\left(\frac{y_{i,j}}{i!j!}\right)^{k_{i,j}}=\frac{A_{m,n,k}}{m!n!},$$

where

$$A_{m,n,k} := A_{m,n,k}\left(y_{0,1}, y_{1,0}, \ldots, y_{m,n}\right),$$

and

$$F\left(t, u, \omega; X\right) = \sum_{m,n\geq 0}\left(\sum_{k=0}^{m+n} A_{m,n,k}\omega^{k}\right)\frac{t^{m}}{m!}\frac{u^{n}}{n!}.$$

From Theorem 7, we obtain

$$F\left(t, u, \omega; X\right) = \prod_{i+j\geq 1}\left(\sum_{k_{i,j}\geq 0}\frac{1}{k_{i,j}!}\left(\omega y_{i,j}\frac{t^{i}}{i!}\frac{u^{j}}{j!}\right)^{k_{i,j}}\right) = \exp\left(\omega\left(\sum_{i+j\geq 1} y_{i,j}\frac{t^{i}}{i!}\frac{u^{j}}{j!}\right)\right).$$

From the two expressions of $F\left(t, u; \omega, X\right)$, we retrieve the exponential partial bi-partitional polynomials:

$$\sum_{m,n\geq 0}\left(\sum_{k=0}^{m+n} A_{m,n,k}\left(y_{0,1}, y_{1,0}, \ldots, y_{m,n}\right)\omega^{k}\right)\frac{t^{m}}{m!}\frac{u^{n}}{n!} = \exp\left(\omega\left(\sum_{i+j\geq 1} y_{i,j}\frac{t^{i}}{i!}\frac{u^{j}}{j!}\right)\right);$$

see [2, pp: 454–457].

5. Universal Multipartition Theorem

More generally, a multipartition of order r of $\mathbf{n} = \left(n_1, \ldots, n_r\right)$, different from $\mathbf{0} = \left(0, \ldots, 0\right)$, of nonnegative integers, is a non-ordered collection of nonnegative integers $\left(x_i^{(1)}, \ldots, x_i^{(r)}\right)$, $i = 1, 2, \ldots,$ whose sum equals $\mathbf{n}$. In a partition of an r-partite number $\mathbf{n}$, let $k_{\mathbf{i}} := k_{i_1,\ldots,i_r}$ be the number of ordered r numbers that are equal to $\mathbf{i} = (i_1, \ldots, i_r) \in \{0, 1, 2, \ldots, n_1\} \times \cdots \times \{0, 1, 2, \ldots, n_r\}$, $(i_1, \ldots, i_r) \neq \mathbf{0}$, such that

$$\sum_{i_1=0}^{n_1}\cdots\sum_{i_r=0}^{n_r} i_j k_{i_1,\ldots,i_r} = n_j, \quad j = 1, \ldots, r. \tag{7}$$

For the partition of $\mathbf{n}$ into k parts, we add

$$\sum_{i_1=0}^{n_1}\cdots\sum_{i_r=0}^{n_r} k_{i_1,\ldots,i_r} = k. \tag{8}$$

Let $\mathbf{p}\left(\mathbf{n}\right)$ be the number of partitions of the r-partite $\mathbf{n}$ with $\mathbf{p}\left(\mathbf{0}\right) = 1$ and $\mathbf{p}\left(\mathbf{n}, k\right)$ the number of partitions of the r-partite number $\mathbf{n}$ into k parts, with $\mathbf{p}\left(\mathbf{0}, 0\right) = 1$.

