



CONGRUENCES OF CONCAVE COMPOSITION FUNCTIONS

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Abstract

Concave compositions are ordered partitions whose parts are decreasing towards a central part. We study the distribution modulo a of the number of concave compositions. Let $c(n)$ be the number of concave compositions of n having even length. It is easy to see that $c(n)$ is even for all $n \geq 1$. Refining this fact, we prove that

$$\#\{n < X : c(n) \equiv 0 \pmod{4}\} \gg \sqrt{X}$$

and also that for every $a > 2$ and at least two distinct values of $r \in \{0, 1, \dots, a-1\}$,

$$\#\{n < X : c(n) \equiv r \pmod{a}\} > \frac{\log_2 \log_3 X}{a}.$$

We obtain similar results for concave compositions of odd length.

1. Introduction and Statement of Results

In their 1967 paper, Parkin and Shanks [5] conjectured that the partition function $p(n)$ takes on even and odd values with equal likelihood. Very little is known about the distribution of the parity of $p(n)$. Recently, Ahlgren [1] (see also the work of Berndt, Yee, and Zaharescu [3] as well as the works referenced therein) proved that the number of integers with an even number of partitions less than X is on the order of \sqrt{X} . This improved on work of Mirsky [4] showing that about $\log \log X$ numbers less than X have partition values in some nonzero residue class modulo any integer a , for the special case $a = 2$.

A combinatorial object similar to the set of partitions is the set of concave compositions — ordered partitions whose summands decrease towards a center summand.

We break up these compositions into the following three types, as defined by Andrews in [2].

Concave compositions of even length are ordered partitions of the form $a_1 + a_2 + \dots + a_m + b_1 + b_2 + \dots + b_m$ where

$$a_1 > a_2 > \dots > a_m = b_m < b_{m-1} < \dots < b_1$$

and $a_m \geq 0$. We denote the number of concave compositions of even length of an integer n by $\text{ce}(n)$.

Concave compositions of odd length of type 1 are ordered partitions of the form $a_1 + a_2 + \dots + a_{m+1} + b_1 + b_2 + \dots + b_m$ where

$$a_1 > a_2 > \dots > a_{m+1} < b_m < b_{m-1} < \dots < b_1$$

and $a_{m+1} \geq 0$. We denote the number of concave compositions of odd length of type 1 of n by $\text{co}_1(n)$.

Finally, *concave compositions of odd length of type 2* are ordered partitions of the form $a_1 + a_2 + \dots + a_{m+1} + b_1 + b_2 + \dots + b_m$ where

$$a_1 > a_2 > \dots > a_{m+1} \leq b_m < b_{m-1} < \dots < b_1$$

and $a_{m+1} \geq 0$. We denote the number of concave compositions of odd length of type 2 of n by $\text{co}_2(n)$. Note that all concave compositions of odd length of type 1 are also of type 2.

It is natural to consider the distribution of these functions modulo a . To this end, we define

$$E_f(r, a; X) = \#\{n < X : f(n) \equiv r \pmod{a}\}.$$

Since the function $\text{co}_1(n)$ is odd exactly when n is a triangular number (see the remark following Lemma 2.1), we define the function $\text{co}'_1(n)$ by subtracting 1 from $\text{co}_1(n)$ if n is triangular and keeping it the same otherwise. Then we have the following theorem.

Theorem 1.1. *The following are true.*

(i) *There exists an explicit constant $c > 0$ such that for sufficiently large X we have*

$$E_{\text{ce}}(0, 4; X) > c\sqrt{X}.$$

(ii) *There exists an explicit constant $c > 0$ and $0 < \alpha < 1$ such that for sufficiently large X we have*

$$E_{\text{co}'_1}(0, 4; X) > \frac{X \log^\alpha X - cX}{(\sqrt{24X + 1} + 1) \log^\alpha X}.$$

(iii) *There exists an explicit constant $c > 0$ such that for sufficiently large X we have*

$$E_{\text{co}_2}(0, 2; X) > c\sqrt{X}.$$

If we consider the more general case of a modulus a , we get a result similar to that of Mirsky [4].

Theorem 1.2. *For every $a > 2$ and at least two distinct values of r among $\{0, 1, \dots, a - 1\}$, we have*

$$E_{ce}(r, a; X) > \frac{\log_2 \log_3 X}{a}, \tag{1.1}$$

$$E_{co_1}(r, a; X) > \frac{\log_2 \log_3 X}{a}, \text{ and} \tag{1.2}$$

$$E_{co_2}(r, a; X) > \frac{\log_2 \log_3 X}{a} \tag{1.3}$$

for X sufficiently large. In the cases co_1 and co_2 this also applies when $a = 2$.

Although the above bound is the best we can prove, we expect the true distributions to be much more balanced.

Conjecture. For any modulus $a \geq 2$, we have

$$E_{co_1}(r, a; X) \sim \frac{X}{a}, \text{ } a \text{ odd}$$

$$E_{co_1}(r, a; X) \sim \frac{2X}{a}, \text{ } a \text{ even, } r \text{ even}$$

$$E_{co_1}(r, a; X) \sim \frac{2\sqrt{2X}}{a}, \text{ } a \text{ even, } r \text{ odd.}$$

For ce , we expect similar asymptotics to hold, but without the last case. For co_2 , we expect uniformity across residue classes for any a .

2. Proofs

2.1. Generating Functions

When faced with combinatorial objects such as concave compositions, it is natural to consider the generating functions for each object. Andrews found q -series expansions for each type of concave composition in Theorems 1-3 of [2], which we restate here.

Lemma 2.1. *Define the generating functions $\mathcal{CE}(q) = \sum_{n=0}^{\infty} ce(n)q^n$, $\mathcal{CO}_1(q) = \sum_{n=0}^{\infty} co_1(n)q^n$, and $\mathcal{CO}_2(q) = \sum_{n=0}^{\infty} co_2(n)q^n$. Then we have*

$$\mathcal{CE}(q) = \frac{1 + \sum_{n=1}^{\infty} \left(-q^{\frac{3n^2-n}{2}} + q^{\frac{3n^2+n}{2}} \right)}{1 + \sum_{n=1}^{\infty} (-1)^n \left(q^{\frac{3n^2-n}{2}} + q^{\frac{3n^2+n}{2}} \right)} = 1 + 2q^2 + 2q^3 + 4q^4 + 4q^5 + \dots,$$

$$\mathcal{CO}_1(q) = \frac{1 + \sum_{n=1}^{\infty} (-q^{6n^2-2n} + q^{6n^2+2n})}{1 + \sum_{n=1}^{\infty} (-1)^n (q^{\frac{3n^2-n}{2}} + q^{\frac{3n^2+n}{2}})} = 1 + q + 2q^2 + 3q^3 + 4q^4 + 6q^5 + \dots,$$

$$\mathcal{CO}_2(q) = \frac{1 + \sum_{n=1}^{\infty} (q^{6n^2-8n+3} - q^{6n^2-4n+1})}{1 + \sum_{n=1}^{\infty} (-1)^n (q^{\frac{3n^2-n}{2}} + q^{\frac{3n^2+n}{2}})} = 1 + 2q + 3q^2 + 4q^3 + 7q^4 + 10q^5 + \dots.$$

Remark. It is easy to show that $\mathcal{CO}_1(q) \equiv \sum_{n=0}^{\infty} q^{n(n+1)/2} \pmod{2}$, as follows. We have by Lemma 2.1 that

$$\mathcal{CO}_1(q) = \frac{1 - \sum_{n=1}^{\infty} (q^{4n(3n-1)/2} - q^{4n(3n+1)/2})}{1 + \sum_{n=1}^{\infty} (-1)^n (q^{n(3n-1)/2} + q^{n(3n+1)/2})} \equiv (q)_{\infty}^3 \pmod{2}$$

where $(q)_{\infty} = \prod_{n=1}^{\infty} (1 - q^n)$. By Lemma 12 of [2], we have that

$$\sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} = (q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n^2} \equiv (q)_{\infty} \left(\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n} \right)^2 \equiv (q)_{\infty}^3 \pmod{2}$$

since the partitions into distinct parts are conjugate to the partitions into $1, 2, \dots, n$ missing nothing. The desired congruence follows.