Theorem 11. *Let* $X = \left(x_{\mathbf{i},s}\right)$, $\mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{N}^r$, $\mathbf{i} \neq \mathbf{0}$, $s = 0, 1, 2, \ldots,$ *be a sequence of indeterminates with* $r + 1$ *indices, with* $x_{\mathbf{0},s} = 0$, $\Pi\left(\mathbf{n}, k\right)$ *the set of all nonnegative integers* $k_{i_1,\ldots,i_r}$ *satisfying (7) and (8) and* $\Pi\left(\mathbf{n}\right) := \bigcup_{k=1}^{n_1+\cdots+n_r}\Pi\left(\mathbf{n}, k\right)$

the set of all nonnegative integers solutions of (7). For every partition of $\mathbf{n}$ *into* k *parts, we set*

$$\mathbf{p}(\mathbf{n};k,X) := \sum_{\Pi(\mathbf{n},k)} \prod_{\mathbf{i}=\mathbf{0}}^{\mathbf{n}} x_{\mathbf{i},k_{\mathbf{i}}}.$$

Then

$$F(\mathbf{t},\omega;X) = \sum_{\mathbf{n}\geq\mathbf{0}} \left(\sum_{k=0}^{\mathbf{n}\cdot\mathbf{1}} p(\mathbf{n},k;X)\,\omega^k \right) \mathbf{t}^{\mathbf{n}} = \prod_{\mathbf{i}\cdot\mathbf{1}\geq 1} \left(\sum_{s=0}^{\infty} x_{\mathbf{i},s} \left(\omega \mathbf{t}^{\mathbf{i}}\right)^s \right).$$

where $\mathbf{t}^{\mathbf{n}} := t_1^{n_1} \dots t_r^{n_r}$, $\mathbf{n}\cdot\mathbf{1} : = n_1 + \cdots + n_r$, $\mathbf{n} \geq \mathbf{0} \Leftrightarrow n_1 \geq 0, \dots, n_r \geq 0.$

Proof. We have the following

$$\begin{aligned} F(\mathbf{t},\omega;X) &= \sum_{\mathbf{n}\geq\mathbf{0}} \left(\sum_{k=0}^{\mathbf{n}\cdot\mathbf{1}} p(\mathbf{n},k;X)\,\omega^k \right) \mathbf{t}^{\mathbf{n}} \\ &= \sum_{\mathbf{n}\geq\mathbf{0}} \sum_{k=0}^{\mathbf{n}\cdot\mathbf{1}} \left(\sum_{\Pi(\mathbf{n},k)} \left(\prod_{\mathbf{i}=\mathbf{0}}^{\mathbf{n}} x_{\mathbf{i},k_{\mathbf{i}}} \right) \omega^{\sum_{i_j\leq n_j} k_{\mathbf{i}}} \prod_{j=1}^{r} t_j^{\sum_{i_j\leq n_j} i_j k_{\mathbf{i}}} \right) \\ &= \sum_{\mathbf{n}\geq\mathbf{0}} \sum_{\Pi(\mathbf{n})} \left(\prod_{\mathbf{i}=\mathbf{0}}^{\mathbf{n}} x_{\mathbf{i},k_{\mathbf{i}}} \left(\omega \mathbf{t}^{\mathbf{i}}\right)^{k_{\mathbf{i}}} \right), \end{aligned}$$

and exploiting $x_{\mathbf{0},0} = 0$, the last expression becomes

$$\prod_{\mathbf{i}\geq\mathbf{0}} \left(\sum_{k_{\mathbf{i}}\geq 0} x_{\mathbf{i},k_{\mathbf{i}}} \left(\omega \mathbf{t}^{\mathbf{i}}\right)^{k_{\mathbf{i}}} \right) = \prod_{\mathbf{i}\cdot\mathbf{1}\geq 1} \left(\sum_{k_{\mathbf{i}}\geq 0} x_{\mathbf{i},k_{\mathbf{i}}} \left(\omega \mathbf{t}^{\mathbf{i}}\right)^{k_{\mathbf{i}}} \right).$$

□

For $x_{\mathbf{i},s} = a_{\mathbf{i},s} \in \{0,1\}$ for all $\mathbf{i} \in \mathbb{N}^r$, $\mathbf{i} \neq \mathbf{0}$, $s = 0,1,2,\dots,$ we obtain:

Corollary 12. *Let* $A = (a_{\mathbf{i},s})$, $\mathbf{i} \in \mathbb{N}^r, \mathbf{i} \neq \mathbf{0}$, $s = 0,1,2,\dots,$ *with* $a_{\mathbf{i},s} = 0$ *or* 1 *and* $\mathbf{p}(\mathbf{n},k;A)$ *be the number of partitions of* $\mathbf{n}$ *into* k *parts whose number* $y_{\mathbf{i}}$ *of parts which are equal to* $\mathbf{i}$ *belongs to the set* $Y_{\mathbf{i}} = \{s : a_{\mathbf{i},s} = 1\}$, $\mathbf{i} \in \mathbb{N}^r$, $\mathbf{i} \neq \mathbf{0}$. *Then*

$$F(t,u,\omega;A) := \sum_{\mathbf{n}\geq\mathbf{0}} \left(\sum_{k=0}^{\mathbf{n}\cdot\mathbf{1}} \mathbf{p}(\mathbf{n},k;A)\,\omega^k \right) \mathbf{t}^{\mathbf{n}} = \prod_{\mathbf{i}\cdot\mathbf{1}\geq 1} \left(\sum_{s\geq 0} a_{\mathbf{i},s} \left(\omega \mathbf{t}^{\mathbf{i}}\right)^s \right).$$

For $x_{\mathbf{i},s} = 1$ for all $\mathbf{i} \in \mathbb{N}^r$, $\mathbf{i} \neq \mathbf{0}$, $s = 0,1,2,\dots,$ we obtain:

Corollary 13. *Let* $\mathbf{p}(\mathbf{n})$ *be the number of partitions of the* r*-partite number* $\mathbf{n}$ *with* $\mathbf{p}(\mathbf{0}) = 1$ *and* $\mathbf{p}(\mathbf{n}, k)$ *the number of partitions of the* r*-bipartite number* $\mathbf{n}$ *into* k *parts, with* $\mathbf{p}(\mathbf{0}, 0) = 1$*. Then*

$$\sum_{\mathbf{n}\geq\mathbf{0}} \left(\sum_{k=0}^{\mathbf{n}\cdot\mathbf{1}} p(\mathbf{n}, k)\, \omega^k \right) \mathbf{t}^{\mathbf{n}} = \prod_{\mathbf{i}\cdot\mathbf{1}\geq 1} \left(1 - \omega \mathbf{t}^{\mathbf{i}}\right)^{-1}.$$

Consequently

$$\sum_{\mathbf{n}\geq\mathbf{0}} p(\mathbf{n})\, \mathbf{t}^{\mathbf{n}} = \prod_{\mathbf{i}\cdot\mathbf{1}\geq 1} \left(1 - \mathbf{t}^{\mathbf{i}}\right)^{-1}.$$

Remark 14. If we take $t_1 = \cdots = t_r = t$, we obtain

$$\begin{aligned} \prod_{i\geq 1} \left(1 - \omega t^i\right)^{-\binom{i+r-1}{r-1}} &= \sum_{n\geq 0} \sum_{k=0}^{n} \left(\sum_{n_1+\cdots+n_r=n} \mathbf{p}(\mathbf{n}, k) \right) \omega^k t^n \\ &= \sum_{n\geq 0} \left(\sum_{n_1+\cdots+n_r=n} \mathbf{p}_{\mathbf{n}}(\omega) \right) t^n, \end{aligned}$$

and more generally, for nonnegative integers $a_1, \ldots, a_r$ and $t_1 = t^{a_1}, \ldots, t_r = t^{a_r}$, we obtain

$$\prod_{i\geq 1} \left(1 - \omega t^i\right)^{-f(i,r)} = \sum_{n\geq 0} \left(\sum_{a_1 n_1+\cdots+a_r n_r=n} \mathbf{p}_{\mathbf{n}}(\omega) \right) t^n,$$

where

$$\mathbf{p}_{\mathbf{n}}(\omega) = \sum_{k=0}^{a_1 n_1+\cdots+a_r n_r} \mathbf{p}(\mathbf{n}, k)\, \omega^k,$$

and where $f(n, r)$ is the number of solutions of the integer equation

$$a_1 n_1 + a_2 n_2 + \cdots + a_r n_r = n.$$

Acknowledgments The authors wish to express their gratitude to the referee for his/her valuable advice and comments which helped to greatly in improving the quality of the paper.

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