Motivated by this congruence, we define $\mathcal{CO}'_1(q) = \sum_{n=0}^{\infty} co'_1(n)q^n = \mathcal{CO}_1(q) - \sum_{n=0}^{\infty} q^{n(n+1)/2}$, so that all of the coefficients of $\mathcal{CO}'_1(q)$ are even. This will ease our study of the series modulo 4. With these, we can now proceed to the proofs of our main theorems.

2.2. Proofs of Theorems 1.1 and 1.2

Combining the expansions given by Andrews [2] with a generalization of Ahlgren’s argument in [1], we prove our main theorem.

Proof of Theorem 1.1. (i) First, we show that the coefficients $ce(n)$ are all even for $n > 1$. There is a natural pairing between non-palindromic concave compositions of even length given by mirroring the sequence. There is furthermore a natural pairing between palindromic compositions given by inserting or removing a pair of zeroes at the center of the sequence. Alternatively, Lemma 2.1 makes it clear that $\mathcal{CE}(q) \equiv 1 \pmod{2}$.

Thus we have that $ce(n)$ is congruent to either 0 or 2 mod 4. From Lemma 2.1, we have the following:

$$\left(1 + \sum_{n=1}^{\infty} (-1)^n \left(q^{\frac{3n^2-n}{2}} + q^{\frac{3n^2+n}{2}} \right) \right) \left(\sum_{n=0}^{\infty} ce(n)q^n \right) = 1 + \sum_{n=1}^{\infty} \left(-q^{\frac{3n^2-n}{2}} + q^{\frac{3n^2+n}{2}} \right).$$

We will denote the right-hand side by $\sum_{n=0}^{\infty} a(n)q^n$. Thus, since $-2 \equiv 2 \pmod{4}$, we have that

$$a(n) \equiv ce(n) + ce(n-1) + ce(n-2) + ce(n-5) + \dots + ce\left(n - \frac{3k^2 \pm k}{2}\right) + \dots \pmod{4}. \tag{2.1}$$

If $a(n)$ is divisible by 4 and there are an odd number of summands, we can conclude that one of the summands is also divisible by 4. It is easy to see that there will be an odd number of summands exactly when $\frac{3k^2+k}{2} < n < \frac{3(k+1)^2-(k+1)}{2}$. As X tends to infinity, it is easy to show that the number of such $n < X$ tends to $\frac{2}{3}X$ from below very quickly, since $\frac{3(k+1)^2-(k+1)}{2}$ is almost exactly two-thirds of the way from $\frac{3k^2+k}{2}$ to $\frac{3(k+1)^2+(k+1)}{2}$.

Thus for approximately $\frac{2}{3}X$ values of n , one of the terms $ce(i)$ must be congruent to 0 modulo 4. These terms may be overcounted by the number of decompositions of the form (2.1) in which they appear. This is bounded above by twice the number of pentagonal numbers less than X , which is $\frac{\sqrt{24X+1}+1}{3}$. Thus we can conclude that we have, for some small constant ϵ ,

$$E_{ce}(0, 4; X) > \frac{(2 - \epsilon) X}{\sqrt{24X + 1} + 1},$$

as desired.

(ii) We first write out the expansion of \mathcal{CO}'_1 as in (i):

$$\begin{aligned} & \left(1 + \sum_{n=1}^{\infty} (-1)^n \left(q^{\frac{3n^2-n}{2}} + q^{\frac{3n^2+n}{2}}\right)\right) \left(\sum_{n=0}^{\infty} co'_1(n)q^n\right) \\ &= 1 + \sum_{n=1}^{\infty} \left(-q^{6n^2-2n} + q^{6n^2+2n}\right) - \sum_{j,k=1}^{\infty} \left(q^{\frac{j^2-j+3k^2-k}{2}} + q^{\frac{j^2-j+3k^2+k}{2}}\right). \end{aligned}$$

Again writing the right-hand side as $\sum_{n=0}^{\infty} a(n)q^n$, we notice that $a(n) = 0$ whenever n is not expressible as $6k^2 \pm 2k$ or as the sum of a triangular and a pentagonal number. To obtain a bound on how many such terms there are, we first notice that there are at most $\frac{\sqrt{6X+1}-1}{3}$ values of $n < X$ expressible as $6k^2 \pm 2k$. To find how many numbers less than X are expressible as the sum of a triangular and a pentagonal number, we use the result that if $Q(x, y)$ is a positive definite binary quadratic form,

$$\#\{n < X : n = Q(x, y) \text{ for some } x, y \in \mathbb{Z}\} \asymp \frac{X}{\log^\alpha X}$$

for some $0 < \alpha < 1$. For a presentation of a similar result, see Section 2 in [6].

Thus we have that there are at least $X \left(\frac{\log^\alpha X - c}{\log^\alpha X} \right)$ numbers $n < X$ such that $a(n) = 0$. Then by an argument analogous to that in (i), we can conclude that

$$E_{\text{co}'_1}(0, 4; X) > \frac{X \log^\alpha X - cX}{(\sqrt{24X + 1} + 1) \log^\alpha X}.$$

(iii) This proof is analogous to the proof of (i), the only difference being that we need to exclude the values of n for which $a(n)$ is nonzero. Thus the bound we get in this case is, for some small constant ϵ ,

$$E_{\text{co}_2}(0, 2; X) > \frac{(2 - \epsilon) X - 4 - \sqrt{6X - 2}}{\sqrt{24X + 1} + 1},$$

implying the desired result. □

We adapt the strategy of Mirsky in [4] to prove Theorem 1.2.

Proof of Theorem 1.2. Let $E_{\text{ce}}^*(r, a; X) = \#\{n \leq X : \text{ce}(n) \not\equiv r \pmod{a}\}$, and define $E_{\text{co}_1}^*(r, a; X)$ and $E_{\text{co}_2}^*(r, a; X)$ similarly. Fix r ; then, in the case of \mathcal{CE} , we claim that $E_{\text{ce}}^*(r, a; X) > \log_2 \log_3 X - C$ for some constant C .

Recall that $(q)_\infty = \prod_{n=1}^\infty (1 - q^n)$. Then we have by Lemma 2.1, considering the coefficient of $q^{\ell(3\ell+1)/2+2}$ in $(q)_\infty \mathcal{CE}(q)$, that

$$\begin{aligned} & \text{ce}(\ell(3\ell + 1)/2 + 2) + \sum_{k=1}^{\ell} (-1)^k \text{ce} \left(\frac{\ell(3\ell + 1)}{2} + 2 - \frac{k(3k - 1)}{2} \right) \\ & + \sum_{k=1}^{\ell-1} (-1)^k \text{ce} \left(\frac{\ell(3\ell + 1)}{2} + 2 - \frac{k(3k + 1)}{2} \right) + (-1)^\ell \text{ce}(2) = 0 \end{aligned}$$

for all $\ell \geq 1$. For $\ell = 2m - 1$ this becomes

$$\text{ce}((2m - 1)(3m - 1) + 2) + \sum_{k=1}^{2m-1} (-1)^k \text{ce}((2m - 1)(3m - 1) + 2 - k(3k - 1)/2) \tag{2.2}$$

$$+ \sum_{k=1}^{2m-2} (-1)^k \text{ce}((2m - 1)(3m - 1) + 2 - k(3k + 1)/2) = 2. \tag{2.3}$$

We claim that some element of $\text{ce}(2m + 1), \text{ce}(2m + 2), \dots, \text{ce}((2m - 1)(3m - 1) + 2)$ is not congruent to $r \pmod{a}$. Suppose otherwise; then the above equation gives

$$0 \equiv r + \sum_{k=1}^{2m-1} (-1)^k r + \sum_{k=1}^{2m-2} (-1)^k r \equiv 2 \pmod{a},$$

which is a contradiction. We would like to construct a sequence m_j so that the terms appearing in Equation 2.2 for m_j and m_i do not overlap for $i \neq j$, hence giving a value of ce not congruent to $r \pmod{a}$ for each j . Since the lowest-indexed term in Equation 2.2 for m_j is $2m_j + 1$ and the highest is $(2m_j - 1)(3m_j - 1) + 2$, it suffices to set $2m_j + 1 > (2m_{j-1} - 1)(3m_{j-1} - 1) + 2$, or $2m_j > 6m_{j-1}^2 - 5m_{j-1} + 2$. Hence we can choose $m_1 = 1$, $m_j = 3m_{j-1}^2$, so that $m_j = 3^{2^j - 1 - 1}$. This gives $E_{ce}^*(r, a; 3^{2^j - 1 - 1}) \geq j$ for all j . Setting $j = \lfloor \log_2 \log_3 X \rfloor$ gives $E_{ce}^*(r, a; X) \geq \log_2 \log_3 X - C$, as desired.

Since $E_{ce}^*(0, a; X) = \sum_{r' \neq 0} E_{ce}(r', a; X)$, we can write

$$\sum_{r' \neq 0} E_{ce}(r', a; X) > \log_2 \log_3 X - C,$$

from which there is some r_1 so that

$$E_{ce}(r_1, a; X) > \frac{1}{a-1} \log_2 \log_3 X - C > \frac{1}{a} \log_2 \log_3 X$$

for X sufficiently large. Then we also have $\sum_{r' \neq r_1} E_{ce}(r', a; X) = E_{ce}^*(r_1, a; X) > \log_2 \log_3 X - C$, giving some $r_2 \neq r_1$ so that $E_{ce}(r_2, a; X) > \frac{1}{a} \log_2 \log_3 X$. This gives us two residues, r_1 and r_2 , with the desired bound in the \mathcal{CE} case.

For \mathcal{CO}_1 and \mathcal{CO}_2 , the same argument applies almost verbatim, with the remark that we use the coefficient of $q^{(2m-1)(3m-1)}$ in $(q)_\infty \mathcal{CO}_1(q)$, $(q)_\infty \mathcal{CO}_2(q)$ respectively, rather than $q^{(2m-1)(3m-1)+2}$. In both cases we can guarantee that this coefficient must equal 0. In the case \mathcal{CO}_1 the difference between $(2m-1)(3m-1)$ and the nearest exponent of a nonzero coefficient is $3m-1$; in the case \mathcal{CO}_2 it is m . Noting that $co_1(0) = co_2(0) = 1 \neq 0$, the rest of the argument follows. \square

References

[1] S. Ahlgren, *Distribution of the Parity of the Partition Function in Arithmetic Progressions*, Indag. Math. N. S. 10 (2), (1999), 172-181.
 [2] G. Andrews, *Concave Compositions*, Electronic J. of Comb. 18 (2011), P6.
 [3] B. C. Berndt, A. J. Yee, A. Zaharescu, *On the Parity of Partition Functions*, Int. J. Math. 14 (2003), 437-459.
 [4] L. Mirsky, *The Distribution of Values of the Partition Function in Residue Classes*, J. of Math. Analysis and Applications 93 (1983), 593-598.
 [5] T. R. Parkin and D. Shanks, *On the distribution of parity in the partition function*, Math. Comp. 21 (1967), 466-480.
 [6] J.-P. Serre, *Divisibilitie de certaines fonctions arithmetiques*, L'Ens. Math. 22 (1976) 227-260.